The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381V: Convex Optimization — Fall 2009

PROBLEM SET TWO SOLUTIONS

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The point of this problem set is to provide more exposure to and exercise with, the geometry of convex sets, and also to play with Helly's theorem. Also, this will fill in holes left during the lecture.[1] **Do any 5** (or more) **of the following 8 problems**. Note that for some of the problems, you will need the material we will cover on Monday and Wednesday, but there should be enough to keep you busy in the mean time.

1. Suppose we have m parallel line segments in the plane with the property that for any collection of three of them, there is a straight line that intersects all three. Show that there is a straight line that intersects all of them.

   Solution: Suppose, rotating coordinates if necessary, that the line segments in the plane are parallel to the y-axis (i.e., vertical) and let the i^{th} segment be given by $\{(x_i, y_i) : y_i \in [\underline{c}_i, \overline{c}_i]\}$ for $\underline{c}_i < \overline{c}_i$. Note that if a vertical line intersects any two of them, then it intersects (in fact contains) all of them, in which case we are done. Therefore assume that no vertical line intersects any two, and that hence the straight lines intersecting any three, are all non-vertical. In particular, these are affine functions, and hence can be parameterized by their slope and intercept: $y = ax + b$. Now define the sets:

   $$H_i = \{(a, b) \in \mathbb{R}^2 : ax_i + b \in [\underline{c}_i, \overline{c}_i]\},$$

   i.e., H_i is the set of all affine functions intersecting line segment i. It is easy to check that the sets $H_i \in \mathbb{R}^2$ are convex. From here, Helly's theorem immediately applies and we conclude that $H_1 \cap \cdots \cap H_m$ is non-empty.

2. Let $C \subseteq \mathbb{R}^d$ be a compact convex set. Show that there is a point $u \in \mathbb{R}^d$ such that $(-1/d)C + u \subseteq C$.

   Solution: For every $x \in C$, define the set:

   $$A_x = \{u \in \mathbb{R}^d : (-1/d)x + u \in C\}.$$

   The sets A_x are convex and compact because C is. Now consider any $(d+1)$ of these sets, A_{x_i}, for $x_1, \ldots, x_{d+1} \in C$. We show that their intersection is non-empty. Indeed, it contains the point:

   $$u = \frac{1}{d} \sum_{i=1}^{d+1} x_i,$$

   which follows by convexity of C. By Helly's theorem, any finite collection has a non-empty intersection. But then since they are all compact, they must all have a common point (this is the finite intersection property of compact sets).

[1] Some of these problem taken from / inspired by the books of Barvinok; Hiriart-Urruty and Lemaréchal; and Boyd and Vandenberghe.

3. It is a fact (you can try to prove this for yourselves) that if $A \subseteq \mathbb{R}^d$ is algebraically open, then it is open (the converse is obvious). Produce an example of a non-convex set A, such that $A \cap L$ is open in L for every straight line L, but A is not open.

   Solution: The intuition here is that if A is not open, then it contains a point that is the limit point of points in its complement. No straight line can contain an infinite number of these points, since otherwise $A \cap L$ would also contain such a limit point, and hence not be open. Therefore we need to construct a set with a point that is not in the interior, that can only be approached along non-straight lines (from the complement). For example, let:

$$B = \{(x, y) \, : \, y = x^2, \; x \neq 0\}.$$

   So B is the parabola with the origin removed. Now let $A = \mathbb{R}^2 \setminus B$. It is not open, since $(0, 0) \in A$, but clearly not in the interior of A. On the other hand, for any straight line L, $A \cap L$ is the intersection of one, two, or three open intervals.

4. Consider the infinite dimensional vector space of all polynomials with real coefficients. Recall that a polynomial has finitely many terms. (You can identify elements of this vector space with elements of "$\mathbb{R}^\infty$" where all but finitely many elements of the infinite vector are zero). Let A be the subset that contains all polynomials whose highest order term has a strictly positive coefficient.

   - Show that A is convex, and that $0 \notin A$.
   - Show that A is not algebraically open.
   - Show that there is no hyperplane H such that $0 \in H$ and such that A is supported by H, i.e., $A \subseteq \overline{H}_-$ or $A \subseteq \overline{H}_+$.

