

**EE381V: Convex Optimization — Fall 2011**

SOME NOTES ON THE RELATIVE INTERIOR

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This note covers some topological aspects of convex sets in  $\mathbb{R}^k$ . The material is drawn for the most part from Hiriart-Urruty and Lemaréchal, and from Rockafellar.

To motivate us, recall the linear program in standard form:

$$\begin{aligned} \min : & \quad c^\top x \\ \text{s.t.} : & \quad Ax = b \\ & \quad x \geq 0. \end{aligned}$$

Let  $C$  be the feasible set of this LP. We have mentioned repeatedly that optimization and duality are intimately related to the idea of boundary, separation, and interior of a set. What is the interior of the feasible set  $C$  above? Unfortunately, it is not particularly interesting: it is empty. But indeed, the interior of  $C$  in the topology of  $\mathbb{R}^k$ , is the wrong object to look at. Much more natural, is to consider the interior of the set in the relative topology of the smallest affine manifold that contains  $C$ , namely, the affine hull of  $C$ .

Let us recall the definition given in class on Wednesday, of affine hull.

**Definition 1.** *The **affine hull** of a set  $C$  is defined as the intersection of all affine manifolds that contain the set  $C$ , or alternatively, it is the set that contains all affine combinations:*

$$\text{aff } C = \left\{ x = \sum_{i=1}^m \alpha_i x_i : \sum_i \alpha_i = 1, x_i \in C \right\}.$$

In the context of the LP example above, the affine hull of  $C$  is defined by the equality constraints, so  $\text{aff } C = \{x \in \mathbb{R}^k : Ax = b\}$ .

This motivates the following.

**Definition 2.** *The **relative interior** of a convex set  $C \subseteq \mathbb{R}^k$ , is denoted  $\text{ri } C$ , and is the interior relative to the topology induced by the affine hull of  $C$ . Thus, a point  $x \in C$  is in the interior of  $C$  iff  $\exists \varepsilon > 0$  such that  $B_\varepsilon(x) \subseteq C$ . A point  $x$  is in the relative interior of  $C$  iff  $\exists \varepsilon > 0$  such that  $B_\varepsilon(x) \cap \text{aff } C \subseteq C$ .*

Furthermore, we have,

**Definition 3.** *The **dimension** of a convex set is the dimension of its affine hull.*

We now give the following important result. Note that this is not true for non-convex sets. We will see later on in the course that this can also fail for infinite dimensional convex sets.

**Proposition 1.** *Let  $C \subseteq \mathbb{R}^k$  be a nonempty convex set. Then it has a nonempty relative interior. Moreover, the dimension of  $\text{ri } C$  is equal to the dimension of the original convex set  $C$ .*

PROOF. Let  $d$  be the dimension of  $\text{aff } C$ . By definition, this means that  $C$  contains a  $d$ -dimensional simplex, i.e.,  $C$  contains  $d+1$  affinely independent points,  $\{x_0, \dots, x_d\}$ . Let  $\Delta$  denote  $\text{conv}\{x_0, \dots, x_d\}$ . Note that  $\Delta \subseteq C$ , and  $\text{aff } \Delta = \text{aff } C$ , since the dimension of  $\text{aff } \Delta$  is  $d$ . We have left to show that  $\text{ri } \Delta$  is nonempty. Let  $\bar{x}$  be the centerpoint of  $\Delta$ , i.e.,

$$\bar{x} = \frac{1}{d+1} \sum_{i=0}^d x_i.$$

Now, the set  $(\text{aff } \Delta - \bar{x})$  is a  $k$ -dimensional subspace (a  $k$ -dimensional affine manifold passing through the origin). In particular, this means that given any  $y \in (\text{aff } \Delta - \bar{x})$ , the set of equations:

$$y = \sum_{i=0}^d \alpha_i x_i, \quad \sum_{i=0}^d \alpha_i = 0,$$

has a unique solution. Therefore the mapping from  $y \in (\text{aff } \Delta - \bar{x})$  to the coefficients  $\alpha_i$  is linear and in particular continuous. In particular, if we choose  $\|y\|$  small enough, we can guarantee that the absolute value of each coefficient,  $\alpha_i$  is as small as we please, say, at most  $1/(d+1)$ . If this is the case, then in fact  $\bar{x} + y \in \Delta$ . To see this last fact, we have:

$$\begin{aligned} \bar{x} + y &= \bar{x} + \sum_i \alpha_i x_i \\ &= \sum_i \left( \frac{1}{d+1} + \alpha_i \right) x_i. \end{aligned}$$

