Applying Predicate Detection to Discrete Optimization Problems

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Motivation

Consider the following problems:

- Shortest Path Problem:
  Input: a weighted directed graph and a source vertex
  Output: Least Cost of reaching any vertex i
  Dijkstra's algorithm for graph with non-negative weights,
  Bellman-Ford algorithm for graphs with no negative cycles
- Stable Marriage Problem:
  Input: ordered preferences of n men and n women
  Output: Man-optimal stable marriage
  Gale-Shapley's algorithm
- Assignment Problem:
  Input: n items, n bidders with valuation for items
  Output: Least market clearing prices
  Hungarian Algorithm (or Gale-Demange-Sotomayor's Auction)

Could there be a single algorithm that solves all of these problems?

Lattice-Linear Predicate (LLP) Algorithm

Other Applications of LLP

- Housing Allocation Problem:
  Input: n agents, n houses, initial endowment, preference list of agents
  Output: allocation of houses such that there is no blocking group
  Gale's Top Trading Cycle Algorithm
- Minimum Spanning Tree Problem:
  Input: undirected weighted graph
  Output: spanning tree with that minimizes sum of weight of edges
  Prim's Algorithm, Boruvka' Algorithm
- Horn Satisfiability:
  Input: A boolean formula in Horn form
  Output: Least satisfying assignment, if any
  Horn's Satisfiability Algorithm

Outline of the Talk

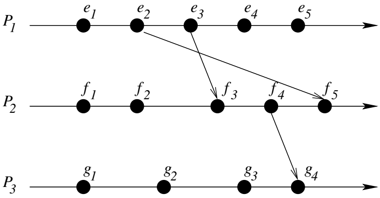
- What are Lattice-Linear Predicates (LLP)?
- LLP Detection Algorithm
- Applications
- Enumerating All Satisfying Global States

Steps of Using LLP Algorithm

- Step 1: Model the underlying search space. A Distributive Lattice of State Vectors. The order on the lattice is based on the optimization objective of the problem.
- Step 2: Define the feasibility predicate B. An element is feasible if it satisfies constraints of the problem
- Step 3: Check whether the feasibility predicate B is Lattice-Linear. If B is lattice-linear, LLP Algorithm will return the optimal feasible solution.

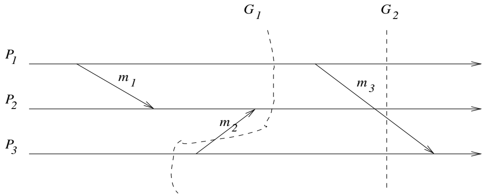
Step 1: Modeling the underlying search space

Model the problem as n processes choosing their component in a vector of size n. The choice for a single process is total ordered.



computation: poset $(E, \rightarrow)$
candidate solution: a possible global state of the system.

Consistent Global State

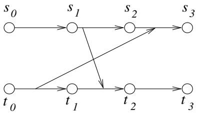


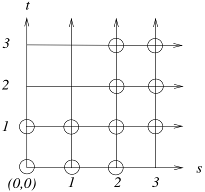
A subset G of E is a consistent global state if

$$\forall e, f \in E : (f \in G) \land (e \rightarrow f) \Rightarrow (e \in G)$$

The set of all consistent global states forms a finite distributive lattice.

Step 1: Order on the underlying space





$(L, \leq)$: Underlying Search Space
L : set of all consistent global state vectors
Order on Global State $G \leq H$ iff for all $i : G[i] \leq H[i]$.
meet of two global states: $K = G \sqcap H = \min(G, H)$
join of two global states: $K = G \sqcup H = \max(G, H)$
meet distributes over join.
$(L, \leq)$ is a distributive lattice.

Step 1: Examples

G: Global State Vector where $G[i]$ is the component for process i.

- Shortest Path: $G[i]$: cost of reaching vertex i from the source vertex
  initially 0
- Stable Marriage: $G[i]$: index in the preference list for man i
  initially 1 // top choice
- Market Clearing Prices: $G[i]$: price of item i
  initially 0

Step 2: Defining Feasibility Predicate B

A global state G satisfies B iff G represents a feasible solution.

