Synchronous Message Passing

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Abstract

This paper studies the characteristics of synchronous ordering of messages. Synchronous ordering of messages defines synchronous communication based on the causality relation rather than time. We present the necessary characteristics of any algorithm providing deadlock-free synchronous ordering of the messages. We also present the sufficient conditions, based on the causality relations, for any algorithm to provide synchronous ordering. The paper proposes an algorithm using acknowledgment messages to implement the sufficient conditions. The acknowledgment messages are used to satisfy the causality relation between the events. The algorithm is deadlock-free, and provides a higher degree of concurrency than the algorithms which define synchronous communication based on time.

1 Introduction

Distributed programs are difficult to design and test due to their non-deterministic nature. That is, a distributed program may exhibit multiple behaviors on the same external input. This non-determinism is caused by a possible reordering of messages in different executions. It is sometimes desirable to control the non-determinism by restricting the possible message ordering in a system. For example, many systems restrict message delivery to FIFO order. This results in simplicity in design of distributed algorithms which may depend on the FIFO assumption [3]. In this paper, we discuss one such possible restriction on message ordering called synchronous message ordering.

The problem of restricting message ordering has received wide attention [1, 7, 12, 13, 10]. Charron-Bost, Mattern, and Tel [1] have recently proposed a strict hierarchy of message ordering - asynchronous, FIFO, causal ordering, and synchronous.

An asynchronous computation does not have any restriction on the message ordering. It is easy to implement, and it permits maximum concurrency; but algorithms based on fully asynchronous communication can be difficult to design. These algorithms are required to work for all ordering of the messages. A FIFO ordered computation is also easy to implement. It is usually implemented by using sequence numbers for messages. Causal ordering is a stronger condition than FIFO. Intuitively, causal ordering requires that a single message should not be overtaken by a sequence of messages. Joseph and Birman [7] have given many examples of problems which are easier to solve if causal ordering is assumed. It was first implemented in ISIS [4]. Many other algorithms for causal ordering have appeared since then [12, 13]. Synchronous ordering is a stronger requirement than causal ordering. Algorithms for synchronous systems are much easier to design than those for causally ordered systems. Unlike the case of causal ordering, there is very little work done on synchronous ordering of messages. Usually, to achieve synchronous ordering of messages, the send is blocking, i.e., the sender waits for an acknowledgment from the receiver to execute the next event. This informal algorithm as presented by Charron-Bost, Mattern, and Tel [1] can result in deadlocks.
In this paper, we present a deadlock free algorithm that ensures synchronous ordering of messages. In addition, our algorithm permits higher degree of concurrency than algorithms based on blocking sends. In our algorithm the sender continues to execute internal events and receive events while waiting for an acknowledgment message.

This paper is organized as follows. In Section 2, we give formal definitions of various message orderings in distributed computations. This section also includes background results on characterization of synchronous computation using crowns [1]. In Section 3, we investigate the requirements of any algorithm that achieves synchronous ordering. Note that message ordering can be restricted, in general, by delaying sends or by delaying receives. For example, the causal ordering algorithm proposed by Raynal, Schiper, and Toueg [12] uses receiver based delays. We show in this section that any algorithm that implements synchronous ordering must include both sender and receiver based delays. In Section 4, we present our algorithm and Section 5 provides the proof of its correctness. Section 6 discusses the overhead associated with the algorithm and presents concluding remarks.

2 Distributed Computations

2.1 Our Model

Let a distributed system be defined as a set of $n$ sequential processes $\{P_i \mid 0 \leq i < n\}$. Each process $P_i$ consists of a set of events $\mathcal{E}$, which are classified into three types: send events $\mathcal{S}$, receive events $\mathcal{R}$, and internal events $\mathcal{I}$. The next event that is executed by process $P_i$ after an event $e$ is given by the relation $<_1$. The events in the same process form a total order under the transitive closure of $<_1$.

We assume that every send event has a corresponding receive event, i.e., every message that is sent is eventually received. We also assume that every receive event has a corresponding send event, i.e., there are no spurious messages in the system. A send event $s$ in process $P_i$ and its corresponding receive event $r$ in process $P_j$ ($j \neq i$) are related by the relation $\sim$. This is represented as $s \sim r$. For convenience, we denote the receive event corresponding to the send event $s_i$ by $r_i$ and vice-versa. The message pair is represented as $(s_i, r_i)$. Thus, $\forall i : s_i \sim r_i$.

The causal ordering of events in distributed systems is based on the well known "happened before (→)" relation [8]. The happened before relation (also called causally precedes relation) is defined as the transitive closure of union of $<_1$ and $\sim$ relations. In other words, $e \rightarrow f$ iff

1. $(e <_1 f) \lor (e \sim f)$, or
2. $\exists h : (e \rightarrow h \land h \rightarrow f)$.

Similarly, the relation $\prec$ is defined as the transitive closure of $<_1$, that is, $e \prec f$ iff

1. $e <_1 f$, or
2. $\exists h : (e \prec h \land h \prec f)$.