   Solution:

   - Obvious.
   - A line is given by $L = \{u + \tau v \, : \, \tau \in \mathbb{R}\}$, for any $u, v \in V$. Let $u = 1$, and $v = x$. Then we have:

$$A \cap L = \{1 + \tau x \, : \, \tau \in [0, \infty)\},$$

     which is a closed interval. Therefore A is not algebraically open.
   - For f a linear functional and r a scalar, the associated hyperplane is given by

$$H_{f,r} = \{v \in V \, : \, f(v) = r\}.$$

     If $0 \in H_{f,r}$, we must have $r = 0$. Observe that since $A \subseteq \overline{H}^+_{f,0}$ (say), then by linearity, for any monomial x^n, we have $f(x^n) = 0$. This is means that $f(v) = 0$ for any $v \in V$.

5. **For the solutions to the below, please see (recycled) solution set part II**

6. Consider the sets

$$\text{(monotone nonnegative cone)} \qquad K_{m+} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n \, : \, x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$
$$\text{(copositive matrix cone)} \qquad K_{cp} \triangleq \{M \in \mathbb{S}^n \, : \, \boldsymbol{x}^\top M \boldsymbol{x} \geq 0, \; \forall \boldsymbol{x} \geq 0\}.$$

   (a) Show that K_{m+} and K_{cp} are both convex cones.

(b) Compute the dual cone for each of K_{m+}, and K_{cp}, where recall the dual cone K^* of a cone K is defined as

$$K^* \triangleq \{\boldsymbol{x} : \langle \boldsymbol{x}, \boldsymbol{y} \rangle \geq 0 \; \forall \boldsymbol{y} \in K\}.$$

The following hints may help: (i) Recall the definition of the inner product for the copositive cone from past homeworks (same as for the semidefinite cone). (ii) Furthermore for the copositive cone, it might help to note that $K_{cp} \supseteq \mathbb{S}_+^n$, (the semidefinite matrices) and therefore $(K_{cp})^* \subseteq (\mathbb{S}_+^n)^* = \mathbb{S}_+^n$. (iii) For the monotone cone, use the identity

$$\sum_{i=1}^n x_i y_i = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \cdots + x_n(y_1 + \cdots + y_n).$$

7. Let $C_1 \subseteq C_2 \subseteq \mathbb{R}^n$ be closed convex sets.

   (a) Show that if C_1, C_2 are subspaces, then $p_{C_1} \circ p_{C_2} = p_{C_1}$, i.e., if you project first onto C_2 and then onto C_1, you get the same result as if you project directly onto the smaller set C_1.

   (b) Show that this still holds when the larger set, C_2 is affine, but now C_1 is an arbitrary closed convex set. In particular this shows that projecting first onto aff$C$ and then onto C is the same as projecting directly onto C.

   (c) Show that the result fails when $C_1 \subseteq C_2$ are closed convex sets.

8. Show that injective (1-to-1) linear mappings preserve extreme points, while general linear maps do not have this property. So, show that if $C \subseteq \mathbb{R}^n$ is a convex set, and $A : \mathbb{R}^n \to \mathbb{R}^m$ an injective affine mapping, then if $x \in \text{Ext}C$, we must have $Ax \in \text{Ext}AC$.

9. Recall that we proved in class the following version of the Hahn-Banach theorem in finite dimensions: If $C \subseteq \mathbb{R}^n$ is closed and convex, and $x \notin C$, then there exists a vector $s \in \mathbb{R}^n$ such that

$$\langle s, x \rangle > \sup\{\langle s, y \rangle \, : \, y \in C\}.$$

Then, we defined the *support function* of a set C to be the right hand side of the above:

$$\sigma_C(s) \triangleq \sup\{\langle s, y \rangle \, : \, y \in C\}.$$

The support function is very important, and we will see it again later in the class. In the meantime:

   (a) Show that the support function is sublinear, i.e., it is convex[2] and positively homogeneous: $\sigma_C(ts) = t\sigma_C(s)$.

   (b) Assume that C is a closed convex set. Compute the function

$$i_C(x) \triangleq \sup_{s \in \mathbb{R}^n} \{\langle s, x \rangle - \sigma_C(s)\}.$$

   Now compute the function

$$i_C^*(s) \triangleq \sup_{x \in \mathbb{R}^n} \{\langle x, s \rangle - i_C(x)\}.$$

[2]We have not officially defined convex functions yet, so here is the definition: a function $f : C \to \mathbb{R}$ is convex if for any $x_1, x_2 \in C$, and any $\lambda \in (0, 1)$, we have $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$.