A Brief Tutorial on Linear and Nonlinear Control Theory

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Outline

- Systems and Two Fundamental Problems/Techniques
- Stabilization of Linear Systems by Linear State Feedback
- Linear Quadratic Regulator
- Real Time State Estimation
- Linear Observers and Kalman Filtering
- Equilibria of Nonlinear Systems and their Linear Approximates
- Feedback Linearization and Backstepping
- Nonlinear Optimal Control
- Linearization by Input/Output Injection
- Extended Kalman Filter, Unscented Kalman Filter
- Nonlinear Filtering, Particle Filtering

Models of Systems

$$\begin{array}{rcl|rcl|l} \dot{x} & = & Fx + Gu & x^{+} & = & Fx + Gu & x^{+}(t) = x(t+1) \\ y & = & Hx + Ju & y & = & Hx + Ju & \end{array}$$

$$\begin{array}{rcl|rcl} \dot{x} & = & f(x,u) & x^{+} & = & f(x,u) \\ y & = & h(x,u) & y & = & h(x,u) \end{array}$$

$$x \in \mathbb{R}^n \quad \Big| \quad u \in \mathbb{R}^m \quad \Big| \quad y \in \mathbb{R}^p$$

The system could be time varying, $F(t), f(t, x, u)$ etc.

It could be an infinite dimensional systems ($n = \infty$), e.g. time delays (delay differential systems) or systems described by PDEs (distributed parameter systems).

The system could be described implicitly, e.g., $f(\dot{x}, x, u) = 0$,

$\boldsymbol{u(t)}$ is the input to the system at time $\boldsymbol{t}$. It is the way that the external world affects the system. It could be a control or a reference signal to be tracked or stochastic noise (e.g. WGN) or deterministic noise or a combination of these.

$\boldsymbol{x(t)}$ is the state of system at time $\boldsymbol{t}$, it is the memory of the net effect of past inputs. Knowledge of the current state and future inputs allows us to predict the future behaviour of the system.

$\boldsymbol{y(t)}$ is the output to the system at time $\boldsymbol{t}$. It is the way that the system affects the external world. It could be a measurement of a state variable(s) or a state variable(s) to be regulated to a set value or a combination of these.

Very flexible model, systems can be combined in various ways, e. g. serial, parallel or feedback interconnections.

Other than State Space Models

Linear Input-Output Models

$$\boldsymbol{y}(t) = \boldsymbol{w}_0(t)\boldsymbol{u}(t) + \int_0^t \boldsymbol{w}_1(t,\tau)\boldsymbol{u}(\tau)\,d\tau$$

For autonomous, stable linear systems (assuming $\boldsymbol{x}^0 = \boldsymbol{0}$)

$$\boldsymbol{w}_0(t) = \boldsymbol{J}, \quad \boldsymbol{w}_1(t,\tau) = \boldsymbol{H}e^{\boldsymbol{F}(t-\tau)}\boldsymbol{G}$$

Nonlinear Input-Output Models

$$\begin{aligned}\boldsymbol{y}(t) &= \boldsymbol{w}_0(t)\boldsymbol{u}(t) + \int_0^t \boldsymbol{w}_1(t,\tau)\boldsymbol{u}(\tau)\,d\tau \\ &\quad + \int_0^t \int_0^{\tau_1} \boldsymbol{w}_2(t,\tau_1,\tau_2)\boldsymbol{u}(\tau_1)\boldsymbol{u}(\tau_2)\,d\tau_2\,d\tau_1 + \ldots\end{aligned}$$

Linear Frequency Domain Models for autonomous, stable linear systems (assuming $\boldsymbol{x}^0 = \boldsymbol{0}$). Laplace transforms.

$$\begin{aligned}\boldsymbol{y}(s) &= \boldsymbol{T}(s)\boldsymbol{u}(s) \\ \boldsymbol{T}(s) &= \boldsymbol{H}(s\boldsymbol{I} - \boldsymbol{F})^{-1}\boldsymbol{G} + \boldsymbol{J}\end{aligned}$$

where $s \in \mathbb{C}$. $\boldsymbol{T}(s)$ is the Transfer Matrix of the system.