For example in Figure 1, the events $a, b, c, e, f$ are related as $a <_1 b, b <_1 e, b \sim e$ and $e <_1 f$; therefore, the events $a$ and $f$ are related as $a \rightarrow f$, and the events $a$ and $c$ are related as $a \prec c$ (therefore, $a \rightarrow c$).
2.2 Hierarchy of Communication Modes

Based on the happened before relation a strict hierarchy of communication modes can be defined [1]. The various communication modes are:

**FIFO** : Any two messages from a process $P_i$ to $P_j$ are received in the same order as they were sent. Formally,

$$s_1 < s_2 \implies \neg (r_2 < r_1). \quad \text{(FIFO)}$$

**Causally Ordered** : Let any two send events $s_1$ and $s_2$ in a distributed computation be related such that the first send causally precedes the second send. Then, the second message cannot be received before the first message by any process [7, 12]. Formally,

$$s_1 \rightarrow s_2 \implies \neg (r_2 < r_1). \quad \text{(CO)}$$

**Synchronous** : A computation is synchronous if its time diagram can be drawn such that all message arrows are vertical [1] (see Figure 2). That is, all external events can be assigned a timestamp such that time increases within a single process and for any message its send and receive are assigned the same timestamp. Formally,

$$\exists T : \mathcal{E} \rightarrow \mathbb{N} : \forall s, r, e, f \in \mathcal{E} \setminus I$$

$$s \sim r \implies T(s) = T(r) \quad \text{(SYNC)}$$

$$e \prec f \implies T(e) < T(f).$$
This definition is different from that given in [1] but it can easily be shown to be equivalent. In the rest of the paper, we assume that there are no internal events, as they do not affect the message ordering; therefore, $E = R \cup S$. It is easy to see that, for any two events $e$ and $f$

$$(e \rightarrow f) \land \neg(e \sim f) \implies T(e) < T(f).$$

(1)

The following theorem [1] is proved for the sake of completeness.

**Theorem 1** The hierarchy associated with the various modes of communications is

$Synchronous \subseteq Causally \text{ Ordered} \subseteq FIFO$.

**Proof:**

Causally Ordered $\subseteq$ FIFO : This is true because,

$$s_1 \prec s_2 \implies s_1 \rightarrow s_2.$$  

Synchronous $\subseteq$ Causally Ordered : We show that if a computation is synchronous then it is also causally ordered. Since the communication is synchronous there exists a function $T$ satisfying SYNC.

For any set of events $s_1, s_2 \in \mathcal{S}$ and $r_1, r_2 \in \mathcal{R}$ such that $s_1 \sim r_1, s_2 \sim r_2$ and $s_1 \rightarrow s_2$:

$$T(s_1) = T(r_1) \quad T(s_2) = T(r_2), \quad \text{and} \quad T(s_1) < T(s_2).$$

It follows that $T(r_1) < T(r_2)$. Therefore, (1) implies

$$\neg(r_2 \rightarrow r_1).$$

\[ \|$}

2.3 Crowns in a Distributed Computation

A computation can also be characterized as synchronous based on absence of a structure called “crown”. The concept of a crown was introduced by Charron-Bost et al., in [1] where they also
prove that a computation is synchronous if and only if it does not contain any crown. In the rest of the section we present the definition of a crown and provide a simpler proof of this property.

**Definition (Crown):** Let $C$ be a computation. A crown (of size $k$) is a sequence $\langle (s_i, r_i), i \in \{0, 1, \ldots, k-1\} : s_i \sim r_i \rangle$ of pairs of corresponding send and receive events such that (see Figure 3)

$$s_0 \rightarrow r_1, s_1 \rightarrow r_2, \ldots, s_{k-2} \rightarrow r_{k-1}, s_{k-1} \rightarrow r_0.$$

**Theorem 2** A computation is synchronous iff there is no crown in it.

**Proof:**

**Synchronous $\implies \neg$Crown:**

Since the computation is synchronous there exists a function $T$ satisfying $\text{SYNC}$, and for any two events $e$ and $f$

$$(e \rightarrow f) \land \neg(e \sim f) \implies T(e) < T(f).$$

Suppose, if possible, the computation has a crown of size $k$,

$$s_0 \rightarrow r_1, s_1 \rightarrow r_2, \ldots, s_{k-2} \rightarrow r_{k-1}, s_{k-1} \rightarrow r_0.$$

Therefore,

$$\forall i \in \{0, 1, \ldots, k-1\} \quad T(s_i) < T(r_{(i+1) \mod k}) \quad (*)$$

$$\forall i \in \{1, 2, \ldots, k-1\} \quad T(s_i) = T(r_i). \quad (**)$$

Therefore, from equations $(*)$ and $(**)$,

$$T(s_0) < T(r_0).$$

which is a contradiction because $\text{SYNC}$ implies that $T(s_0) = T(r_0)$.

**$\neg$Crown $\implies$ Synchronous:**

Given a computation, we form a directed graph $G(V, E)$, as follows. The vertex set $V$ consists of all messages in the computation. Thus, each vertex $v_i$ represents a set of two events: the send event $s_i$ and the corresponding receive event $r_i$. That is,

$$v_i = \{s_i, r_i\}.$$

There is an edge from $v_i$ to $v_j$ if there is an event $e \in v_i$ and an event $f \in v_j$ such that $e \rightarrow f$. Thus, $(v_i, v_j) \in E$ iff $(s_i \rightarrow s_j) \lor (s_i \rightarrow r_j) \lor (r_i \rightarrow s_j) \lor (r_i \rightarrow r_j)$. It is easy to see that each of the four disjuncts implies $s_i \rightarrow r_j$. Hence, $(v_i, v_j) \in E$ iff $s_i \rightarrow r_j$.