- Shortest Path: All nodes most have a parent node. For every vertex j (except source): there exists a vertex i such that $G[j] \geq G[i] + w[i, j]$.
- Stable Marriage: Every man must be matched to a different woman and there must not be any blocking pair.
- Market Clearing Prices: There is no overdemanded item at that pricing vector.

Step 2: Defining Feasibility Predicate Formally

- Shortest Path: Every non-source node has a parent. For any node $j \neq 0$,

$$\exists i \in pre(j) : G[j] \geq G[i] + w[i,j]$$

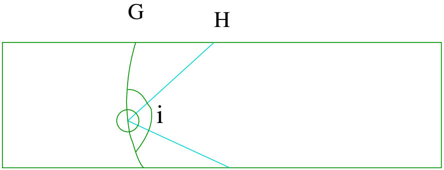
- Stable Marriage: Every man must be matched to a different woman and there must not be any blocking pair. For any man j, let $z = mpref[j][G[j]]$; //current woman assigned to man j

$$\neg\exists i : \exists k \leq G[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j]))$$

- Market Clearing Prices: There is no overdemanded item at that pricing vector. For any item j,

$$\neg\exists J : minimalOverDemanded(J, G) \wedge (j \in J)$$

Lattice-Linearity for Predicate Detection



Forbidden State The state at P_i is forbidden at G with respect to B if unless P_i is advanced B cannot become true.

$$\text{forbidden}(G, i, B) \equiv \forall H : G \subseteq H : (G[i] = H[i]) \Rightarrow \neg B(H)$$

Lattice-Linear Predicates A predicate B is lattice-linear if for all consistent cuts G,

$$\neg B(G) \Rightarrow \exists i : \text{forbidden}(G, i, B).$$

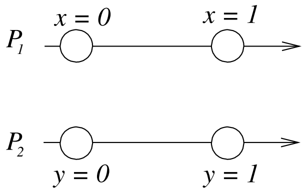
Examples: Conjunctive Predicates: $l_1 \wedge l_2 \wedge ... \wedge l_n$, Feasible Path, Stable Matching, Market Clearing Prices, Minimum Spanning Tree, Housing Core, Horn Formulas

Examples of Lattice-Linear Predicates

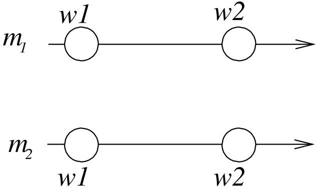
- A conjunctive predicate
  $l_1 \wedge l_2 \wedge \ldots \wedge l_n$, where l_i is local to P_i.
  Suppose G is not feasible. Then, there exists j such that l_j is false in G. The index j is forbidden in G.
- Shortest Path
  Any j such that v_j does not have a parent,
  $(\forall i \in pre(j) : G[j] < G[i] + w[i,j])$ is forbidden in G.
- Stable Marriage
  j is forbidden in G if
  $\exists i : \exists k \leq G[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j]))$
- Market Clearing Price
  $(\neg \exists J : minimalOverDemanded(J, G) \wedge (j \in J))$
  Any j in a minimal overDemanded set is forbidden.

Example of Predicates that are not Lattice-Linear

Example 1: $B(G) \equiv x + y \geq 1$



Example 2: $B(G) \equiv G$ is a matching.



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Detecting Lattice-Linear Predicates

(Advancement Property) There exists an efficient (polynomial time) algorithm to determine the forbidden state.

Theorem

[Chase and Garg 95] Any lattice-linear predicate that satisfies advancement property can be detected efficiently.

LLP Algorithm

How much to advance: j is forbidden in G until α iff

$$\forall H \in L : H \geq G : (H[j] < \alpha) \Rightarrow \neg B(H).$$

```
 vector function getLeastFeasible(T: vector, B: predicate)
//T: top element of the lattice
 var G: vector of reals initially ∀i : G[i] = 0;
 while ∃j: forbidden(G, j, B)  do
      for all j such that forbidden(G, j, B) in parallel:
           if (α(G, j, B) > T[j]) then return null;
           else G[j] := α(G, j, B);
 endwhile;
 return G; // the optimal solution
```

LLP Algorithm: Stable Marriage Problem

P_j:

var G: array[1..n] of 1..n;

input: $mpref[i,k]$: int for all i,k; // men preferences

$rank[k][i]$: int for all k,i; // women ranking

init: $G[j] := 1$;

always: $w = mpref[j][G[j]]$;

forbidden:

$(\exists i : \exists k \leq G[i] : (w = mpref[i][k]) \wedge (rank[w][i] < rank[w][j]))$

advance: $G[j] := G[j] + 1$;

Slightly more general than Gale-Shapley Algorithm:
instead of starting from $(1, 1, \ldots, 1)$, can start from any choice vector.