Two Fundamental Problems/Techniques

- Stabilization by State Feedback
- Real Time State Estimation (Filtering)

Stabilization by State Feedback
Equilibrium of a system

$$\begin{aligned} \dot{x} &= f(x, u) \\ 0 &= f(x^0, u^0) \end{aligned}$$

Often we shall assume that $x^0 = 0, u^0 = 0$

Find a (continuous) (smooth) state feedback $u = \kappa(x)$ so the closed loop system

$$\dot{x} = f(x, \kappa(x))$$

is (locally) asymptotically stable to $x^0 = 0$.

If a linear system

$$\dot{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{u}$$

can be stabilized by state feedback then it can be stabilized by linear state feedback

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x}$$

Closed Loop System

$$\dot{\boldsymbol{x}} = (\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K})\boldsymbol{x}$$

Choose $\boldsymbol{K}$ so that $\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}$ is Hurwitz, i.e., $\boldsymbol{\sigma}(\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}) < \mathbf{0}.$

Invertible state feedback, $\boldsymbol{M}$ invertible,

$$\boldsymbol{u} = \boldsymbol{K}\boldsymbol{x} + \boldsymbol{M}\boldsymbol{v}$$

Theorem The spectrum of $\boldsymbol{F}+\boldsymbol{G}\boldsymbol{K}$ can be placed arbitrarily (up to cc) by choice of $\boldsymbol{K}$ iff the smallest $\boldsymbol{F}$ invariant subspace containing the range of $\boldsymbol{G}$ is $\mathbb{R}^n$ or equivalently

$$\operatorname{rank}\left[\, \boldsymbol{G}\ \boldsymbol{F}\boldsymbol{G}\ \ldots\ \boldsymbol{F}^{n-1}\boldsymbol{G} \,\right] = n$$

If this holds then $\boldsymbol{F}, \boldsymbol{G}$ is called a controllable pair and the system is said to be controllable.

If the system is not controllable then there is a linear change of coordinates so that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ G_2 \end{bmatrix} u$$

where $\boldsymbol{F}_{22}, \boldsymbol{G}_2$ is a controllable pair on $\boldsymbol{x}_2$ space.

We can change the spectrum of $\boldsymbol{F}_{22}$ by state feedback but we can't change the spectrum of $\boldsymbol{F}_{11}$.

If $\boldsymbol{F}_{11}$ is Hurwitz the system is said to be stabilizable.

Where do we place the spectrum of $\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}$?

Far to the left.

- Requires large $\boldsymbol{K}$ which leads to large $\boldsymbol{u}$ which may not be feasible.
- Leads to peaking in the transient behaviour.
- Very sensitive to errors in $\boldsymbol{x}$.

Close to the imaginary axis.

- Stabilization is very slow.
- Very sensitive to noise in the dynamics and modeling error.

$\boldsymbol{\epsilon}$ pseudo spectrum of $\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}$.

Values of $\boldsymbol{s}$ where $\boldsymbol{s}\boldsymbol{I} - (\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K})$ is within $\boldsymbol{\epsilon}$ of being singular.

This is where $(\boldsymbol{s}\boldsymbol{I} - (\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}))^{-1}$ is very large, $\boldsymbol{O}(1/\boldsymbol{\epsilon})$.

This is related to the fact that for large $\boldsymbol{K}$, the closed loop dynamics $\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}$ is usually far from being normal.

Linear Quadratic Regulation (LQR)

Use an optimization criterion to set the poles.
Choose $Q \geq 0, R > 0$ and

$$\min_{u} \int_0^\infty x'Qx + u'Ru \; dt$$

subject to

$$\begin{aligned} \dot{x} &= Fx + Gu \\ x(0) &= x^0 \end{aligned}$$

The optimal cost is quadratic and the optimal feedback is linear

$$(x^0)'Px^0, \quad u = Kx$$

where P satisfies the algebraic Riccati equation

$$0 = F'P + PF + Q - PGR^{-1}G'P$$

$$K = -R^{-1}G'P$$

Since the Riccati equation is quadratic there are many solutions but if F, G is stabilizable and $Q^{1/2}, F$ is detectable then there is a unique nonnegative definite solution.

If $P > 0$ then a Lyapunov-LaSalle argument shows closed loop stability

$$\frac{d}{dt} x' P x = -x'(Q + PGR^{-1}G'P)x$$

LQR has guaranteed gain and phase margins.