Since the computation does not have any crown, it follows that the graph $G$ is acyclic. This means that $G$ can be topologically sorted [2]. Therefore, there exists a function $F: E \rightarrow \mathbb{N}$ such that,

$$s \sim r \implies F(s) = F(r) \quad \text{and}$$

$$e \prec f \implies F(e) < F(f).$$

Therefore, the computation is synchronous.
In this paper, we also use a structure called strong crown. We define a strong crown as,

**Definition (Strong Crown):** Let $C$ be a computation. A strong crown (of size $k$) is a sequence $\langle (s_i, r_i), i \in \{0, 1 \ldots, k-1\} : s_i \prec r_i \rangle$ of pairs of corresponding send and receive events such that

$$s_0 \prec r_1, s_1 \prec r_2, \ldots, s_{k-2} \prec r_{k-1}, s_{k-1} \prec r_0.$$

Note that a strong crown is a crown, but not all crowns are strong (see Figure 3). Later to prove the safety of the algorithm we show that, any distributed computation satisfying the conditions of the algorithm, has a strong crown if it has a crown.

### 3 Some Impossibility Results

In this section we present the necessary characteristics of any algorithm that ensures synchronous ordering of the messages. The assumptions for any protocol are:

1. A protocol for any message is restricted to the processes executing the send event and the corresponding receive event. If there is a message $(s, r)$ from process $P_i$ to $P_j$, the protocol should not contain any control messages from $P_k$ ($k \neq i, k \neq j$) to $P_i$ or $P_j$, or vice-versa.

2. A process can take a decision at $t = t_0$ about any event $e$ (i.e., whether to delay it or to execute it) only based on the past. The past of any event $e$ at time $t$ is defined as:

$$\text{past}(e) = \{ f | f \rightarrow e \}.$$

The $\rightarrow$ includes the causality formed by the control messages.

A protocol consists of a receiver part and the sender part. A protocol is defined as *bounded delay send* if there exists an upper bound on the time to complete the sender part. Similarly, a protocol is *bounded delay receive* if there exists an upper bound on the time to complete the receiver part. Intuitively, a protocol is bounded delay receive if on receiving a message $(s, r)$ at time $t = t_0$, the process commits (completes the protocol) the message by time $t = t_1 + \delta t$.

For example, consider FIFO ordering in a two process system. The usual protocol to implement FIFO is a *bounded delay send* but not a *bounded delay receive*. If a process intends to send a message, then it can immediately execute the send and end the sender part. Assume that there are two messages $(s_1, r_1)$ and $(s_2, r_2)$, and $s_1 \prec s_2$. If the receiver process receives $r_2$ at time $t_0$ before the message $r_1$, then it cannot assure the commit of the message in a bounded time. This unbounded time execution of the protocol is due to the uncertainty in the amount of time the message $r_1$ will take. In effect, the process is delaying the receive until some other event has taken place.

In this section we prove that any protocol that implement synchronous ordering must be asymmetric and must include sender and receiver based delays. We assume without loss of generality, that the upper bound on time is $\delta t$ on the completion of the sender part or the receiver part, if there exist one. We also assume, without loss of generality, that to implement the protocol for the message $(s_i, r_i)$, the first message of the protocol between two processes is represented as the message $(s_i, r_i)$. 


**Theorem 3** Any protocol that implements SYNC cannot be

1. symmetric, or
2. have bounded delay sends, or
3. have bounded delay receives.

**Proof:**

1. We show that given a symmetric protocol with respect to the processes, there exists a distributed computation where synchronous ordering is not possible.

   Consider the distributed computation shown in Figure 4(a). Both the processes intend to send a message to each other at the same time $t_0$. Since the past for each of the sends is an empty set, none of the processes can delay the send. It is obvious that the system is in a symmetric state with respect to $P_1$ and $P_2$. If this symmetric state is the input to any symmetric protocol, the resulting output cannot be asymmetric. This rules out the possibility of the asymmetric ordering of messages i.e., $(s_1 < r_2 \land r_1 < s_2)$ or $(s_2 < r_1 \land r_2 < s_1)$, which are the only possible synchronous message orderings.

   Therefore, either the resulting computation does not satisfy SYNC, or the processes will never send the message.

2. *Bounded delay receive* protocol:

   Consider the distributed computation as shown in Figure 4(b). Each of the processes $P_1$, $P_2$ and $P_3$ sends a message at time $t_0$. Since the past for each of the sends is an empty set, none of the processes can delay the send. The message $(s_1, r_1)$ is received by process $P_2$ at time $t_1$. The process $P_2$ completes the protocol for message $(s_1, r_1)$ by time $t_1 + \delta t$ since the protocol is *bounded delay receive*. At the completion of the protocol, process $P_2$ may not have any information about the event $r_2$. This is because the time taken for message $(s_2, r_2)$ may have been greater than $t_1 + \delta t - t_0$.

