ON STEINER’S PROBLEM WITH RECTILINEAR DISTANCE*

M. HANAN†

1. Introduction. This paper is concerned with the following type of problem. Given \( n \) cities, construct a network of roads of minimum total length so that a traveler can get from one city to any other. Roads may cross each other outside of the city limits and these points are called junction points. (Roads which cross within the cities, however, will not be referred to as junctions.) It is assumed that junction points add no extra cost to the construction of the network so that there may be as many as necessary to minimize the total length. Usually, the roads are straight-line connections and the distance between two points is the Euclidean distance. In this paper, however, the rectilinear distance is used. The rectilinear distance \( d(p_1, p_2) \) between two points \( p_1 \) and \( p_2 \) is defined as

\[
d(p_1, p_2) = |x_1 - x_2| + |y_1 - y_2|,
\]

where \((x_i, y_i)\) are the coordinates of \( p_i \).

Rectilinear distance has application in printed circuit technology where \( n \) electrically common points must be connected with the shortest possible length of wire and the wires must run in the horizontal and vertical directions. The junction points of the wires are analogous to the above-mentioned road junctions.

Actually this is a well-known problem due to Steiner (cf. [2] and [6]) and is now formally stated.

**Steiner’s Problem.** Given \( n \) points in the plane find the shortest tree(s) whose vertices contain these \( n \) points.

A tree with \( m \) vertices is a connected graph with \( m - 1 \) edges. (For graph-theoretic terminology see Berge [1].) Several necessary conditions about the solution of this problem are known when the distance between two points is taken to be the Euclidean distance. In this paper several necessary conditions are given for any \( n \), using rectilinear distance. Some of these conditions are analogous to the problem with Euclidean distance, some hold only for rectilinear distance, and some are invariant with respect to the metric. Exact solutions are constructed for \( n \leq 5 \).

Since rectilinear distance is not invariant with respect to rotations in the plane, the statement of Steiner’s problem must be properly interpreted. Hence it is assumed throughout this paper that when \( n \) points in the plane are given, a Cartesian coordinate system is also given and the rectilinear distance is defined with respect to this coordinate system.

We now state two problems which are related to Steiner’s problem and whose solutions we will have occasion to use in this paper. To distinguish these we refer to Steiner’s problem as \( S \) and we now define \( P \) and \( T \).

\( P \) : Given \( n \) points \((p_1, \ldots, p_n)\) in the plane, find a point \( q \) such that the sum of the distances from \( q \) to \( p_i \), \( i = 1, \ldots, n \), is a minimum.

\( T \) : Given \( n \) points in the plane, find the shortest tree whose vertices are these \( n \) points.

(We have departed slightly from the notation used by Melzak [6].) The \( P \) problem has been solved for both Euclidean distance (cf. [7]) and rectilinear distance (cf. [3]). The \( T \) problem has also been solved (cf. [5] and [8]) and the method of solution is independent of the metric used.

2. Steiner’s problem with three points.

2a. Euclidean distance. Given three points in the plane, let \( T \) be the triangle whose vertices are these three points. If every angle of \( T \) is less than 120°, then the point \( q \) of \( P \) lies inside \( T \) and the lines from \( p_i \) to \( q \), \( i = 1, 2, 3 \), meet at 120° at \( q \). If an angle of \( T \) is greater than or equal to 120°, then \( q \) coincides with that vertex. (See [4] for a proof and a construction of the solution.) It is not difficult to see that \( P \) yields the same solution as \( S \). Also, if an angle of \( T \) is greater or equal to 120°, then the solution of \( S \) is identical to the solution of \( T \). See Fig. 1.

2b. Rectilinear distance. Using rectilinear distance, the solution to \( S \) (or equivalently \( P \)) is simpler to construct than the corresponding problem using Euclidean distance. In place of the triangle \( T \), we consider the enclosing rectangle \( R \) which we now define, in general, for \( n \) points.

**Definition 1.** Given \( n \) points in the plane the enclosing rectangle \( R \) is the smallest rectangle whose sides are parallel to the \( x \) and \( y \) axes and which includes the \( n \) points either within or on its boundary.

We refer to the solution of the three-point problem throughout this paper and therefore state the result as a separate theorem.