Design parameters $Q \geq 0, R > 0$.

If R is not invertible then the optimal solution is difficult to find and can require extremely large u (impulses).

The larger Q is as compared with R , the further to the left is the spectrum of $F + GK$.

Real Time State Estimation (Filtering)

From inexact knowledge of $\boldsymbol{x^0}$ and the past inputs and outputs estimate the current value of the state.

State Estimate $\hat{\boldsymbol{x}}(\boldsymbol{t})$, Estimation Error $\tilde{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t}) - \hat{\boldsymbol{x}}(\boldsymbol{t})$

Linear system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{F}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{H}\boldsymbol{x} + \boldsymbol{J}\boldsymbol{u} \end{aligned}$$

Linear observer

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \boldsymbol{F}\hat{\boldsymbol{x}} + \boldsymbol{G}\boldsymbol{u} - \boldsymbol{L}(\boldsymbol{y} - \hat{\boldsymbol{y}}) \\ \hat{\boldsymbol{y}} &= \boldsymbol{H}\hat{\boldsymbol{x}} + \boldsymbol{J}\boldsymbol{u} \end{aligned}$$

Linear error dynamics

$$\dot{\tilde{\boldsymbol{x}}} = (\boldsymbol{F} + \boldsymbol{L}\boldsymbol{H})\tilde{\boldsymbol{x}}$$

Choose $\boldsymbol{L}$ so that $\boldsymbol{F} + \boldsymbol{L}\boldsymbol{H}$ is Hurwitz. Output injection $\boldsymbol{L}\boldsymbol{y}$.

Theorem The spectrum of $\boldsymbol{F} + \boldsymbol{L}\boldsymbol{H}$ can be assigned arbitrarily (up to cc) by choice of $\boldsymbol{L}$ iff the largest $\boldsymbol{F}$ invariant subspace in the kernel of $\boldsymbol{H}$ is $\mathbf{0}$ or equivalently

$$\text{rank}\begin{bmatrix} \boldsymbol{H} \\ \vdots \\ \boldsymbol{H}\boldsymbol{F}^{n-1} \end{bmatrix} = \boldsymbol{n}$$

If the rank is $\boldsymbol{n}$ then $\boldsymbol{H}, \boldsymbol{F}$ are said to be an observable pair. If not, after a linear change of coordinates the system is

$$\begin{bmatrix} \dot{\boldsymbol{x}}_1 \\ \dot{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_{11} & \mathbf{0} \\ \boldsymbol{F}_{21} & \boldsymbol{F}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \begin{bmatrix} \boldsymbol{G}_1 \\ \boldsymbol{G}_2 \end{bmatrix} \boldsymbol{u}$$

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{H}_1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} + \boldsymbol{J}\boldsymbol{u}$$

where $\boldsymbol{H}_1, \boldsymbol{F}_{11}$ is a observable pair on $\boldsymbol{x}_1$ space so we can change the spectrum of $\boldsymbol{F}_{11}$ in the dynamics of $\tilde{\boldsymbol{x}}_1$ by output injection but we can't change the spectrum of $\boldsymbol{F}_{22}$ in the dynamics of $\tilde{\boldsymbol{x}}_2$. If $\boldsymbol{F}_{22}$ is Hurwitz the system is said to be detectable.

Where do we place the spectrum of $\boldsymbol{F} + \boldsymbol{L}\boldsymbol{H}$?

Far to the left.

- Requires large $\boldsymbol{L}$.
- Leads to peaking in the transient behaviour of the error.
- The observer approximates a bank of differentiators.
- Very sensitive to noise in the observations.
- Observation noise leads to poor steady state performance.

Close to the imaginary axis.

- Convergence of the estimate is very slow.
- Very sensitive to modeling errors and noise in the dynamics.

The transfer matrix from noise in the plant to errors in the estimate has the factor

$$(\boldsymbol{sI} - (\boldsymbol{F} + \boldsymbol{LH}))^{-1}$$

Hence the $\boldsymbol{\epsilon}$ pseudo spectrum of $\boldsymbol{F} + \boldsymbol{LH}$ plays a crucial role in the performance of the observer.