   Consider message $(s_1, r_1)$; the following are true:

   (a) Processes taking part in the protocol are $P_1$ and $P_2$.
   (b) Process $P_2$ completes the protocol at time $t_1 + \delta t$. 

---

Figure 4: Non-synchronous ordering for impossibilities
(c) Process $P_2$ cannot send any more messages to $P_1$ after time $t_1 + \delta t$, because of Bounded delay receive.

(d) When the process $P_1$ completes the protocol at say time $t_2$, it knows its past (i.e., $t \leq t_2$) and the past of the process $P_2$ before the completion of event $r_1$, i.e., $t \leq (t_1 + \delta t)$.

As a result of statement (d), the process $P_1$ has no knowledge of event $r_2$ at time $t_2$.

Now consider the message $(s_4, r_4)$, which the process $P_1$ wants to execute at $t = t_3 > t_2$. From the assumption 2 of the protocol, the process $P_1$ executes the event $s_4$ based on its past, i.e.,

$$\text{past}(s_4) \subseteq \{s_1, s_2, r_1\}.$$ 

Based on the past, the process $P_1$ cannot delay the send of event $s_4$. Let the process $P_1$ receive the message $(s_4, r_4)$ at time $t_4$. The process $P_4$ completes the protocol at time $t_4 + \delta t$ since the protocol is bounded delay receive.

If the message $(s_3, r_3)$ takes more than $t_4 + \delta t - t_0$ units of time, then the process $P_1$ orders the receive events $r_3, r_4$ such that $r_4 < r_3$. The resulting distributed computation is non-synchronous as there is a crown.

3. Bounded delay send protocol:

Again, consider the distributed computation as shown in figure 4.(b), each process $P_1$, $P_2$, and $P_3$ send a message at time $t_0$.

Assume the process $P_1$ wants to send another message $(s_4, r_4)$ at time $t_1$. Since the protocol is bounded delay send, the process $P_1$ executes $s_4$ and completes the send part of the protocol before $t = t_1 + \delta t$. If the message $(s_1, r_1)$ takes more than $t_1 + \delta t - t_0$ units of time, then the message $(s_4, r_4)$ carries no information about the event $r_1$. Therefore,

$$\text{past}(s_4) = \{s_1\}.$$ 

Let the message $(s_4, r_4)$ reach the process $P_4$ at time $t = t_2$. On receiving the message $r_4$, the process has no knowledge of the message $(s_3, r_3)$ since,

$$\text{past}(r_3) = \{s_1, s_4\}.$$ 

Therefore, the only possible action the process $P_4$ can take is to commit the receive of the message $r_3$ at time $t_2$. The message ordering results in a non-synchronous computation.

4 Algorithm

4.1 Commit Point of a Message

To implement synchronous ordering in the traditional algorithm, a process sends a message $s_1 \sim r_1$ and waits for an acknowledgment $s_2 \sim r_2$ before executing other events. Therefore, the message transaction is completed on the receive of the acknowledgment. In these systems, the
interval from \( s_1 \) to \( r_2 \) is atomic. That is, the sender does not execute any event between \( s_1 \) and \( r_2 \). In our algorithm, the send of a message and the receive of the corresponding acknowledgment result in completion of the message transaction, but the events \( s_1 \) and \( r_2 \) are not atomic. This gives a process flexibility to order the message as if it was sent at either \( s_1 \) or \( r_2 \). For example, in figure 5 the process starts the message transaction at event \( s_1 \) and ends the transaction at event \( r_2 \). If the process \( P_2 \) commits the message at \( s_1 \) the message causally precedes the event \( e \), whereas if the process commits the message at \( r_2 \) the event \( e \) causally precedes the message.

### 4.2 Protocol

The algorithm to implement synchronous ordering of messages has three components: the priority rule (PR), the send condition (SC) and receive condition (RC). As shown in Section 3, any protocol that achieves synchronous ordering must be asymmetric with respect to processes. We introduce this asymmetry by the priority rule. The algorithm introduces control messages. These control messages are denoted by \( s^c \) and \( r^c \) and belong to the event set \( E^c \). Based on the event sets \( E \) and \( E^c \), we define the relations \( \rightarrow_E \) and \( \leftarrow \), subsets of \( \rightarrow \) as,

**Definition of \( \rightarrow_E \):**

\[ e \rightarrow_E f \iff
(e, f \in E) \land
((e \sim f) \lor (e \prec f) \lor (\exists s_1 \in E : e \prec s_1 \land r_1 \rightarrow_E f)). \]

In other words two events are related by \( \rightarrow_E \) iff the causality chain is formed by the events in \( E \).