**Theorem 1.** Let \((x_i, y_i)\) be the coordinates of the given points \( p_i \), \( i = 1, 2, 3 \). The \( q \)-point of \( P \) is located at \((x_m, y_m)\) where \( x_m \) and \( y_m \) are the medians of \([x_i]\) and \([y_i]\), respectively.

As stated earlier, this special case of \( P \) is solved in [3]. The following theorem relates \( P \), \( S \), and \( T \). Let \( d_{e}, d_{r}, \) and \( d_{s} \) be the total (rectilinear) distance in the solutions of \( S \), \( P \), and \( T \), respectively.

---

* Received by the editors March 15, 1965.
† Thomas J. Watson Research Center, International Business Machines Corporation, Yorktown Heights, New York.

1 We would like to thank the referee for suggesting two other references to this paper: E. Goursat, *A Course in Mathematical Analysis*, vol. 1, Dover, New York, 1939, p. 130; H. S. M. Coxeter, *Introduction to Geometry*, John Wiley, New York, 1961, p. 21.


Copyright © 1966 by the Society for Industrial and Applied Mathematics.
Theorem 2. The solutions of $S_1$ and $P_3$ are identical; in fact,
\begin{equation}
    d_{S_1} = d_{P_3} = \frac{1}{2}P(R) \leq d_{P_1},
\end{equation}
where $P(R)$ is the perimeter of the enclosing rectangle $R$. The equality sign holds in (1) only if $q$ is coincident with some $p_i$, i.e., $(x_i, y_i) = (x_j, y_j)$ for some $i = 1, 2, 3$.

The proof of Theorem 2 is straightforward. It follows from Theorem 1 and the fact (which we prove later for $S_1$, in general) that the minimum tree solution to $S_1$ can have either zero or one additional vertex. See Fig. 2.

3. Necessary conditions on a solution to $S_n$. We use the following notation: $p_i$ are the given $n$ points and $q_i$, $i = 1, \cdots, k$, are the additional $k$ vertices in the solution $G$ of $S_n$. When we are referring to the vertices of $G$, we speak of $p$-vertices or $q$-vertices. When we are referring to the location of these vertices in the coordinate system we speak of $p$-points or $q$-points. We use the notation $p_i$ (or $q_i$) interchangeably for a vertex of $G$ or the location of that vertex. Its meaning should be clear from the context. We let $P$ be the set of $p$-points or $q$-vertices and $Q$ be the set of $q$-points or $q$-vertices. When we speak of a vertex $a_i$, $i = 1, \cdots, n + k$, we mean either $p_i$ or $q_i$. We let $w(a_i)$ be the local degree of the vertex $a_i$, that is, the number of vertices adjacent to $a_i$ and $C(a_i)$ be a set of vertices. (Two vertices are adjacent if they have an edge in common.) The following essentially sums up the present knowledge about the solution to $S_n$ using Euclidean distance (cf. [2] and [8]).

(1) $w(q_i) = 3$, $1 \leq i \leq k$,
(2) $w(p_i) \leq 3$, $1 \leq i \leq n$,
(3) $0 \leq k \leq n - 2$,
(4) each $q_i$, $1 \leq i \leq k$, is the $q$-point of $C(q_i)$.

These conditions are easy to prove. In fact (4) can be replaced by the stronger statement that every connected subtree of a solution $G$ of $S_n$ is a minimum tree of those $m \leq n$ points.

The analogous necessary conditions on a solution $G$ of $S_n$ using rectilinear distance are:

(1) $w(q_i) = 3$ or $4$, $1 \leq i \leq k$,
(2) $1 \leq w(p_i) \leq 4$, $1 \leq i \leq n$,
(3) $0 \leq k \leq n - 2$.

Conditions (1) and (2) are almost obvious. In fact if $w(a_i) = 4$, then two pairs of vertices of $C(a_i)$ must be collinear and $a_i$ is at the intersection of the straight lines connecting those pairs. See Fig. 3.

To prove the inequality on the right side of (3), assume that there are $k$ $q$-vertices in $G$ and find the least number of $p$-vertices possible. Assume the worst case, that is, $w(q_i) = 3$ and the $q$-vertices form a subtree with $k - 1$ edges. Since each edge counts twice in the total degree of the $q$-vertices,

$$n \geq 3k - 2(k - 1) = k + 2,$$

or

$$k \leq n - 2.$$

To show that zero is a true lower bound, we can easily construct an example where $k = 0$. Since $\frac{1}{2}P(R)$ is a lower bound for $d_{S_1}$ and, in the example shown in Fig. 4, $d_{S_1} = \frac{1}{2}P(R)$, we have found a minimum tree with $k = 0$.

