PS2 Solutions
Thursday, October 11, 2007
7:39 PM

1. $C_1 \subseteq C_2 \subseteq \mathbb{R}^n$

(a) We have to show that if C_1, C_2 are subspaces, then

$$P_{C_1} \circ P_{C_2} = P_{C_1}.$$

pf: Take an o.n.b. for C_1: $v_1, \ldots, v_m$
and extend it to form an o.n.b. for C_2: $v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+r}$.

Extend this to form an onb for $\mathbb{R}^n$, and then observe that P_{C_1}, P_{C_2} are simultaneously diagonalized for this basis, and have the form:

$$M_{P_{C_1}} = n\left\{\begin{bmatrix} \underbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}_{m} & \\ & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} \end{bmatrix}\right. \qquad M_{P_{C_2}} = \begin{bmatrix} \underbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}_{m+r} & \\ & \begin{matrix}0 & & \\ & \ddots & \\ & & 0\end{matrix} \end{bmatrix}.$$

The result follows.

(b) Next we assume that the larger set C_2 is affine, but the set C_1 is an arbitrary closed convex set.

pf: This follows by the Pythagorian theorem.

WLOG, let C_2 be the subspace spanned by the first k standard basis vectors in $\mathbb{R}^n$ $(k \leq n)$.

$$\Rightarrow y \in C_1 \Rightarrow y = (y_1, 0), \quad y_1 \in \mathbb{R}^k.$$

Then, for $x \notin C_2$, (if $x \in C_2$, the result is immediate)
let us write: $x = (x_1, x_2)$, where $x_1 \in \mathbb{R}^k$, $x_2 \in \mathbb{R}^{n-k}$.

$$\Rightarrow P_{C_1}(x) = \operatorname{argmin} \|(x_1, x_2) - (y_1, 0)\| \quad \text{st}: (y_1, 0) \in C_2$$

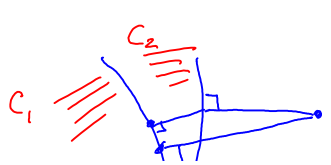
$$= \operatorname{argmin} \|x_1 - y_1\| \quad \text{st}: (y_1, 0) \in C_2$$

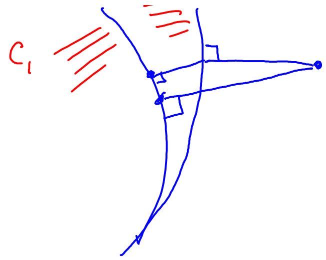
$$= \operatorname{argmin} \|P_{C_2}(x_1, x_2) - y_1\| \quad \text{st} \ (y_1, 0) \in C_2$$

$$= P_{C_1} \circ P_{C_2}(x).$$

(c) For $C_1 \subseteq C_2 \subseteq \mathbb{R}^n$ closed convex sets, the result fails in general.

A picture will suffice to illustrate this point





(2) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be an injective map.

Let C be a convex set in $\mathbb{R}^n$, with $x \in \text{ext}\, C$. Then we show that $Ax \in \text{ext}(AC)$.

~~pf~~: Suppose not. Then $\exists\, y_1, y_2 \in AC \subseteq \mathbb{R}^m$ with $\frac{1}{2}(y_1+y_2) = Ax$.

But $y_1, y_2 \in AC \Rightarrow \exists\, x_1, x_2 \in C$ with $Ax_i = y_i$, $i = 1,2$. $(x_1 \neq x_2)$

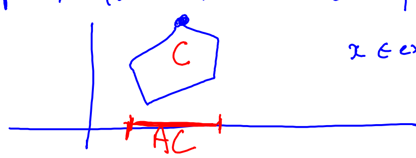
Since A is injective, $Ax' = Ax''$ iff $x' = x''$

$\Rightarrow \frac{1}{2}(y_1+y_2) = A\left(\frac{1}{2}(x_1+x_2)\right) = Ax$ iff $\frac{1}{2}(x_1+x_2) = x$

which is a contradiction.

For an example where the result fails in the absence of the injective assumption, let A be a projection operator.

Ex:



$x \in \text{ext}\, C$, $Ax \notin \text{ext}(AC)$.

(3) The support function of a convex set C

$\sigma_C(s) \triangleq \sup\{\langle s, x\rangle \mid x \in C\}$

(i) First we show it is sublinear.

It is clearly +'vely homogeneous, by the linearity of the inner product.