**Definition of \( \leftarrow \):**

\[ e \leftarrow f \iff
(e, f \in E) \land
((e \sim f) \lor (e \prec f) \lor (\exists s_1 \in E : e \prec s_1 \land r_1 \rightarrow f)). \]

The relations is motivated from the fact that a computation restricted to the event set \( E \) is not synchronous if there exists a crown, such that causality chains are formed by the relation \( \rightarrow_E \). That
is, a computation is synchronous if and only if there is no crown such that,
\[ s_0 \rightarrow E \ r_1, \cdots, s_{k-2} \rightarrow E \ r_{k-1}, s_{k-1} \rightarrow E \ r_0. \]

It is easy to see that if there exist a crown (of size \( k \)) restricted to the event set \( E \) then there exists a crown,
\[ s_0 \leftarrow r_1, s_1 \leftarrow r_2, \cdots, s_{k-2} \leftarrow r_{k-1}, s_{k-1} \leftarrow r_0. \]

### 4.2.1 Priority Rule (PR)

We define a total order among the processes of a distributed system as \( P_i < P_j \) iff \( i < j \). We also define the function \( P : E \rightarrow \mathbb{N} \), such that \( P(e) = i \) iff \( e \) is an event in process \( P_i \). Based on the function \( P \), any message \((s \rightarrow r)\) can be classified into two types (assuming that a process cannot send a message to itself):

**Type 1** A message to a smaller process, i.e., \( P(s) > P(r) \), and

**Type 2** A message to a bigger process, i.e., \( P(s) < P(r) \).

The messages of type 1 are committed at the send of the message by the process that is executing the send. The receiver commits the message as soon as it receives the message. In case of the message of type 2, the smaller process sends a request message to the bigger process. The bigger process, when in a position to commit the message, executes the message and sends the message (with the same content) to the smaller process. The smaller process commits the message on receiving. For example,

**Type 1** \( s \rightarrow r \): As shown in figure 6, the process \( P_2 \) commits on the message at event \( s \) and process \( P_1 \) commits at event \( r \).

**Type 2** \( s \rightarrow r' \): As shown in figure 6, the process \( P_2 \) commits the message at the event \( r' \), and the process \( P_3 \) commits the message at event \( s' \). Therefore, the process \( P_2 \) orders its event such that event \( e \) causally precedes the message (or the send of the message \( s \rightarrow r \)).

Therefore, every message is committed by the participating process at the send and receive of a message from the bigger process to the smaller process. Keeping this in mind we can assume the system has messages from bigger to smaller processes only. Therefore,

\[ \forall (s, r) \in E : P(s) > P(r). \]

### 4.2.2 Send Condition (SC)

The send condition of a protocol delays the send event. The send condition for message \((s_2, r_2) \in E\) is formally stated as
\[ s_1 < s_2 \implies r_1 \rightarrow r_2 \]

Informally, the condition restricts a process from sending a message until it has the knowledge of receives of all the previous sends. That is, there exists a sequence of messages in \( E \cup E^c \) such that \( r_1 \rightarrow r_2 \) holds.
4.2.3 Receive Condition (RC)

The receive condition of the delays the receive of a message. It is formally stated for the receive of message \((s_2, r_2) \in E\) as,

\[ s_1 \prec r_2 \implies \neg (r_2 \rightarrow r_1). \]

Informally, the condition delays the receive of a message until it is sure that there cannot exist a message in \(E \cup E^c\) such that \(r_2 \rightarrow r_1\).

4.3 Implementation

In this section we describe how acknowledgment messages can be used to satisfy the send and receive conditions.

In the algorithm, every message \(e \sim f\) has a underlying acknowledgment message and is represented as \(e.ack \sim f.ack\), as shown in figure 7, where \(e, f \in E\) and \(e.ack, f.ack \in E^c\) (the control message). The acknowledgment messages are used to implement the Send Condition and the Receive Condition.

A process \(P_i\) can be in one of the two states: active or passive. The initial state of every process is active. A process changes its state according to the following rules:

- On sending a message, the process changes its state from active to passive,
- On receiving an acknowledgment, the process changes its state from passive to active.

Let us consider a message \(e \sim f\), were \(e \in P_i\) and \(f \in P_j\). On receiving the message at \(f\), process \(P_j\) executes a send of an acknowledgment \(f.ack\), such that \(f \prec f.ack\) and process \(P_i\) executes a receive of the acknowledgment \(e.ack\), such that \(f.ack \sim e.ack\) and \(e \prec e.ack\) (see figure 7).

4.3.1 Send Protocol (SP)

The Send Protocol prohibits a process from executing a send event \((s \in E)\) when it is passive. For example in figure 7, the process \(P_i\) was active just before the event \(e\), and passive until the event.
\textit{e.ack}. As the event \(e\) can be enabled only if the process is \textit{active}, the last send from the process \(P_i\) must have been acknowledged before \(e\).

Formally, we can state SP as,

To send a message \((s_1 \sim r_1)\) from \(P_i\) to \(P_j\) \((i > j)\), wait until it is \textit{active} (i.e., wait for an acknowledgment for the previous send). Therefore,

\[
s_1 < s_2 \implies s_1.\text{ack} < s_2.
\]

**Theorem 4** \(SP\) is sufficient to implement \(SC\).

**Proof:** Let \(s_1 < s_2\). We need to show that \(r_1 \rightarrow r_2\).

From SP,

\[
s_1 < s_2 \implies s_1.\text{ack} < s_2. \tag{2}
\]

By the conditions of acknowledgment messages, \(r_1 < r_1.\text{ack}\) and \(r_1.\text{ack} \sim s_1.\text{ack}\). Therefore, \(r_1 \rightarrow s_1.\text{ack}\). From (2), we get that \(r_1 \rightarrow s_2\). This in turn implies \(r_1 \rightarrow r_2\) since \(s_2 \sim r_2\). \(||\)

4.3.2 Receive Protocol (RP)

The Receive Protocol requires that a process send an acknowledgment for the receive of a message only if it is \textit{active} (i.e., it has received an acknowledgment for the previous send). For example, in figure 7 the process \(P_j\) can execute the event \(f.\text{ack}\) when the process is \textit{active}, i.e., the last send from the process \(P_j\) has been acknowledged.

