Maximal Antichain Lattice Algorithms for Distributed Computations

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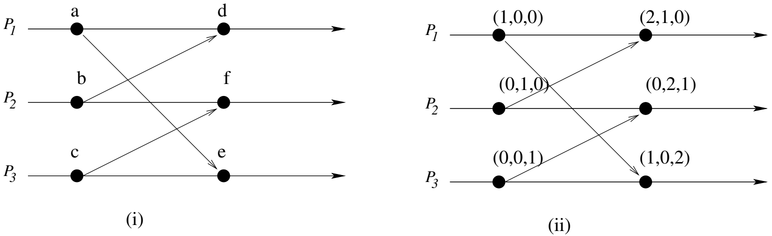
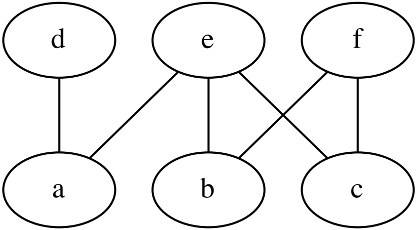
Model of a Distributed Computation: Poset

distributed computation = poset (partially ordered set)
$(E, \rightarrow)$ where
$E =$ is the set of events, and
$\rightarrow$ is (happened-before) relation.



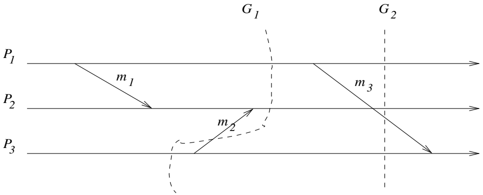
Events can be timestamped in an online fashion using Vector Clocks such that $e \rightarrow f \equiv V(e) < V(f)$.

Computing Meet and Join



- Meet of a subset of events
  meet of {d, e}
  meet of {a,b}
  meet of {e,f}
- Join of a subset of events
- Lattice: a poset in which all finite subsets have meets and joins.

Consistent Cuts of a Distributed System



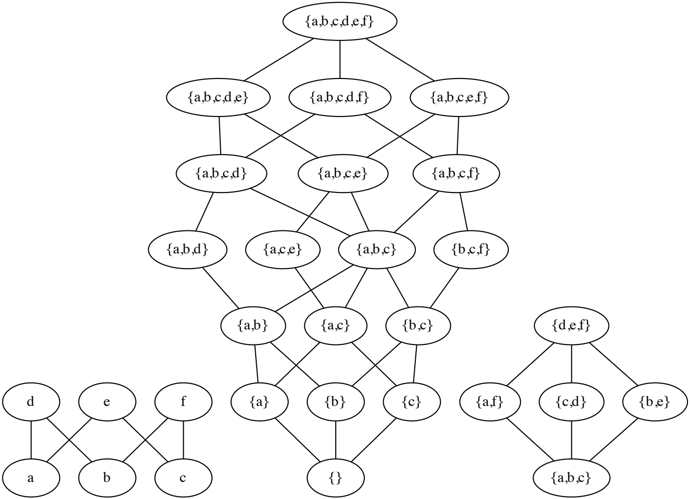
Consistent cut = set of events executed so far

A subset G of E is a consistent cut (consistent global state) if

$$\forall e, f \in E : (f \in G) \wedge (e \to f) \Rightarrow (e \in G)$$

Same as the *order* ideal of the partial order $(E, \to)$.

Motivation: Detecting Global Conditions in Distributed Systems

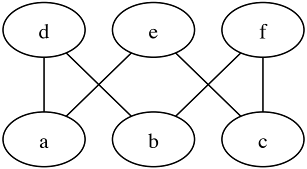


- Traversing a significantly smaller lattice of maximal antichains rather than consistent cuts for certain predicates

Outline of the Talk

- Motivation
- Incremental Lattice Algorithms
- Lattice Enumeration Algorithms
- Applications to Global Predicate Detection
- Conclusions and Future Work

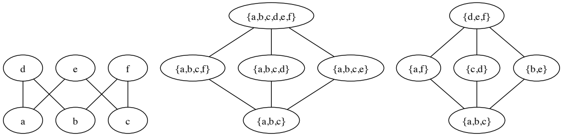
Ideals and Antichains



Poset $P = (X, \leq)$

- Ideal: $Q \subseteq X$ is an ideal $\equiv$ if f is in Q and $e \leq f$, then e is also in Q.
- Antichain: $Y \subseteq X$ is called an *antichain*, if every distinct pair of elements from Y are incomparable.