If  $|\alpha_i| \leq 1/(d+1)$ , then  $(\frac{1}{d+1} + \alpha_i) \geq 0$ , and since  $\sum_i \alpha_i = 0$ , the coefficients sum to 1 and therefore  $(\bar{x} + y)$  is a convex combination of the points  $\{x_0, \dots, x_m\}$ , i.e.,  $\bar{x} + y \in \Delta$ .

Thus we have shown: there exists  $\varepsilon > 0$  (this is what “choose  $\|y\|$  small enough” means) such that  $B_\varepsilon(\bar{x}) \cap \text{aff } \Delta \subseteq \Delta$ .  $\square$

In problem 3 of Problem set 1, you are asked to show that the interior of a convex set is again convex, i.e., if you move from one interior point to the other, you stay within the interior of the set. We can refine this further: If we move from an interior point to a closure point, then that entire interval (except perhaps for the final point) is in the interior of the set. That is, we have the following:

**Proposition 2.** *Let  $C \subseteq \mathbb{R}^k$  be a convex set. For  $x_1 \in \text{cl } C$ , and  $x_2 \in \text{ri } C$ , then*

$$\{\alpha x_1 + (1 - \alpha)x_2 : 0 \leq \alpha < 1\} \subseteq \text{ri } C.$$

PROOF. The proof is left as a homework exercise.  $\square$

This result is particularly useful for proving the following two results.

**Proposition 3.** *Let  $C \subseteq \mathbb{R}^k$  be a convex set. Then the sets  $\text{ri } C \subseteq C \subseteq \text{cl } C$  have the same affine hull (proven above), the same relative interior, and the same closure.*

This says, in particular, that if two convex sets  $C_1$  and  $C_2$  have the same closure, then they must also have the same relative interior. Note that this is certainly not true in the absence of convexity. (Come up with an example to convince yourself of this).

The proof is left as a voluntary exercise.

**Proposition 4.** *Let  $C_1, C_2 \subseteq \mathbb{R}^k$  be convex. If  $\text{ri } C_1 \cap \text{ri } C_2 \neq \emptyset$ , then we have*

$$\begin{aligned}\text{ri}(C_1 \cap C_2) &= \text{ri } C_1 \cap \text{ri } C_2 \\ \text{cl}(C_1 \cap C_2) &= \text{cl } C_1 \cap \text{cl } C_2.\end{aligned}$$

The proof of this is left as a voluntary exercise.

Consider Proposition 1 above. The bulk of the proof is devoted to showing that  $\Delta$  has a nonempty relative interior. Suppose, however, that we could assume that the points  $\{x_0, x_1, \dots, x_d\}$  were in fact the points  $\{0, e_1, \dots, e_d\}$ , i.e., the origin and the first  $d$  standard basis vectors. Then deducing a nonempty relative interior would be a straightforward matter. Indeed, what is needed to take such a simplified approach, is the result that affine mappings do not change the relative interior. This is exactly the content of the next result.

**Proposition 5.** *Let  $C \subseteq \mathbb{R}^n$  be a convex set. For  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  an affine mapping, we have*

$$\text{ri}[A(C)] = A(\text{ri } C).$$

PROOF. The proof makes heavy use of the results given above. First, we prove that  $\text{ri}[A(C)] \subseteq A(\text{ri } C)$ . We do this by showing that  $A(\text{ri } C)$  and  $A(C)$  have the same closure, and hence by the proposition above, they must also have the same interior. Therefore  $\text{ri } A(C) = \text{ri}[A(\text{ri } C)] \subseteq A(\text{ri } C)$ .

Thus we have to show that  $A(\text{ri } C)$  and  $A(C)$  have the same closure. Since  $A$  is continuous, we have  $A(\text{cl } U) \subseteq \text{cl}[A(U)]$ , for any set  $U$ . Then take  $U = \text{ri } C$  to conclude the desired result.

The reverse inclusion is left as a voluntary exercise.

Also a voluntary exercise is the same result for  $A$  replaced by the inverse mapping. □