Formally, we can state RP as,

On receiving a message \((s_1 \sim r_1)\) from \(P_i\), \(P_j\) \((i > j)\), commits on the receive of the message and sends an acknowledgment \((r_1.\text{ack} \sim s_1.\text{ack})\) back when it is \textit{active} (i.e., the acknowledgment of the last send of the message from \(P_j\) has been received). Therefore,

\[
s_1 < r_2 \implies s_1.\text{ack} < r_2.\text{ack}.
\]

**Theorem 5** If a system satisfies \(SP\), then \(RP\) is sufficient to implement \(RC\).
\(P_i::\)

\[
\begin{array}{l}
\text{var} \\
\text{messageQueue} :: \text{Queue of messages} \\
\text{ackQueue} :: \text{Queue of messages} \\
\text{state} :: \text{Boolean initially active}
\end{array}
\]

\[\square \text{SendIntent} ((m, P_i, P_j))\]

\[
\begin{array}{l}
\text{if } (i < j) \\
\quad \text{/* message to bigger process PR */} \\
\quad \text{send } ((m, P_i, P_j)) \\
\text{else} \\
\quad \text{if } (\text{state} = \text{active}) \\
\quad \quad \text{/* commit and send the message */} \\
\quad \quad \text{state} = \text{passive} \\
\quad \quad \text{send } ((m, P_i, P_j)) \\
\quad \quad \text{commit } ((m, P_i, P_j)) \\
\quad \text{else} \\
\quad \quad \text{/* wait for the ack message SP */} \\
\quad \text{Enqueue } (\text{messageQueue}, (m, P_i, P_j))
\end{array}
\]

\[\square \text{Receive} ((m, P_j, P_i))\]

\[
\begin{array}{l}
\text{if } (i > j) \\
\quad \text{/* message from smaller process PR */} \\
\quad \text{SendIntent } ((m, P_i, P_j)) \\
\text{else} \\
\quad \text{Commit } ((m, P_j, P_i)) \\
\quad \text{if } (\text{state} = \text{active}) \\
\quad \quad \text{send } ((\text{ack}, P_i, P_j)) \\
\quad \text{else} \\
\quad \quad \text{/* wait for the previous ack SP */} \\
\quad \quad \text{Enqueue } (\text{ackQueue}, (\text{ack}, P_i, P_j))
\end{array}
\]

\[\square \text{Receive} ((\text{ack}, P_i, P_j))\]

\[
\begin{array}{l}
\text{state} = \text{active} \\
\text{while } (\text{notEmpty} (\text{ackQueue})) \\
\quad \text{/* send acks if active SP */} \\
\quad \text{\langle ack, P_i, P_j \rangle = Dequeue } (\text{ackQueue}) \\
\quad \text{send } ((\text{ack}, P_i, P_j)) \\
\text{if } (\text{notEmpty} (\text{messageQueue})) \\
\quad \text{state} = \text{passive} \\
\quad \text{\langle m, P_i, P_j \rangle = Dequeue } (\text{messageQueue}) \\
\quad \text{send } ((\langle m, P_i, P_j \rangle) \\
\quad \text{commit } ((\langle m, P_i, P_j \rangle)
\end{array}
\]

Figure 8: Algorithm to Implement Synchronous Ordering of Messages
Proof: Let $s_1 \prec r_2$. We need to show that $\neg(r_2 \rightarrow r_1)$.
From $s_1 \prec r_2$, we get that $s_1.ack \prec r_2.ack$ using RP. Now consider $s_1.ack$ and $r_2$. There are two cases possible:

Case 1: $s_1.ack \prec r_2$:
Since $r_1 \rightarrow s_1.ack$ and $s_1.ack \prec r_2$, we get that $r_1 \rightarrow r_2$.
Therefore, $\neg(r_2 \rightarrow r_1)$ holds.

Case 2: $r_2 \prec s_1.ack$:
Since $r_1 \rightarrow s_1.ack$,

$$r_2 \rightarrow r_1 \implies \exists s \in \mathcal{E} \cup \mathcal{E}^c : r_2 \prec s \prec s_1.ack \land r \rightarrow r_1.$$ The send $s \notin \mathcal{E}$ otherwise SP is violated. The send $s' \notin \mathcal{E}$ otherwise RP is violated. Therefore there cannot exists a send in the interval starting at $r_2$ and ending at $s_1.ack$.
Therefore, $\neg(r_2 \rightarrow r_1)$ holds.