For large observer gains $\boldsymbol{L}$, usually $\boldsymbol{F} + \boldsymbol{LH}$ is far from being normal.

Kalman Filtering

We add noises to the model

$$\begin{aligned} \dot{x} &= Fx + Gu + Bw \\ y &= Hx + Ju + Dv \end{aligned}$$

where $w(t), v(t)$ are independent WGNs and D is invertible.

Steady State Kalman Filter

$$\begin{aligned} \dot{\hat{x}} &= F\hat{x} + Gu - L(y - \hat{y}) \\ \hat{y} &= H\hat{x} + Ju \end{aligned}$$

$$\begin{aligned} L &= -PH'(DD')^{-1} \\ 0 &= FP + PF' + BB' - PH'(DD')^{-1}HP \end{aligned}$$

$$\hat{x}(t) = E(x(t)|y(0:t)), \quad P = \lim_{t\to\infty} E(\tilde{x}(t)\tilde{x}'(t))$$

If H, F is detectable then $P < \infty$.
If F, B is stabilizable then $0 \leq P$
If $P > 0$ then the gain L stabilizes $F + LH$ for

$$\frac{d}{dt}\tilde{x}'P^{-1}\tilde{x} = -\tilde{x}'(P^{-1}BB''P^{-1} + H(DD')^{-1}H)\tilde{x}$$

Design Parameters of Kalman Filter

$BB' \geq 0$ driving noise covariance

$DD' > 0$ observation noise covariance

If D is not invertible then the optimal filter would differentiate the noise free part of y . The larger BB' is relative to DD' , the further to the left the spectrum of $F + LH$.

If the noises are colored then linear prefilters driven by WGN can be added to the model. This increases n .

Closed Loop Synthesis

Plant

$$\dot{x} = Fx + Gu$$
$$y = Hx + Ju$$

Compensator

$$\dot{\hat{x}} = F\hat{x} + Gu - L(y - \hat{y})$$
$$\hat{y} = H\hat{x} + Ju$$
$$u = K\hat{x}$$

Closed loop system in $x, \tilde{x}$ coordinates

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} F + GK & -GK \\ 0 & F + LH \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

Stable if both $F + GK, F + LH$ are Hurwitz.

If $\boldsymbol{K}$ chosen via LQR and $\boldsymbol{L}$ chosen via Kalman filtering then this is called a Linear Quadratic Gaussian Controller (LQG).
Also called $\boldsymbol{H}^{\mathbf{2}}$ control.

Very widely used but there are no guaranteed gain and phase margins.

May fail to be robust so often an $\boldsymbol{H}^{\boldsymbol{\infty}}$ controller is used instead.

Linear H^∞ Control

Linear Plant

$$\begin{aligned} \dot{x} &= Fx + Gu + Bw \\ y &= Hx + Ju + Dv \\ z &= Cx \end{aligned}$$

There are three inputs, $u(t)$ is the control and $w(t), v(t)$ are deterministic driving and observation noises, unknown $L^2[0,\infty)$ functions.
There are two outputs, the observations $y(t)$ and the variable $z(t)$ to be regulated to zero.
Linear Compensator

$$\begin{aligned} \dot{\xi} &= \alpha\xi + \beta y \\ u &= \gamma\xi + \delta y \end{aligned}$$

We assume that the initial conditions are $0, 0$.

The result is a linear operator

$$\Sigma_{cl} : \begin{bmatrix} w(\cdot) \\ v(\cdot) \end{bmatrix} \mapsto \begin{bmatrix} z(\cdot) \\ u(\cdot) \end{bmatrix}$$

The goal is to choose the compensator to minimize the L^2 induced norm of this mapping,

$$\|\Sigma_{cl}\| = \sup \frac{\left\| \begin{bmatrix} w(\cdot) \\ v(\cdot) \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} z(\cdot) \\ u(\cdot) \end{bmatrix} \right\|_2}$$

This is equivalent to minimizing the H^∞ norm of the closed loop transfer matrix

$$\begin{bmatrix} z(s) \\ u(s) \end{bmatrix} = T_{cl} \begin{bmatrix} w(s) \\ v(s) \end{bmatrix}$$

$$\|T_{cl}\|_\infty = \sup_\omega \{\sigma_{\max}(T_{cl}(i\omega))\}$$

This problem is hard even for linear systems so instead one chooses and attenuation level γ and seeks a compensator that achieves this level,

$$\|\Sigma_{cl}\| \leq \gamma$$

If this is possible we lower γ and try again, if not...