We state condition (4) as a separate lemma for future reference.

Lemma 1. Given $n$ points in the plane, let $G$ be a solution of $S_n$. If $G'$ is a connected subgraph of $G$ with $m$ vertices, then $G'$ is a solution of $S_m$.

The following theorem has an analog in Euclidean geometry where the triangle $T$ replaces the rectangle $R$. However, we have not seen it stated in the literature.

Theorem 3. If $q$ is any $q$-vertex of $G$ with degree three, then $q$ can be the only vertex of $G$ inside the enclosing rectangle $R$ of $C(q)$.

We note first that there may be vertices (including $q$ itself) on the boundary of $R$. To prove the theorem, assume the contrary, that is, assume that there is another vertex $a_m$ inside $R$. There must exist some path from
a, to one of the vertices $a_i \in C(q)$, say $a_1$. Now consider the problem $S_2$ consisting of the three points $\{a_0, a_1, a_2\}$ and let $R_1$ be the enclosing rectangle of these three points. Since $a_0$ lies inside $R$, $\frac{1}{2}P(R_1) < \frac{1}{2}P(R)$, so that we can replace the subtree on the vertices $(a_1, a_2, a_3, q)$ with the new subtree on $(a_0, a_2, a_3, q_1)$, where $q_1$ is the new q-point. This subtree is connected to the rest of the graph by the path from $a_0$ to $a_1$. Hence we have found a new graph $G_1$ with a smaller total distance than $G$, contradicting the hypothesis that $G$ is a solution of $S_2$.

In general, a solution $G$ to $S_n$ is not unique, that is, there is more than one set $Q$ which yields a minimum tree. Let $N(n, k)$ be the number of sets $Q$. The main result of Meikle [6] is that $N(n, k)$ is finite for all $n$ and $k$ in Euclidean geometry and there exists a finite sequence of Euclidean constructions yielding all minimizing trees of the problem $S_n$. In rectilinear geometry this is not true. (We will give an example in §4 where $N(4, 2)$ is infinite.) Hence we cannot guarantee that we can find (by construction) all solutions $G$ to $S_n$. However, we now prove a theorem which does guarantee finding a finite subset of solutions. The theorem proves, in effect, that there always exists a solution $G$ such that all the vertices in the set $Q$ are located at a predetermined finite set of possible locations.

**Theorem 1.** Let $|x_0|$ and $|y_0|$ be the sets of $x$ and $y$ coordinates of the given $n$ p-points. If $(x_{qj}, y_{qj})$ are the coordinates of any vertex $q_j \in Q$, then there exists a solution $G$ to the problem $S_n$ such that $x_{qj} \in |x_0|$ and $y_{qj} \in |y_0|$ for all $j = 1, 2, \ldots, k < n - 2$.

If straight lines are drawn parallel to the $x$ and $y$ axes through all the given points, a grid is imposed on the plane. Theorem 1 states that there exists a solution $G$ such that all the $q$-vertices are on the intersections of these grid lines. Let $I$ be the set of intersection points or, when referring to the vertices, the set of vertices of $G$ which are located at these intersections. By definition, $P \subseteq I$. For any $q \in Q$, if $C(q)$ contains only p-vertices then $q \notin I$. This last statement is an immediate consequence of Theorem 1 and Lemma 1 when $w(q) = 3$ and is obvious when $w(q) = 4$.

To facilitate the proof of Theorem 4, we first prove two lemmas.

**Lemma 2.** Let $G$ be any solution of $S_n$ and let $q_j$ be any vertex in $Q$ such that $C(q_j)$ contains two vertices in the set $I$, say $i_1$ and $i_2$. If $q_j \notin I$, then a solution $G'$ can be obtained from $G$ such that there is no vertex of $G'$ located at the point $q_j$ and if a vertex $q_j'$ of $G'$ is connected to both $i_1$ and $i_2$ then $q_j \in I$.