It is the supremum of affine functions, and hence it is convex.

Note that in general, the supremum of convex functions is also convex:

$f_S(x) \triangleq \sup\{f_\alpha(x) \mid \alpha \in S\}$, where f_α is convex.

Then $\forall\, \lambda \in (0,1)$, $x_1, x_2 \in \text{dom}\, f_S$, we have:

$$
\begin{aligned}
f_S(\lambda x_1 + (1-\lambda)x_2) &= \sup\{f_\alpha(\lambda x_1 + (1-\lambda)x_2) \mid \alpha \in S\} \\
&\le \sup\{\lambda f_\alpha(x_1) + (1-\lambda) f_\alpha(x_2) \mid \alpha \in S\} \\
&\le \sup\{\lambda f_\alpha(x_1) + (1-\lambda) f_\beta(x_2) \mid \alpha, \beta \in S\} \\
&= \lambda f_S(x_1) + (1-\lambda) f_S(x_2).
\end{aligned}
$$

$$= \lambda f_S(x_1) + (1-\lambda) f_S(x_2).$$

(ii) $$i_C(x) \triangleq \sup_{s \in \mathbb{R}^n} \{ \langle s, x \rangle - \sigma_C(s) \}$$

$$= \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{if } x \notin C \end{cases}$$

Indeed, if $x \in C$, $\langle s, x \rangle \leq \sup \{ \langle s, y \rangle \mid y \in C \}$

On the other hand, if $x \notin C$, since C is closed we can strictly separate $\{x\}$ from C, and the result follows.

Now we compute the Fenchel transform of $i_C(x)$

$$i_C^*(s) \triangleq \sup_{x \in \mathbb{R}^n} \{ \langle s, x \rangle - i_C(x) \} = \sup_{x \in C} \{ \langle s, x \rangle - i_C(x) \}$$

$$= \sup_{x \in C} \{ \langle s, x \rangle \} = \sigma_C(s).$$

4 The monotone nonnegative cone and the copositive matrix cone.

The monotone nonnegative cone is given as

$$K_{m+} \triangleq \{ x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0 \}$$

K_{m+} is <u>convex</u> because nonnegative combinations preserve inequalities:

$$x, y \in K_{m+}, \quad \lambda \in (0,1)$$

$$z = \lambda x + (1-\lambda) y$$

$$x_i \geq x_{i+1}$$
$$y_i \geq y_{i+1}$$

$$\Rightarrow z_i = \lambda x_i + (1-\lambda) y_i \geq \lambda x_{i+1} + (1-\lambda) y_{i+1} = z_{i+1}.$$

It is a <u>cone</u> because $x_i \geq x_{i+1}$, $\lambda > 0$ $\Rightarrow \lambda x_i \geq \lambda x_{i+1}$.

Next we compute the <u>dual cone</u>, using the identity:

$$\sum x_i y_i = (x_1 - x_2) y_1 + (x_2 - x_3)(y_1 + y_2) + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n (y_1 + \cdots + y_n).$$

$$(K_{m+})^* \triangleq \{ y \mid \langle x, y \rangle \geq 0 \quad \forall x \in K_{m+} \}$$

$$= \{ y \mid (x_1 - x_2) y_1 + \cdots + x_n (y_1 + \cdots + y_n) \geq 0 \quad \forall x \in K_{m+} \}$$

$= \{ y \mid y_1 \geq 0,\ y_1 + y_2 \geq 0,\ \ldots,\ y_1 + \cdots + y_n \geq 0 \}$.

The copositive matrix cone is given as:

$$K \triangleq \{ M \in S^n \mid x^T M x \geq 0 \quad \forall x \in \mathbb{R}^n_+ \}$$

Conic convexity follows by linearity:

$M_1, M_2 \in K, \quad \lambda \in (0,1), \quad x \in \mathbb{R}^n_+$

$$\Rightarrow x^T (\lambda M_1 + (1-\lambda) M_2) x$$

$$= \lambda \underbrace{(x^T M_1 x)}_{\geq 0} + (1-\lambda) \underbrace{(x^T M_2 x)}_{\geq 0} \geq 0.$$

Also, for $\alpha > 0$, $x^T (\alpha M) x = \alpha (x^T M x) \geq 0$.

$$K^* = \{ N = \textstyle\sum y_i y_i^T \mid y_i \in \mathbb{R}^n_+ \}.$$