The result is a differential game, with two players, the controller who chooses $u(\cdot)$ and nature who chooses the noises $w(\cdot), v(\cdot)$.

The controller's goal is

$$\inf_{u} \sup_{w,v} \left\{ \int_0^\infty |z|^2 + |u|^2 - \gamma^2 \left(|w|^2 + |v|^2 \right) \, dt \right\}$$

As with LQG the problem splits into two parts. First find the optimal state feedback assuming that the state can be measured exactly. Then construct an optimal observer to estimate the state.

The attenuation level γ is achievable iff there are positive definite solutions to two algebaic Riccati equations

$$0 = F'P + PF + P\left(\frac{1}{\gamma^2}BB' - GG'\right)P + C'C$$

$$0 = \bar{F}'Q + Q\bar{F} + \frac{1}{\gamma^2}QBB'Q - \gamma^2 H'(DD')^{-1}H + K'K$$

where the optimal state feedback gain is

$$K = -G'P$$

the optimal observer gain is

$$L = \gamma^2 Q^{-1} H'$$

and the dynamics with the least desirable driving noise is

$$\bar{F} = F + \frac{1}{\gamma^2}BB'P$$

If it is known that the noises are in a given frequency band, then a linear prefilter can added to accentuate the input in this band.

If we wish to attenuate more in a given frequency band than a postfilter can be added to accentuate the output in this band.

Both of these techniques increase the dimension of the state of the model and result in a compensator of larger dimension.

Nonlinear Control around an Equlibrium

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{aligned}
$$

$$
\begin{aligned}
0 &= f(x^0, u^0) \\
y^0 &= h(x^0, u^0)
\end{aligned}
$$

Linear Approximating System

$$
\delta x = x - x^0, \quad \delta u = u - u^0, \quad \delta y = y - y^0
$$

$$
\begin{aligned}
\dot{\delta x} &= F\delta x + G\delta u \\
y &= H\delta x + J\delta u
\end{aligned}
$$

$$F = \frac{\partial f}{\partial x}(x^0, u^0) \quad G = \frac{\partial f}{\partial u}(x^0, u^0)$$

$$H = \frac{\partial h}{\partial x}(x^0, u^0) \quad J = \frac{\partial h}{\partial u}(x^0, u^0)$$

If this linear system is stabilizable and detectable then we can design a linear compensator and impliment it on the nonlinear system

$$\begin{aligned}
\dot{x} &= f(x, u^0 + K\hat{\delta x}) \\
y &= h(x, u^0 + K\hat{\delta x}) \\
\dot{\hat{\delta x}} &= f(x^0 + \hat{\delta x}, u^0 + K\hat{\delta x}) - L(y - \hat{y}) \\
\hat{y} &= h(x^0 + \hat{\delta x}, u^0 + K\hat{\delta x})
\end{aligned}$$

The closed loop system is locally asymptotically stable because its linear approximation is asymptotically stable.
If the linear system is not stabilizable and detectable ...?

Frequently there are multiple operating points (equilbria) and one designs a linear controller for each.

Transitions between operating points are done in a feedforward fashion, frequently by an operator.

Gain scheduling in the regions between operating points?

All of this can be done around a reference trajectory, $\boldsymbol{x^0(t), u^0(t), y^0(t)}$, then the linear approximating system is time varying,

$$\begin{aligned} \dot{\delta x} &= F(t)\delta x + G(t)\delta u \\ y &= H(t)\delta x + J(t)\delta u \end{aligned}$$

$$F(t) = \frac{\partial f}{\partial x}(x^0(t), u^0(t)) \ldots$$

Henceforth $\boldsymbol{x^0 = 0,\ u^0 = 0,\ y^0 = 0}$

Feedback Linearization, Dynamic Inversion

A controllable linear system can always be transformed to Brunovsky form by a linear change of coordinates and invertible linear feedback. Suppose $m = 1$ then Brunovsky form is just a string of integrators

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v$$

Nonlinear change of coordinates and invertible feedback

$$z = \phi(x), \qquad v = \alpha(x) + \beta(x)u, \quad \beta(x) \neq 0$$

(locally) transforms the linear system into a nonlinear one

$$\dot{x} = f(x) + g(x)u$$

Suppose $\boldsymbol{v} = \boldsymbol{K}\boldsymbol{z}$ stabilizes the linear system then

$$\boldsymbol{u} = \frac{1}{\boldsymbol{\beta}(\boldsymbol{x})}\left(\boldsymbol{K}\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{\alpha}(\boldsymbol{x})\right)$$

stabilizes the nonlinear system. Automatic gain scheduling.