Maximal Antichains



- Maximal Antichain: An antichain A is maximal if every element not in A is comparable to some element in A.
  $\{d, e\}$ is an antichain but not a maximal antichain
  $\{d, e, f\}$ is a maximal antichain
- Maximal Ideal: An ideal Q is maximal antichain ideal if its maximal elements forms a maximal antichain.
  $\{a, b, c, d\}$ and $\{a, b, c, e\}$ are maximal ideals
  $\{a, b, c, d, e\}$ is not a maximal ideal

Important Lattices Associated with a Poset

Lattice of	Interpretation in DC	References
Ideals	consistent global states	[Mattern88, CM91,..]
Normal Cuts	Smallest lattice containing P	[Garg OPODIS12]
Maximal Ideals	State for maximal antichain	[JRJ94, this paper]

Table: Summary of Lattices for Distributed Computations modeled as a poset P

Related Work: Incremental Algorithms

Elements of the computation arrive in a sorted order

Input: poset P, its lattice of maximal antichains L, element x

Output: $L' :=$ lattice of maximal antichains of $P \cup \{x\}$

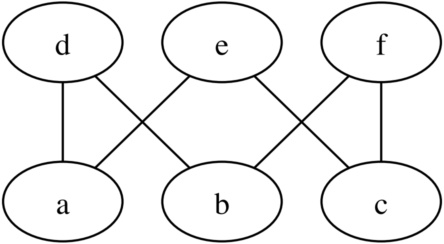
Incremental Algorithms	Time Complexity	Space Complexity
[Jourdan, Rampon, Jard 94]	$O(w^3 m)$	$O(mn \log n)$
[Nourine and Raynaud 99, 02]	$O(mn)$	$O(mn \log n)$
Algorithm ILMA [this paper]	$O(wm \log m)$	$O(mw \log n)$
Algorithm OLMA [this paper]	$O(m_x w^2 \log w_L))$	$O(w_L w \log n)$

Symbol	Definition	Symbol	Definition
n	size of the poset P	m	size of the lattice L
w	width of the poset P	m_x	# (strict ideals $\geq D(x)$)
w_L	width of the lattice L		

Strict Ideals

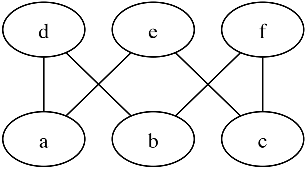
$D(A)$ = the set of elements strictly smaller than A

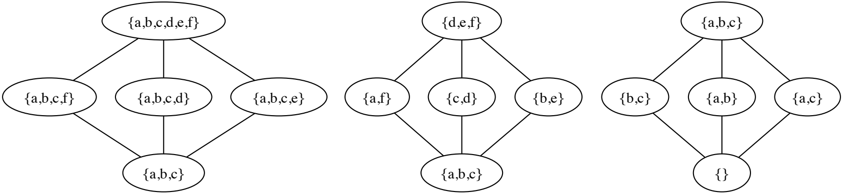
Strict Ideal: A set Y is a strict ideal if there exists an antichain A such that $D(A) = Y$.



Example: $\{a, b\}$ is a strict ideal. $\{a, b, c, d\}$ is not.

Equivalence of maximal ideals and strict ideals





Lattices of maximal ideals, maximal antichains, and strict ideals are isomorphic.

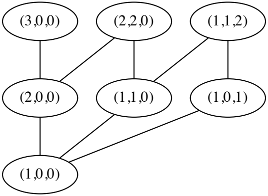
Mapping from strict ideals to maximal antichains: S is mapped to the minimal elements of the complement of S.