Lemma 2 states, in effect, that given a tree $G$, certain of the q-vertices can always be "moved" to new locations which are at the intersections of the grid. We first note that if $w(q_j) = 4$ then clearly $q_j \in I$, so we assume that $w(q_j) = 3$. Let $i_1, i_2, q_j$ and $q_j'$ be the three vertices of $C(q_j)$. If $a \in I$ then $q_j \in I$ (Theorem 2 and Lemma 1) so that $a \in Q$. Hence let us call this vertex $q_j$. The locations of these vertices with respect to $q_j$ must be essentially as shown in Fig. 5. (We have drawn the connection from $q_j$ to $i_1$ in the way shown for future use.)

By Lemma 1 and Theorem 1, at least one of the vertices $i_1$ or $i_2$ must be on the horizontal line through the point $q_j$. (Clearly the figure can be rotated through an angle of $m\pi/2$, $m = 1, 2, 3$. There is, of course, no loss of generality in assuming this configuration.) Assuming that $i_1$ is the vertex on this horizontal line, then $i_1$ can be anywhere in the quadrant $x > x_j$, $y > y_j$. Again, by Lemma 1 and Theorem 1, at least one vertex of $C(q_j)$, say $a_j$, must lie on the line $y = y_j$. Without loss of generality, assume that $a_j$ is to the right of $q_j$. There are now two possibilities which must be considered: (i) $a_j$ is to the right of $i_1$, and (ii) $a_j$ is in the interval between $q_j$ and $i_1$.

Considering (i) first, the line joining $q_j$ to $q_j'$ can be moved parallel to itself to the line $x = x_j$, as indicated in Fig. 6. By making this move, the new graph $G'$ is also a connected tree and its length is the same as $G$. Hence $G'$ is also a solution to $S_n$. Clearly $q_j' \notin I$ and there is no vertex of

![Fig. 5](image-url)
The proof of Theorem 4 now follows immediately by successively applying Lemmas 2 and 3. Given a solution $G_1$, partition the vertices into two disjoint sets $Q_1$ and $I_1$. If $Q_1$ is not empty, then, by Lemma 3, at least one vertex in $Q_1$ has two adjacent vertices in $I_1$. By Lemma 2 this vertex can be moved to a new position which is in $I_1$. Partition the vertices of this new solution tree $G_2$, generated by this move, into two disjoint sets $Q_2$ and $I_2$. If $Q_2$ is empty, the theorem is proved. If $Q_2$ is nonempty, apply Lemma 3. By continuing this process, $Q_1$ must be empty for some finite $l$ since the set of points $I$ is certainly finite.

4. The cases $n = 4$ and $n = 5$. In this section we concentrate on the solutions implied by Theorem 4, that is, those solutions which have all their vertices in $I$, although many of the statements made here are applicable to all solutions of $S_4$ and $S_5$.

We begin the study of $S_4$ by first solving the special case where the given four points are located on the corners of the enclosing rectangle $R$. By Theorem 4, there exists a solution $G$ with no $q$-vertices since the four intersection points are occupied by $p$-vertices. Hence, in this case, there exists a solution to Steiner's problem which is the same as the solution to the minimum spanning tree problem. We state this as a separate lemma for future reference.

**Lemma 4.** If the four points of $S_4$ are located at the corners of the enclosing rectangle $R$, then

$$d_{x} = d_{x} = l + 2w,$$

where $l$ and $w$ are the length and width of $R$.

This is the simplest example where the number of sets $Q$ is infinite. For, referring to Fig. 8, the line joining $q_1$ to $q_2$ can be moved parallel to itself anywhere in the interval $y_{m} \leq y \leq y_{e}$, which implies an infinite number of possible locations for $q_1$ and $q_2$.

---

1. We assume throughout this paper that $w \leq I$, i.e., we say the width, by definition, is the smaller of the two numbers.
We now show that when the four points of \( S_4 \) are located anywhere in the plane, the problem can always be reduced either to the above case or to a Steiner problem with less than four points. We first note that there cannot be any \( q \)-vertices on any side of \( R \) unless there are two \( p \)-points on this same side. (This can be deduced easily by Theorem 1 and Lemma 1.) Therefore, if there is a side \( a_i \) of \( R \) with only one vertex on it, that vertex can be "moved" perpendicularly to \( a_i \) to the closest intersection point. It is not difficult to see that, in doing this, we have reduced the general four-point Steiner problem to one where Lemma 4 is applicable (see Fig. 9a) or to a Steiner problem with less than four points (see Fig. 9b). In order to state these results more succinctly, some new terminology is introduced.