What nonlinear systems can be (locally) linearized by change of state coordinates and invertible state feedback?

The answer to this question is found by observing that the function $\boldsymbol{\psi}(\boldsymbol{x}) = \boldsymbol{z}_1 = \boldsymbol{\phi}_1(\boldsymbol{x})$ satisfies

$$\begin{aligned}
\boldsymbol{L_g}(\boldsymbol{\psi})(\boldsymbol{x}) &= \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \\
\boldsymbol{L_g}(\boldsymbol{L_f}(\boldsymbol{\psi}))(\boldsymbol{x}) &= \boldsymbol{L_g}\left(\frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{x}}\boldsymbol{f}\right)(\boldsymbol{x}) = \boldsymbol{0} \\
&\vdots \\
\boldsymbol{L_g}(\boldsymbol{L_f^{n-1}}(\boldsymbol{\psi}))(\boldsymbol{x}) &\neq \boldsymbol{0}
\end{aligned}$$

This is an $\boldsymbol{n^{th}}$ degree system of PDEs for $\boldsymbol{\psi}$ but it is equivalent to a first degree system of PDEs.

The integrability conditions are given by the Frobenius Theorem.

Unfortunately most nonlinear systems are not feedback linearizable.

Moreover the technique is not applicable to a nonlinear system whose linear approximation is not controllable.

Also it is a local technique.

But fortunately Hamiltonian systems with one actuator for each degree of freedom are frequently feedback linearizable.

Example of of feedback linearizable system.

$$\begin{aligned}\dot{x}_1 &= x_2 + f_1(x_1)\\ &\vdots\\ \dot{x}_i &= x_{i+1} + f_i(x_1, \ldots, x_i)\\ &\vdots\\ \dot{x}_n &= u + f_n(x_1, \ldots, x_n)\end{aligned}$$

Such a system is said to be in strict feedback form.

Backstepping is a technique of using x_{i+1} as a pseudo control to stabilize x_i .

Let $z_1 = x_1,\ z_2 = x_2 - \alpha_1(z_1),\ V_1(z) = \frac{1}{2}z_1^2$ then

$$\begin{aligned}\dot{z}_1 &= z_2 + \alpha_1(z_1) + f_1(z_1)\\ \dot{V}_1 &= z_1\left(z_2 + \alpha_1(z_1) + f_1(z_1)\right)\end{aligned}$$

Choose $\alpha_1(z_1)$ so that $\alpha_1(z_1) + f_1(z_1) = -z_1$ then

$$\dot{V}_1 = -z_1^2 + z_1 z_2$$

Next define $z_3 = x_3 - \alpha_2(z_1, z_2)$, $V_2(z) = \frac{1}{2}(z_1^2 + z_2^2)$ then

$$\begin{aligned} \dot{z}_2 &= z_2 + \alpha_2(z_1, z_2) + f_2(z_1, z_2) \\ \dot{V}_2 &= -z_1^2 + z_2 \left(z_1 + z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2)\right) \\ \bar{f}_2(z_1, z_2) &= f_2(z_1, z_2) - \frac{d}{dt}\alpha_1(z_1) \end{aligned}$$

Choose $\alpha_2(z_1, z_2)$ so that $\alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2) = -z_2$ then

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3$$

Etc. until u stabilizes z_n .

$$
\begin{aligned}
V_n &= \frac{1}{2}\left(z_1^2 + \ldots + z_n^2\right) \\
\dot{V}_n &= -\left(z_1^2 + \ldots + z_n^2\right)
\end{aligned}
$$

- Backstepping is feedback linearization at the level of the Lyapunov function.
- If $\bar{f}_i(z_1, \ldots, z_i)$ is already stabilizing e.g., $\bar{f}_i = -z_i$, don't cancel it out.
- Backstepping can be done robustly when the dynamics is not exactly known.
- It is applicable only to feedback linearizable systems.
- The change of coordinates from x to z is usually very poorly conditioned.