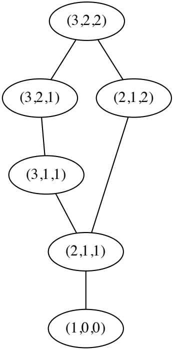
Input: P: a finite poset as a list of vector clocks
L: lattice of maximal antichains of vector clocks
x: new element
Output: $L' :=$ Lattice of maximal antichains of $P \cup \{x\}$ initially L

// Step 1: Compute the set $D(x)$
Let V be the vector clock for x on process P_i;
$S := V$; $S[i] := S[i] - 1$;
// Step 2:
if $S \notin L$ then
    $L' := L' \cup \{S\}$;
    forall vectors $W \in L$:
        if $max(W, S) \notin L$ then $L' := L' \cup max(W, S)$;

Example

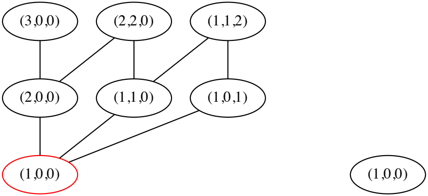
Poset and its lattice of maximal antichains





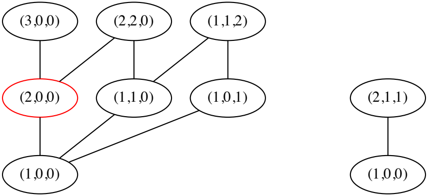
Example

Step 1: $D[x] = (1, 0, 0)$
$D(x) = (0, 0, 0)$, strict ideals added: $(0, 0, 0)$
Set of Maximal Antichains $= \{(1, 0, 0)\}$



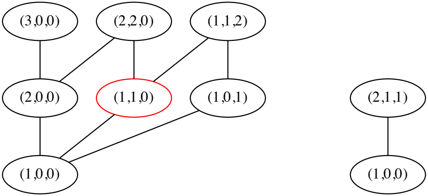
Example

Step 2: $D[x] = (2, 0, 0)$
$D(x) = (1, 0, 0)$, strict ideals added: $(1, 0, 0)$
Set of Maximal Antichains $= \{(1, 0, 0), (2, 1, 1)\}$



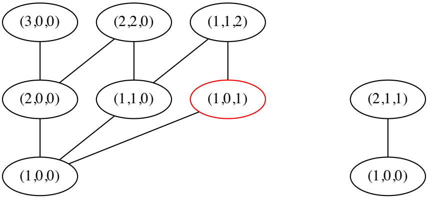
Example

Step 3: $D[x] = (1, 1, 0)$
$D(x) = (1, 0, 0)$, strict ideals added: ϕ
Set of Maximal Antichains $= \{(1, 0, 0), (2, 1, 1)\}$



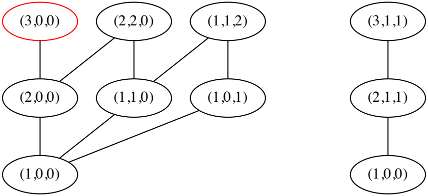
Example

Step 4: $D[x] = (1, 0, 1)$
$D(x) = (1, 0, 0)$, strict ideals added: ϕ



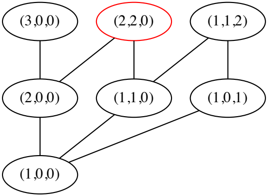
Example

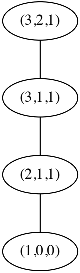
Step 5: $D[x] = (3, 0, 0)$
$D(x) = (2, 0, 0)$, strict ideals added: $(2, 0, 0)$
Maximal antichain added: $(3, 1, 1)$



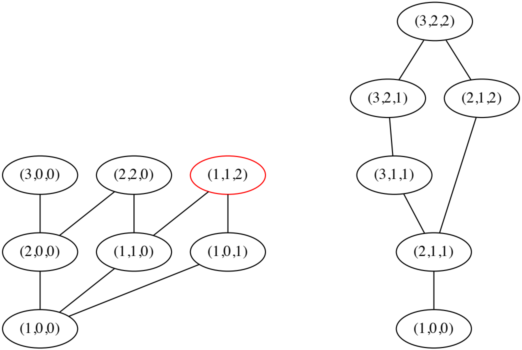
Example

Step 6: $D[x] = (2, 2, 0)$
$D(x) = (2, 1, 0)$, strict ideals added: $(2, 1, 0)$
Maximal antichain added: $(3, 2, 1)$





