

**EE381V: Convex Optimization — Fall 2011**

PROBLEM SET THREE

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Due: Thursday, October 27, 2011.

The point of this problem set is to provide more exposure to and exercise with, the geometry of convex sets, projection, and separation, as well as with conjugate functions. Also, this will fill in holes left during the lecture.<sup>1</sup>

1. Separation under polyhedral assumptions.

Show that if  $C_1, C_2$  are two nonempty convex subsets of  $\mathbb{R}^n$ , and moreover  $C_2$  is polyhedral:

$$C_2 = \{x \in \mathbb{R}^n \mid a_i^\top x \leq b_i, i = 1, \dots, m\},$$

then  $C_1$  and  $C_2$  can be separated *properly* by a hyperplane that does not contain  $C_1$ , if and only if  $\text{ri}(C_1) \cap C_2 = \emptyset$ . Note that this may not hold if  $C_2$  is not polyhedral. Moreover, there may not exist a hyperplane properly separating  $C_1$  and  $C_2$  that does not contain  $C_2$ .

2. Cones and the Farkas Lemma.

(a) Consider a cone  $K$  generated by vectors  $a_1, \dots, a_m \in \mathbb{R}^n$ :

$$\begin{aligned} K &= \text{cone}\{a_1, \dots, a_m\} \\ &= \left\{ \sum_i \lambda_i a_i \mid \lambda_i \geq 0 \right\}. \end{aligned}$$

Compute the polar cone  $K^\circ$  in terms of the  $a_i$ .

(b) Consider vectors  $x, e_1, \dots, e_m, a_1, \dots, a_r \in \mathbb{R}^n$ . Show that  $\langle x, y \rangle \leq 0$  for all  $y \in \mathbb{R}^n$  such that  $\langle y, e_i \rangle = 0$ , and  $\langle y, a_j \rangle \leq 0$ , if and only if

$$x = \sum_{i=1}^m \lambda_i e_i + \sum_{j=1}^r \mu_j a_j,$$

for  $\lambda_i, \mu_j \in \mathbb{R}^n$  and  $\mu_j \geq 0$ .

3. Prove the result that was stated in class:

For  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , the following are equivalent:

- (a) The function  $f$  is lower semicontinuous.
- (b) The epigraph of  $f$  is closed.
- (c) The sublevel sets  $S_r(f)$  are closed for all  $r \in \mathbb{R}$  (remember that an empty set is closed).

<sup>1</sup>Some of these problem taken from / inspired by Hiriart-Urruty and Lemaréchal, and Boyd and Vandenberghe.

4. Consider the function:

$$f_m : \mathbb{S}^n \rightarrow \mathbb{R}$$

$$A \mapsto f_m(A) = \sum_{i=1}^m \lambda_i(A),$$

i.e.,  $f_m$  maps the symmetric (not necessarily positive definite) matrix  $A$  to the sum of its  $m$  largest eigenvalues. Show that this function is convex. Then show that the sum of the  $k$  smallest eigenvalues is a concave function.

(Hint: For the first part, it might be helpful to write an outer description of  $f_m$ , i.e., write  $f_m$  as the pointwise supremum of affine functions. For the second part, recall that  $f_n$  is actually an affine function.)

5. Computing Fenchel transforms.

- (a) Let  $C$  be a closed convex set, and let  $f(\cdot) = \sigma_C(\cdot)$  denote its support function. Compute the Fenchel transform,  $f^*$ .
- (b) Let  $f(x) = \frac{1}{2}\langle x, Qx \rangle + \langle b, x \rangle$ , for  $Q \in \mathbb{S}_{++}^n$ , i.e.,  $Q$  is (strictly) positive definite. Compute  $f^*$ .
- (c) Let  $f(x) = e^x$ . Compute  $f^*$ .
- (d) Let  $f(x) = (1/p)|x|^p$ , where  $1 < p < \infty$ .

6. Prove that the function  $f(A) = \log \det(A^{-1})$  is indeed a convex function of  $A$ . Note that a function  $f(\cdot)$  is convex, if and only if it is convex on any interval in its domain.

7. Consider a linear optimization problem:

$$\begin{aligned} \text{minimize :} & \quad \mathbf{c}^\top \mathbf{x} \\ \text{subjectto :} & \quad A\mathbf{x} \geq \mathbf{b}. \end{aligned}$$

- (a) Compute the dual problem explicitly.
- (b) Assuming that the primal problem has an optimal solution  $\mathbf{x}^*$  with a finite value  $f^* = \mathbf{c}^\top \mathbf{x}^*$ , prove that there is no duality gap. Note that you cannot appeal to the theorem proved in class on Tuesday, since we do not assume the existence of a point  $\bar{\mathbf{x}}$  that satisfies  $A\bar{\mathbf{x}} < \mathbf{b}$ .

8. The Minimum Distance Problem: Consider two sets,

$$C \triangleq \{\mathbf{x} : f(\mathbf{x}) \leq 0\}$$

$$D \triangleq \{\mathbf{x} : g(\mathbf{x}) \leq 0\},$$

where (possibly vector-valued) functions  $f$  and  $g$  are convex.

- (a) Show that the problem of finding the minimum distance between  $C$  and  $D$ , measured in some norm  $\|\cdot\|$  (say, Euclidean norm), is a convex problem.

(b) Defining a variable  $\mathbf{w} = \mathbf{x} - \mathbf{y}$ , rewrite the problem as

$$\begin{aligned} \text{minimize :} & \quad \|\mathbf{w}\| \\ \text{subject to :} & \quad f(\mathbf{x}) \leq 0 \\ & \quad g(\mathbf{y}) \leq 0 \\ & \quad \mathbf{w} = \mathbf{x} - \mathbf{y}. \end{aligned}$$

Compute the dual function,  $q(\mu_1, \mu_2, \lambda)$ , and its domain.

(c) Consider the dual optimization problem. Show that if there exists a dual feasible solution with strictly positive  $q$  value, then  $C$  and  $D$  are strictly separable, and using the dual solution, explicitly construct a separating hyperplane.