Stabilization through Optimization

Minimize

$$\int_0^\infty l(x,u)\, dt$$

subject to

$$\dot{x} = f(x,u)$$

Hamilton-Jacobi-Bellman (HJB) equation for the optimal cost $\pi(x)$ and optimal feedback $\kappa(x)$.

$$\frac{\partial \pi}{\partial x}(x) f(x,\kappa(x)) = -l(x,\kappa(x))$$

Under suitable conditions $\pi(x)$ is a Lyapunov function for the closed loop system.

Two approaches to solve.

- Discretize and solve HJB equation directly. But $\boldsymbol{\pi}$ is not smooth everywhere and HJB only holds in the viscosity sense.
- Discretize the optimal control problem and solve.

A local solution is possible sometimes. Suppose

$$\begin{aligned} \boldsymbol{f(x,u)} &= \boldsymbol{Fx + Gu + O(x,u)^2} \\ \boldsymbol{l(x,u)} &= \boldsymbol{x'Qx + u'Ru + O(x,u)^2} \end{aligned}$$

and the LQR problem is nice. Then

$$\begin{aligned} \boldsymbol{\pi(x)} &= \boldsymbol{x'Px + O(x)^3} \\ \boldsymbol{\kappa(x)} &= \boldsymbol{Kx + O(x)^2} \end{aligned}$$

where $\boldsymbol{P, K}$ are the solutions to the LQR problem. The higher degree terms in the Taylor series of $\boldsymbol{\pi(x)}$, $\boldsymbol{\kappa(x)}$ satisfy a sequence of linear equations which depend on lower degree terms.

Receding Horizon Control

Discrete time dynamics

$$x^{+} = f(x, u)$$

Minimize

$$\pi(x(t+T)) + \sum_{t}^{t+T-1} l(x, u)$$

Use first order necessary conditions (Pontryagin Maximum Principle) to solve this for the current state $x(t)$. This requires the solution of TPBVP for a system of ODEs.

Then impliment the resulting optimal control $u(t)$ for one time step. Repeat the optimization at time $t + 1$.

The function $\pi(x)$ must be chosen carefully to ensure stability.

This technique is widely used in chemical engineering where the dynamics is slow relative to computational capabilities.

Linearization of the Error Dynamics by Output Injection

$$\dot{x} = f(x) = Fx + O(x)^2$$
$$y = h(x) = Hx + O(x)^2$$

Given $\beta(y) = By + O(y)^2$ where $A = F + BH$ is Hurwitz, find $z = \theta(x)$ satisfying

$$\frac{\partial \theta}{\partial x}(x) f(x) = A\theta(x) - \beta(h(x))$$

for then

$$\dot{z} = Az + \beta(y)$$
$$\dot{\hat{z}} = A\hat{z} + \beta(y)$$
$$\dot{\tilde{z}} = A\tilde{z}$$

If $\hat{\boldsymbol{z}} = \boldsymbol{\theta}(\hat{\boldsymbol{x}})$ then

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{f}(\hat{\boldsymbol{x}}) - \left(\frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{x}}(\hat{\boldsymbol{x}})\right)^{-1} (\boldsymbol{\beta}(\boldsymbol{y}) - \boldsymbol{\beta}(\boldsymbol{h}(\boldsymbol{x})))$$

Automatic observer gain scheduling.

The PDE is locally solvable for almost all real analytic systems where $\boldsymbol{H}, \boldsymbol{F}$ is an observable pair.

The Taylor series of $\boldsymbol{\theta}(\boldsymbol{x})$ can be found degree by degree.

Unfortunately when there are inputs, additional integrability conditions must be satisfied.

What do we do when $\boldsymbol{H}, \boldsymbol{F}$ is not an observable pair?

Extended Kalman Filtering (EKF)

Add WGNs to the nolinear model

$$\begin{aligned} \dot{x} &= f(x,u) + Bw \\ y &= h(x,u) + Dv \end{aligned}$$

and linearize around the estimated trajectory.

$$F(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \quad H(t) = \frac{\partial h}{\partial x}(\hat{x}(t), u(t))$$

Then build a Kalman filter for the linear system and implment it on the nonlinear one.