Step 6: $D[x] = (1, 1, 2)$
$D(x) = (1, 1, 1)$, strict ideals added: $(1, 1, 1)$, $(2, 1, 1)$
Maximal antichains added: $\{(2, 1, 2), (3, 2, 2)\}$



Analysis of the Incremental Algorithm 1

1. Simple Algorithm
2. To check if $max(S, W) \in L$, maintain L as a binary search tree
3. Requires storage of the the entire lattice (exponential in size of the poset in the worst case)

Space Efficient Incremental Algorithm

Input: a finite poset P, x maximal element in $P' = P \cup \{x\}$

Output: enumerate M such that $L_{MA}(P') = L_{MA}(P) \cup M$

$S :=$ the vector clock for x on process P_i;
$S[i] := S[i] - 1$;
if S is not a strict ideal of P then
// $BFS(S)$: Do Breadth-First-Search traversal of M
    $\mathcal{T}$:= set of vectors initially $\{S\}$;
    while $\mathcal{T}$ is nonempty do
        $H :=$ delete the smallest vector from $\mathcal{T}$;
        enumerate H;
        foreach process k with next event e do
            explore H using e;
        endfor;
    endwhile;
endif;

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- Global predicate detection requires enumeration not construction of the lattice
- Lattice of maximal antichains may be exponential in the number of processes

Related Work: Enumeration Algorithms

Input: a nonempty finite poset P
Output: enumerate all elements of $L :=$ DM-completion of P

Algorithm	Time	Space
[Jourdan, Rampon, Jard 94]	$O((n+w^2)wm)$	$O(mn \log n)$
[Nourine and Raynaud 99, 02]	$O(mn^2)$	$O(mn \log n)$
Algorithm ILMA [this paper]	$O(nwm \log m)$	$O(mw \log n)$
BFS-MA [this paper]	$O(mw^2 \log m)$	$O(w_L w \log n)$
DFS-MA [this paper]	$O(mw^4)$	$O(nw \log n)$
Lexical by [Ganter84]	$O(mn^3)$	$O(n \log n)$

The parameters are:
n: size of the poset P,
m: size of the lattice L of normal cuts of P,
w: width of the poset P,
w_L: width of the lattice L.

- *closure*(Y) = smallest maximal antichain ideal that contains Y. The operator *closure*(Y) is monotone, extensive and idempotent.
- Idea: View the lattice of maximal antichains as a directed graph and enumerate the nodes of the graph using the closure operator.
- Difficulty: Usual DFS on graph cannot be employed as the graph cannot be stored. Cannot mark which nodes have been visited before.

Depth First Search Enumeration of Maximal Antichain Ideals

Input: a finite poset P, starting state G
Output: DFS Enumeration of Maximal Antichain Ideals of P
(1) output(G);
(2) **foreach** event e enabled in G do
(3) $K := closure(G \cup \{e\})$;
(4) if (K covers G) and (not visited before) then
(5) DFS(K);

How to avoid revisiting cuts?

Visit a state only from the maximum predecessor.

(4) if K does not cover G then go to the next event;
(5) $M :=$ get-Max-predecessor(K) ;
(6) if $M = G$ then
(7) DFS-MaximalIdeals(K);

- To check whether K covers G: use the efficient characterization provided by [Reuter 91].
- To choose the maximum predecessor Expand the nodes of the dual poset and choose the biggest predecessor

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Application to Global Predicate Detection

Definition (Antichain-Consistent Predicate)

A global predicate is an *antichain-consistent predicate* if

1. its evaluation depends only on maximal events of a consistent global state and
2. if it is true on a subset of processes, then presence of additional processes does not falsify the predicate.

Examples of antichain-consistent predicates

- Violation of mutual exclusion: "there is more than one process in the critical section."
- Violation of resource usage: "there are more than k concurrent activation of certain service,"
- Global Control Point: The predicate, B, "Process P_1 is at line 35 and P_2 is at line 23 concurrently,"

Theorem

There exists a consistent global state that satisfies an antichain-consistent predicate B iff there exists a maximal ideal that satisfies B.

Conclusions and Future Work

- An Incremental Algorithm to generate the lattice of maximal antichains
- BFS and DFS enumeration of maximal antichains
- Applications to global predicate detection

Question: Is there a space-efficient algorithm with time complexity $O(mw \log n)$?