

**EE381V: Convex Optimization — Fall 2009**

PROBLEM SET THREE

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Due: Wednesday, September 30, 2009.

The point of this problem set is to provide more exposure to and exercise with, the geometry of convex sets, projection, and separation, as well as with conjugate functions. Also, this will fill in holes left during the lecture.<sup>1</sup>

1. Separation under polyhedral assumptions.

Show that if  $C_1, C_2$  are two nonempty convex subsets of  $\mathbb{R}^n$ , and moreover  $C_2$  is polyhedral:

$$C_2 = \{x \in \mathbb{R}^n \mid a_i^\top x \leq b_i, i = 1, \dots, m\},$$

then  $C_1$  and  $C_2$  can be separated *properly* by a hyperplane that does not contain  $C_1$ , if and only if  $\text{ri}(C_1) \cap C_2 = \emptyset$ . Note that this may not hold if  $C_2$  is not polyhedral. Moreover, there may not exist a hyperplane properly separating  $C_1$  and  $C_2$  that does not contain  $C_2$ .

2. Cones and the Farkas Lemma, part I (part II next week).

(a) Consider a cone  $K$  generated by vectors  $a_1, \dots, a_m \in \mathbb{R}^n$ :

$$\begin{aligned} K &= \text{cone}\{a_1, \dots, a_m\} \\ &= \left\{ \sum_i \lambda_i a_i \mid \lambda_i \geq 0 \right\}. \end{aligned}$$

Compute the polar cone  $K^\circ$  in terms of the  $a_i$ .

(b) Consider vectors  $x, e_1, \dots, e_m, a_1, \dots, a_r \in \mathbb{R}^n$ . Show that  $\langle x, y \rangle \leq 0$  for all  $y \in \mathbb{R}^n$  such that  $\langle y, e_i \rangle = 0$ , and  $\langle y, a_j \rangle \leq 0$ , if and only if

$$x = \sum_{i=1}^m \lambda_i e_i + \sum_{j=1}^r \mu_j a_j,$$

for  $\lambda_i, \mu_j \in \mathbb{R}^n$  and  $\mu_j \geq 0$ .

3. Prove the result that was stated in class:

For  $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ , the following are equivalent:

- (a) The function  $f$  is lower semicontinuous.
- (b) The epigraph of  $f$  is closed.
- (c) The sublevel sets  $S_r(f)$  are closed for all  $r \in \mathbb{R}$  (remember that an empty set is closed).

<sup>1</sup>Some of these problem taken from / inspired by Hiriart-Urruty and Lemaréchal, and Boyd and Vandenberghe.

4. Consider the function:

$$f_m : \mathbb{S}^n \rightarrow \mathbb{R}$$
$$A \mapsto f_m(A) = \sum_{i=1}^m \lambda_i(A),$$

i.e.,  $f_m$  maps the symmetric (not necessarily positive definite) matrix  $A$  to the sum of its  $m$  largest eigenvalues. Show that this function is convex. Then show that the sum of the  $k$  smallest eigenvalues is a concave function.

(Hint: For the first part, it might be helpful to write an outer description of  $f_m$ , i.e., write  $f_m$  as the pointwise supremum of affine functions. For the second part, recall that  $f_n$  is actually an affine function.)

5. Complexity Theory: Read the complexity theory handout by Kleinberg and Tardos, which you can find on the course web page in the calendar section. The next two parts of this exercise are due on Wednesday, October 7th.

(a) In this exercise you will prove that the problem MAX-CUT is NP-complete.<sup>2</sup> We will look at MAX-CUT again in detail later on in the class.

Given a graph  $G = (V, E)$ , a cut is a partition of  $V$  into two disjoint sets,  $V = V_1 \cup V_2$ . The value of the cut, is equal to the number of edges joining points in  $V_1$  to points of  $V_2$ . The MAX-CUT problem is that of finding the cut of highest value. Let us define:

$$\text{MAX-CUT} \triangleq \{\langle G, k \rangle \mid G \text{ has a cut of size } k \text{ or more}\}.$$

Show that under this definition, MAX-CUT is NP-complete.

Hint: Given a 3cnf-formula  $\phi$ , a  $\neq$ -assignment is one where no clause has three true literals. It turns out (you may assume this!) that  $\neq$ -SAT is NP-complete. Obtain a polynomial-time reduction of MAX-CUT to this problem.

(b) (More difficult...) Recall from the reading that a 2cnf-formula is an AND of clauses, where each clause is an OR of exactly two (not necessarily distinct) literals. Show that computing a satisfying assignment, or showing no such assignment exists, can be done in polynomial time, i.e., show that 2SAT  $\in P$ .

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<sup>2</sup>This exercise is taken from M. Sipser's Theory of Computation book, which, incidentally, is excellent.