The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381V: Convex Optimization — Fall 2009

Problem Set One

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The point of this problem set is to provide more exposure to and exercise with, the geometry of convex sets. Also, this will fill in holes left during the lecture. Problems marked by a '$*$' *do not need to be handed in.*

1. $*$ Let $f_n : \mathbb{R}^k \to \mathbb{R}$ converge pointwise and uniformly to a continuous function f. Let $\boldsymbol{x}_n \in \mathbb{R}^k$ converge to $\boldsymbol{x} \in \mathbb{R}^k$. Show that $f_n(\boldsymbol{x}_n)$ converges to $f(x)$.

2. $*$ If $C \subseteq U \subseteq \mathbb{R}^k$ is a compact set, and $f : U \to \mathbb{R}$ a continuous function, show that $f(C) \subseteq V$ is also compact; then show that f attains its minimum and maximum on C.

3. $*$ Let $\{C_n\}_{n=1}^{\infty}$ be a countable collection of compact subsets of $\mathbb{R}^k$, endowed with the standard topology. Show that if $C_{n_1} \cap \cdots \cap C_{n_k} \neq \emptyset$, for any finite subcollection $(n_1, \ldots, n_k)$, then in fact
$$\bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$
Show by example that this need not hold if the sets C_i are not compact.

4. Dual Cones
   (a) Let $L \subseteq \mathbb{R}^k$ be a subspace. Define
   $$L^* = \{\boldsymbol{u} : \langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq 0 \quad \forall \boldsymbol{v} \in L\}.$$
   Characterize the set L^*, and show that $(L^*)^* = L$.
   (b) Set $\mathbb{S}^n$ denote the set of $n \times n$ symmetric matrices, and $\mathbb{S}^n_+ \subseteq \mathbb{S}^n$ the set of positive semidefinite symmetric matrices. Define an inner product on $\mathbb{S}^n$, by
   $$\langle A, B \rangle = \mathrm{Tr}(AB) = \sum_{i,j} A_{ij} B_{ij}.$$
   Compute $(\mathbb{S}^n_+)^*$, defined as in the first part of the question above.

5. $*$ Convexity and Topology:
   (a) Show that if $S \subseteq \mathbb{R}^d$ is compact, then so is $\mathrm{conv}(S)$.
   (b) Show that if $S \subseteq \mathbb{R}^2$ is closed, then $\mathrm{conv}(S)$ need not be closed.
   (c) Show that if $S \subseteq \mathbb{R}^d$ is open, then $\mathrm{conv}(S)$ is also open.

6. Let $S \subseteq V$ be a set, and take two points $u, v \notin \mathrm{conv}(S)$. Show that if $u \in \mathrm{conv}(S \cup \{v\})$, and $v \in \mathrm{conv}(S \cup \{u\})$, then $u = v$.

7. Consider a univariate degree k polynomial $f(z)$, with $k \geq 1$, $f(z)$. Suppose $f(z)$ has roots $z_1, \ldots, z_k \in \mathbb{C}$ (not necessarily distinct) where we write $z_d = x_d + iy_d$. Show that if $\hat{z} = \hat{x} + i\hat{y}$ is a root of the $(k-1)$-degree polynomial $f'(z)$, then $(\hat{x}, \hat{y}) \in \text{conv}\{(x_1, y_1), \ldots, (x_k, y_k)\}$.[1]

8. $*$ For A a $n \times n$ square matrix, recall that the characteristic polynomial is defined as:

$$p_A(\lambda) \stackrel{\triangle}{=} \det(\lambda I - A),$$

where I is the $n \times n$ identity matrix. Prove the Cayley-Hamilton theorem, that states that A satisfies its own characteristic equation, i.e., show that $p_A(A)$ is the all-zeros matrix.

Hint: using the Jordan canonical form should help...

9. (Boyd and Vandenberghe, Ex. 2.10) Consider the set

$$C = \{x \in \mathbb{R}^n \, : \, x^\top A x + b^\top x + c \leq 0\},$$

where $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

   (a) Show that if $A \in \mathbb{S}^n_+$ (i.e., A is positive semidefinite) then the set C is convex.

   (b) Consider the set obtained by intersecting C with a hyperplane:

$$C_1 = C \cap \{x \, : \, g^\top x + h = 0\}.$$

   Show that C_1 is convex if there exists $\lambda \in \mathbb{R}$ such that $(A + \lambda g g^\top) \in \mathbb{S}^n_+$.

10. $*$ In class we claimed that there are several natural operations on sets, that preserve convexity. Convince yourselves that the following all preserve convexity.

   (a) Cartesian product: If $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ are convex sets, then the set

$$C \stackrel{\triangle}{=} C_1 \times \cdots \times C_m = \{(x_1, \ldots, x_m), \; x_i \in C_i\}$$

   is convex.

   (b) Affine and inverse maps: For $C \subseteq \mathbb{R}^n$ convex, and $A : \mathbb{R}^n \to \mathbb{R}^m$ a linear operator (i.e., an $m \times n$ matrix) then **show that** the following two sets are convex:

$$\begin{aligned} D_1 &\stackrel{\triangle}{=} \{Ax \, : \, x \in C\} \\ D_2 &\stackrel{\triangle}{=} \{x \, : \, Ax \in C\}. \end{aligned}$$

   (c) Minkowski sum: If $C_1, C_2 \subseteq \mathbb{R}^n$ are convex, **show that**

$$C \stackrel{\triangle}{=} C_1 + C_2 \stackrel{\triangle}{=} \{x = x_1 + x_2 \, : \, x_1 \in C_1, x_2 \in C_2\}$$

   is convex.

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[1] This exercise taken from Barvinok's *A Course in Convexity.*

11. The Asymptotic Cone, also called the Recession Cone.

    Unbounded convex sets are not that much more difficult to deal with than compact convex sets. The fundamental reason for this is that convex sets can only be unbounded in very particular ways. This exercise explores this.

    Let $C \subseteq \mathbb{R}^d$ be a closed convex set. We define the directions of recession from a particular point $x \in C$ as follows:

$$C_\infty(x) \overset{\triangle}{=} \{d \in \mathbb{R}^n \,:\, x + td \in C,\, \forall t > 0\} = \bigcap_{t>0} \frac{C - x}{t}.$$

    (a) Show that $C_\infty(x)$ is a closed convex cone.
    (b) Now, show that $C_\infty(x)$ does not depend on $x \in C$. One way to do this is to take $x_1, x_2 \in C$, and then show that for $d \in C_\infty(x_1)$ and $t > 0$, we have $x_2 + td \in C$. (Note that C is closed).
    (c) Consider the convex set

$$C = \{(x, r) \in \mathbb{R}^2 \,:\, r \geq x^2\}.$$

    Find C_∞.

    Therefore, thanks to the result of part (b) of this exercise, we can simply define: The set C_∞ is defined as the *asymptotic cone* of the closed convex set C.

12. Show that a closed convex set C is compact if and only if $C_\infty = \{0\}$.

13. The Relative Interior of a convex set $C \subseteq \mathbb{R}^d$.

    (Read the handout on relative interiors). Prove the following important refinement to Problem 3 above (Proposition 2 in the handout):

    Let $C \subseteq \mathbb{R}^k$ be a convex set. For $x_1 \in \operatorname{cl} C$, and $x_2 \in \operatorname{ri} C$, then

$$\{\alpha x_1 + (1 - \alpha)x_2 \,:\, 0 \leq \alpha < 1\} \subseteq \operatorname{ri} C.$$

14. $*$ Fill in the missing proofs in the handout on the relative interior.