

EE381V: Convex Optimization — Fall 2009

PROBLEM SET TWO

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Due: Monday, September 21, 2009.

The point of this problem set is to provide more exposure to and exercise with, the geometry of convex sets, and also to play with Helly's theorem. Also, this will fill in holes left during the lecture.¹ **Do any 5 (or more) of the following 8 problems.** Note that for some of the problems, you will need the material we will cover on Monday and Wednesday, but there should be enough to keep you busy in the mean time.

1. Suppose we have m parallel line segments in the plane with the property that for any collection of three of them, there is a straight line that intersects all three. Show that there is a straight line that intersects all of them.
2. Let $C \subseteq \mathbb{R}^d$ be a compact convex set. Show that there is a point $u \in \mathbb{R}^d$ such that $(-1/d)C + u \subseteq C$.
3. It is a fact (you can try to prove this for yourselves) that if $A \subseteq \mathbb{R}^d$ is algebraically open, then it is open (the converse is obvious). Produce an example of a non-convex set A , such that $A \cap L$ is open in L for every straight line L , but A is not open.
4. Consider the infinite dimensional vector space of all polynomials with real coefficients. Recall that a polynomial has finitely many terms. (You can identify elements of this vector space with elements of " \mathbb{R}^∞ " where all but finitely many elements of the infinite vector are zero). Let A be the subset that contains all polynomials whose highest order term has a strictly positive coefficient.
 - Show that A is convex, and that $0 \notin A$.
 - Show that A is not algebraically closed.
 - Show that there is no hyperplane H such that $0 \in H$ and such that A is supported by H , i.e., $A \subseteq \overline{H}_-$ or $A \subseteq \overline{H}_+$.
5. Consider the sets

$$\begin{aligned} \text{(monotone nonnegative cone)} \quad K_{m+} &\triangleq \{\mathbf{x} \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\} \\ \text{(copositive matrix cone)} \quad K_{cp} &\triangleq \{M \in \mathbb{S}^n : \mathbf{x}^\top M \mathbf{x} \geq 0, \forall \mathbf{x} \geq 0\}. \end{aligned}$$

- (a) Show that K_{m+} and K_{cp} are both convex cones.
- (b) Compute the dual cone for each of K_{m+} , and K_{cp} , where recall the dual cone K^* of a cone K is defined as

$$K^* \triangleq \{\mathbf{x} : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0 \forall \mathbf{y} \in K\}.$$

¹Some of these problem taken from / inspired by the books of Barvinok; Hiriart-Urruty and Lemaréchal; and Boyd and Vandenberghe.

The following hints may help: (i) Recall the definition of the inner product for the copositive cone from past homeworks (same as for the semidefinite cone). (ii) Furthermore for the copositive cone, it might help to note that $K_{cp} \supseteq \mathbb{S}_+^n$, (the semidefinite matrices) and therefore $(K_{cp})^* \subseteq (\mathbb{S}_+^n)^* = \mathbb{S}_+^n$. (iii) For the monotone cone, use the identity

$$\sum_{i=1}^n x_i y_i = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \cdots + x_n(y_1 + \cdots + y_n).$$

6. Let $C_1 \subseteq C_2 \subseteq \mathbb{R}^n$ be closed convex sets.
- Show that if C_1, C_2 are subspaces, then $p_{C_1} \circ p_{C_2} = p_{C_1}$, i.e., if you project first onto C_2 and then onto C_1 , you get the same result as if you project directly onto the smaller set C_1 .
 - Show that this still holds when the larger set, C_2 is affine, but now C_1 is an arbitrary closed convex set. In particular this shows that projecting first onto $\text{aff}C$ and then onto C is the same as projecting directly onto C .
 - Show that the result fails when $C_1 \subseteq C_2$ are closed convex sets.
7. Show that injective (1-to-1) linear mappings preserve extreme points, while general linear maps do not have this property. So, show that if $C \subseteq \mathbb{R}^n$ is a convex set, and $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ an injective affine mapping, then if $x \in \text{Ext}C$, we must have $Ax \in \text{Ext}AC$.
8. Recall that we proved in class the following version of the Hahn-Banach theorem in finite dimensions: If $C \subseteq \mathbb{R}^n$ is closed and convex, and $x \notin C$, then there exists a vector $s \in \mathbb{R}^n$ such that

$$\langle s, x \rangle > \sup\{\langle s, y \rangle : y \in C\}.$$

Then, we defined the *support function* of a set C to be the right hand side of the above:

$$\sigma_C(s) \triangleq \sup\{\langle s, y \rangle : y \in C\}.$$

The support function is very important, and we will see it again later in the class. In the meantime:

- Show that the support function is sublinear, i.e., it is convex² and positively homogeneous: $\sigma_C(ts) = t\sigma_C(s)$.
- Assume that C is a closed convex set. Compute the function

$$i_C(x) \triangleq \sup_{s \in \mathbb{R}^n} \{\langle s, x \rangle - \sigma_C(s)\}.$$

Now compute the function

$$i_C^*(s) \triangleq \sup_{x \in \mathbb{R}^n} \{\langle x, s \rangle - i_C(x)\}.$$

²We have not officially defined convex functions yet, so here is the definition: a function $f : C \rightarrow \mathbb{R}$ is convex if for any $x_1, x_2 \in C$, and any $\lambda \in (0, 1)$, we have $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$.