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u) - L(t)(y - h(\hat{x}, u)) \\ L(t) &= -P(t)H(t)'(DD')^{-1} \\ \dot{P} &= F(t)P(t) + P(t)F(t)' + BB' \\ &\quad -P(t)H(t)'(DD')^{-1}H(t)P(t) \end{aligned}$$

Interpertation

$$x(t) \approx N(\hat{x}(t), P(t))$$

- The EKF is the most widely used nonlinear filter.
- It generally performs well but can diverge.
- Recently it has shown to be locally convergent for a broad class of nonlinear systems in both continuous and discrete time.

Unscented Kalman Filter

In discrete time, it uses a deterministic sample of $2n + 1$ points from $N(\hat{x}(t), P(t))$, the noisy dynamics and Bayes rule to obtain $\hat{x}(t+1),\ P(t+1)$

Nonlinear Filtering

Ito equations

$$\begin{aligned} dx &= f(x,u)dt + Bdw \\ dy &= h(x,u)dt + dv \end{aligned}$$

Suppose the density of $x(0)$ is $p^0(x)$, compute the conditional density $p(x,t)$ of $x(t)$ given $y(0:t),\ u(0:t)$.

Zakai equation

$$dq = -\sum \frac{\partial q f_i}{\partial x_i} dt + \frac{\partial^2 q}{\partial x_i \partial x_j}(BB')_{ij} dt + q \sum h_i dy_i$$

$$\begin{aligned} q(x,0) &= p^0(x) \\ p(x,t) &= \frac{q(x,t)}{\int q(z,t)\, dz} \end{aligned}$$

The Zakai equation is stochastic PDE in the Ito sense driven by the observations. It has to be solved in its Stratonovich form.

It is very difficult to solve if $n > 1$. It can't be solved implicitly and it is parabolic so if the spatial step is small, the temporal step is extremely small.

The function $x \mapsto q(x,t)$ can be thought of as the state at time t of an infinite dimensional observer with inputs $u,\ y$.

Many nonlinear estimators are infinite dimensional.

Monte Carlo Filtering, Particle Filtering

$$p(x,t) \approx \sum \alpha_k(t)\delta(x - x^k(t))$$

Sample $p(x,t)$, use the noisy system and the Bayes formula to compute $\alpha_k(t+1),\ x^k(t+1)$

There are several different implimentations of this basic philosophy including replacing the point masses with Gaussians.

Can be viewed as a multiscale approach. The small scale is the local dynamics of $x^k(t)$. The large scale is the evolution of $p(x,t)$.

Nonlinear H^∞ Control

$$\begin{aligned}\dot{x} &= f(x,u) + b(x)w\\ y &= h(x,u) + d(x)v\\ z &= c(x)\end{aligned}$$

The goal again is to find a compensator that minimizes the $L^2[0,\infty)$ induced norm of the closed loop mapping

$$\Sigma_{cl} : \begin{bmatrix} w(\cdot) \\ v(\cdot) \end{bmatrix} \mapsto \begin{bmatrix} z(\cdot) \\ u(\cdot) \end{bmatrix}$$

This is too hard to do directly so instead we choose an attenuation level γ and try to achieve it.

$$\inf_{u} \sup_{w,v} \left\{ \int_0^\infty |z|^2 + |u|^2 - \gamma^2 \left(|w|^2 + |v|^2 \right) \, dt \right\}$$

Again this breaks into two problems, first find a state feedback that achieves the attenuation level γ for the mapping

$$\boldsymbol{w}(\cdot) \mapsto \begin{bmatrix} \boldsymbol{z}(\cdot) \\ \boldsymbol{u}(\cdot) \end{bmatrix}$$

Then one tries to design an observer such that the overall system has attenuation level γ .

Both these problems reduce to finding postive definite solution of a Hamilton-Jacobi-Isaacs (HJI) partial differential inequality. In the linear case these reduce to the Riccati equations seen before.

The first HJI for the state feedback does not involve the second but the form of the second HJI for the observer depends on the first.

All our previous remarks about the difficulty of solving HJB and Zakai equations hold and then some!