5 Proof of correctness

The correctness of the above algorithm is proved in the usual two steps: safety and liveness.

5.1 Proof of Safety

Our strategy of the proof is to show that if a distributed computation satisfies SC and AC then it will not have a crown formed by causality of events in the set $\mathcal{E}$. We first show that if there exists a crown (of size $k$) in a distributed run (or computation) then, there also exists a strong crown of size $k'$ where $k' \leq k$. The second part of the proof shows that a synchronous distributed run cannot have a strong crown. We first prove an elementary lemma.

**Lemma 1** If $(e \leftarrow f) \land \neg(e \sim f)$ then, 

$$(e \sim f) \lor \exists(s_1, r_1) \in \mathcal{E} : (e \prec s_1 \land r_1 \rightarrow f) \lor \exists g : (e \sim g \land g \rightarrow f).$$

**Proof**: The lemma states that if two events are causally related, then either they are in the same process, or there exists a send event in the same process (and corresponding receive) in the causality chain.

The next lemma is used to reduce the crown to a strong crown. If two events $s_1 \in \mathcal{S}$ and $r_2 \in \mathcal{R}$ are causally related as $s_1 \leftarrow r_2$, then either both the events are in the same process or there is a send event $s_3 \in \mathcal{E}$ in the causality chain such that $s_1 \prec s_3$.

**Lemma 2** Let $(s_1 \sim r_1)$ and $(s_2 \sim r_2)$. If $s_1 \leftarrow r_2$ and $SC$, then 

$$r_1 \rightarrow r_2 \lor s_1 \prec r_2.$$
**Proof:** If $s_1 \rightarrow r_2$ then (by lemma 1),

(i) $s_1 \prec r_2 \lor$

(ii) $s_1 \sim r_1 \land r_1 \rightarrow r_2 \lor$

(iii) $\exists s_3 \in \mathcal{E} : s_1 \prec s_3 \land r_3 \rightarrow r_2$.

The first two cases directly satisfy the lemma. In the third case, as $s_1 \prec s_3$, therefore

$$r_1 \rightarrow r_3$$

by SC.

and as $r_3 \rightarrow r_2$, we have $r_1 \rightarrow r_2$.

We now show that if a crown exists then there also exists a strong crown in the distributed computation.

**Lemma 3** If a distributed computation satisfies SC and RC, and it does not have a strong crown of size 2 formed by the events in $\mathcal{E}$, then it does not have a crown (of size 2) such that,

$$s_1 \rightarrow r_2, \ s_2 \rightarrow r_1.$$

**Proof:** Assume that a distributed computation has a crown of size 2.

$$s_1 \rightarrow r_2, \ s_2 \rightarrow r_1.$$

Since the crown is not strong (without loss of generality) assume that $\neg(s_1 \prec r_2)$. From lemma 2, we get that

$$r_1 \rightarrow r_2.$$  \hspace{1cm} (3)

From $(r_1 \rightarrow r_2)$ by using RC we get that $\neg(s_2 \prec r_1)$.

Applying lemma 2 again to $\neg(s_2 \prec r_1)$ and $s_2 \rightarrow r_1$, we get that

$$r_2 \rightarrow r_1,$$

which is a contradiction to equation 3.

**Lemma 4** If a distributed computation satisfying AC and SC has a crown (of size $k$),

$$s_0 \rightarrow r_1, s_1 \rightarrow r_2, \cdots, s_{k-2} \rightarrow r_{k-1}, s_{k-1} \rightarrow r_0,$$

then it also contains a strong crown of size $(k' \leq k)$ such that,

$$s_0' \prec r_1', s_1' \prec r_2', \cdots, s_{k-2}' \prec r_{k-1}', s_{k-1}' \prec r_0'.$$
**Proof:** If \( k \leq 2 \), then the theorem follows directly from lemma 3. Assume \( k > 2 \). Pick any part of the crown starting from any index \( i \mod k \),

\[
s_{i-1} \leftrightarrow r_i, s_i \leftrightarrow r_{i+1} \quad (\star)
\]
such that, \( \neg(s_i < r_{i+1}) \), such an \( i \) exists otherwise the crown is already strong. Since \( s_i \leftrightarrow r_{i+1} \) (by lemma 2),

\[ r_i \rightarrow r_{i+1}. \]

Since \( s_{i-1} \rightarrow r_i \) and \( r_i \rightarrow r_{i+1} \), equation (\star) can be reduced to

\[ s_{i-1} \leftrightarrow r_{i+1}. \]

Therefore, in the sequence \((k > 2)\),

\[
s_0 \leftrightarrow r_1, s_1 \leftrightarrow r_2, \ldots, s_{k-2} \leftrightarrow r_{k-1}, s_{k-1} \leftrightarrow r_0.
\]

If \( \neg(s_i < r_{i+1}) \) then the sequence can be reduced by removing the \((s_i, r_i)\) message. On repeating the process the resulting sequence can be:

- A strong crown of size \( k' \leq k \):

\[
s'_0 \prec r'_1, s'_1 \prec r'_2, \ldots, s'_{k-2} \prec r'_{k-1}, s'_{k-1} \prec r'_0.
\]