First order both (separately) the \( x \)- and \( y \)-coordinates of the given \( p \)-points in increasing order. (In doing this \((x_1, y_1)\) no longer corresponds to the point \( p_1 \).) Then by drawing lines parallel to the \( y \)-axis through \( x_1 \) and \( x_2 \) and lines parallel to the \( x \)-axis through \( y_1 \) and \( y_2 \), this defines, in general, four points in \( R \) which we call \( c_1, \ldots, c_4 \). The rectangle which has these four points at its corners is called the inner rectangle \( R_1 \). Consider the four quadrants \( U_{c_i} \), exterior to \( R_1 \), formed by the extended lines of \( R_1 \) and each of the \( c_i \). If there is a point \( p_i \) in a quadrant \( U_{c_i} \), then we say that \( p_i \) is transferred to the point \( c_i \). By construction, the \( p \)-vertex may, of course, be at the point \( c_i \). The inner rectangle \( R_1 \) may degenerate to a straight line. There may or may not be vertices of \( G \) located at the points \( c_i \).

A solution to \( S_4 \) can be found by applying Lemma 4 and the following theorem.

**Theorem 5.** Given four points in the plane, let \( l \) and \( w \) be the length and width of the enclosing rectangle \( R \) and let \( w_1 \) be the width of the inner rectangle \( R_1 \). If the \( p \)-vertices are transferred to four distinct points in \( \{c_i\} \), then

\[ d_{S_4} = l + w + w_1. \]

and if they are transferred to less than four distinct points in \( \{c_i\} \), then

\[ d_{S_4} = l + w. \]

It is not difficult to see that, in general, \( d_{S_4} \) has a lower bound of \( \frac{1}{2} \) the perimeter of the enclosing \( R \), that is,

\[ d_{S_4} \geq \frac{1}{2} P(R) = l + w. \]

In fact, the following can easily be deduced from Theorem 5 and Lemma 4.

**Corollary 1.** If \( d_{S_4} = l + w \) then the solution to \( S_4 \) is unique; and if \( d_{S_4} > l + w \) then there exists an infinity of solutions.

The five-point Steiner problem can be treated in essentially the same way as the four-point problem. In this case, lines are drawn parallel to the \( y \)-axis through \( x_2, x_3, \) and \( x_4 \) and parallel to the \( x \)-axis through \( y_2, y_3, \) and \( y_4 \), so that there are in general nine \( c \)-points. The inner rectangle \( R_1 \) is the largest rectangle defined by these nine lines. The \( p \)-vertices are transferred to the \( c \)-points in a manner similar to the above, except that the concept of quadrants must be generalized to include the points \( c_i \) which are not at the corners of \( R_1 \). Hence, by transferring the \( p \)-points, we can always reduce the problem \( S_5 \) to the case where at least two \( p \)-vertices are on each side of the enclosing rectangle. The following theorem (the proof of which we omit) can then be used to find a solution to \( S_5 \) when the points are located anywhere in the plane.

**Theorem 6.** Given five points in the plane with at least two points on each side of the enclosing rectangle \( R \). If four of the five points are at the corners of
STEINER'S PROBLEM WITH RECTILINEAR DISTANCE

\[ d_{s_4} = d_{s_4} = l + 2w \leq d_{r_4}, \]

where \( S_4 \) is the Steiner problem with these four corner points. If all five points are on the boundary of \( R \), then

\[ d_{s_5} = d_{r_5} \leq l + 2w, \]

where \( l \) and \( w \) are the length and width of \( R \).

5. General comments. An algorithm which incorporates several of the necessary conditions stated in §3 has been developed. It yields "good" approximate solutions to the \( n \)-point problem and exact solutions for \( n \leq 4 \). The algorithm is easy and fast to do both by hand and on a computer.

The algorithm is rather elementary in concept and it is anticipated that a more sophisticated algorithm can be devised which incorporates almost all the results presented in this paper. For example, Theorem 4 states that there always exists a solution to Steiner's problem where the \( q \)-vertices are located at a predetermined finite set of points. Therefore, if the number of points \( n \) is not too large, we can find an exact solution by an exhaustive search procedure. These ideas will be investigated in the future.

Acknowledgment. The author wishes to thank P. H. Oden for many interesting discussions relating to this problem.

REFERENCES