- Or the resulting crown is of size 2, such that

\[ s_1 \leftrightarrow r_2, s_2 \leftrightarrow r_1. \]

By lemma 3, the two crown is strong.

\[ \|
\]

**Theorem 6 (Safety)** Any distributed run that satisfies SC, AC and PR is synchronous.

**Proof:** The proof is by contradiction. Let a distributed run be not synchronous. Then by Theorem 2 there exists a crown. The conditions of lemma 4 are true, i.e., SC and AC. Therefore by lemma 4, there will exist a strong crown,

\[
s_0 \prec r_1, s_1 \prec r_2, \ldots, s_{k-2} \prec r_{k-1}, s_{k-1} \prec r_0.
\]

From the crown we get

\[
\forall i \in \{0, 2, 3, \ldots, k - 1\} : \quad P(s_i) = P(r_{i+1 \mod k}), \quad (\star)
\]

and (by PR)

\[
\forall i \in \{1, 2, 3, \ldots, k - 1\} : \quad P(s_i) > P(r_i). \quad (**)
\]

Combining (\star) and (**) we get,

\[ P(r_0) > P(s_0), \]

which is a contradiction to PR.

\[ \|
\]
**Theorem 7 (Safety)** Any distributed computation that satisfies SP and RP is synchronous.

**Proof:** The proof is by contradiction. Let a distributed run be not synchronous. Then by Theorem 2 there exists a crown. By lemma 4, there will exist a strong crown,

\[ s_0 \prec r_1, s_1 \prec r_2, \ldots, s_{k-2} \prec r_{k-1}, s_{k-1} \prec r_0. \]

If \( s_i \prec r_{i+1} \) then by RP, \( s_i.ack \rightarrow r_{i+1}.ack \). Therefore, from the crown we get

\[ \forall i \in \{1, 2, 3, \ldots, (k-1)\} \quad s_i.ack \rightarrow r_{i+1}.ack, \quad (*) \]

and (by definition)

\[ \forall i \in \{1, 2, 3, \ldots, k\} \quad r_i.ack \rightarrow s_i.ack. \quad (**) \]

Combining (*) and (**) we get,

\[ r_1.ack \rightarrow s_k.ack, \]

but according to the strong crown, we have

\[ s_k \prec r_1 \implies s_k.ack \prec r_1.ack, \]

which is a contradiction.

\[ || \]

**5.2 Proof of Liveness**

**Theorem 8 (Liveness)** In a distributed computation that satisfies SP, RP and PR, every process \( P_k \) that wants to send a message will eventually be able to send it.

**Proof:** By induction on \( k \).

Case \( k = 1 \): The smallest process \( P_1 \) does not send any message therefore it is always active. It sends an acknowledgment as soon as it gets a message. Therefore on receiving a message \((s \sim r)\),

\[ r \prec r.ack. \]

Now on applying induction, given that \( k \) smallest processes eventually be in active state, then \((k+1)\)th process if passive will eventually be active. The process \( P_{k+1} \) is passive at time \( t \) if

1. there exists a send of message \((s_1, r_1)\) at time \( t_0 < t \) and

2. the process is passive between the time interval from \( t_0 \) to \( t \).

Therefore, there exists an acknowledgment message \((r_1.ack, s_1.ack)\) from a process \( P(r_1) \) to \( P_{k+1} \) such that,

1. the acknowledgment is in transit, or

2. send of the acknowledgment will eventually be executed when the process \( P(r_1) \) is active.

If the message is in transit then process \( P_{k+1} \) will eventually receive \( s_1.ack \) and become active. If the second condition is true, then as \( P(r_1) < P_{k+1} \) therefore, \( P(r_1) \) will eventually turn active and execute send of acknowledgment message. Therefore, process \( P_{k+1} \) will eventually be active.  

\[ || \]
6 Conclusions

The algorithm consists of three components: the priority rule, send condition, and the receive condition. The priority rule results in one control message and a delay of less than $2t$ time units (assuming that the upper bound in the delay between any two processes is $t$ time units) for every message $(s, r) \in \mathcal{E}$ if $P(s) < P(r)$. In the case of any message $(s, r) \in \mathcal{E}$ where $P(s) > P(r)$ there are no control message or delay introduced. However, note that during the delay introduced due to PR, the process can continue to execute any other event.

The send condition introduces only one control message of every message and the receive condition introduces a delay of which is upper bounded by $nt$, were $n$ is the number of processes in the system. Therefore, the message complexity is $1.5$ control messages ($\in \mathcal{E}^c$) for every message $(s, r) \in \mathcal{E}$.

On comparing with the informal algorithm for synchronous ordering, where the sender waits until the receive of an acknowledgment, the resulting algorithm has a higher degree of concurrency. This can be easily seen as both the send condition and the receive conditions are satisfied by informal algorithm.

In this paper we studied the characteristics of a synchronous ordering of messages. The necessary characteristics, i.e., asymmetric and both sender and receiver based protocol, for any algorithm to ensure synchronous ordering were presented. The conditions sufficient (PR, SP, and RP) to implement synchronous ordering were presented based on the necessary characteristics to ensure safety properties. Further, an algorithm was presented based on acknowledgment messages to satisfying SP and RP conditions.

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References