LLP Algorithm: Shortest Path Problem

input: $pre(j)$: list of $1..n$;
$w[i,j]$: positive int for all $i \in pre(j)$
$s : 1..n$; // source node;
init: $G[j] := 0$;
always:
$parent[j,i] = (i \in pre(j)) \wedge (G[j] \geq G[i] + w[i,j])$;
$fixed[j] = (j = s) \vee (\exists i : parent[j,i] \wedge fixed[i])$
$Q = \{(G[i] + w[i,k]) | (i \in pre(k)) \wedge fixed(i) \wedge \neg fixed(k)\}$;

forbidden: $\neg fixed[j]$
advance: $G[j] := \max\{\min Q, \min\{G[i] + w[i,j] \mid i \in pre(j)\}\}$

By ignoring the second part of advance, we can get Dijkstra's algorithm.

LLP Algorithm: Shortest Path Problem Revisited

Assume no negative cost cycle.

input: $pre(j)$: list of $1..n$;
    $w[i,j]$: int for all $i \in pre(j)$
init: if $(j = s)$ then $G[j] := 0$ else $G[j] :=$ maxint;
forbidden: $G[j] > \min\{G[i] + w[i,j] \mid i \in pre(j)\}$
advance: $G[j] := \min\{G[i] + w[i,j] \mid i \in pre(j)\}$

Lattice is reversed: the bottom element is $(maxint, maxint, \ldots, maxint)$
This is just Bellman-Ford's algorithm.

LLP Algorithm: Market Clearing Prices

input: $v[b, i]$: int for all b, i
init: $G[j] := 0;$
always: $E = \{(k, b) \mid \forall i : (v[b, k] - G[k]) \geq (v[b, i] - G[i])\};$
$\quad demand(U') = \{k \mid \exists b \in U' : (k, b) \in E\};$
$\quad overDemanded(J) \equiv \exists U' \subseteq U : (demand(U') = J) \wedge (|J| < |U'|)$

forbidden: $\exists J : minimal - OverDemanded(J) \wedge (j \in J)$
advance: $G[j] := G[j] + 1;$

This is just Demange-Gale-Sotomayor exact auction algorithm.

Properties of LLP Predicates

Lemma

Let B be any boolean predicate defined on a lattice L of vectors.

- *Let $f : L \to \mathrm{R}_{\geq 0}$ be any monotone function defined on the lattice L of vectors of $\mathrm{R}_{\geq 0}$. Consider the predicate $B \equiv G[i] \geq f(G)$ for some fixed i. Then, B is lattice-linear.*
- *Let L_B be the subset of the lattice L of the elements that satisfy B. If B is lattice-linear then L_B is closed under meets.*
- *If B_1 and B_2 are lattice-linear then $B_1 \wedge B_2$ is also lattice-linear.*

Constrained Optimization

- least stable marriage such that regret of Peter is less than or equal to regret of John
- least feasible path such that the cost of reaching x equals cost of reaching y
- least clearing prices such that $item_1$ is priced at least 5 more than $item_2$.

All of the additional constraints are also lattice-linear.

Lemma

LLP can be adapted to find the least vector G that satisfies $B_1 \wedge B_2$ for any lattice-linear predicates B_1, B_2.

Proof.

The algorithm LLP can be used with the following changes: $forbidden(G, j, B_1 \wedge B_2) \equiv forbidden(G, j, B_1) \vee forbidden(G, j, B_2)$, and $\alpha(G, j, B_1 \wedge B_2) = \max\{\alpha(G, j, B_1), \alpha(G, j, B_2)\}$.

□

Meet Closure of Feasible Predicates

Theorem

[Chase and Garg 95] A predicate B is lattice-linear implies that it is meet-closed (in the lattice of all consistent cuts).

Lattice-Linearity implies

- If G and H are feasible cost vectors, then so is $G \sqcap H$.
- If G and H are stable marriage choice vectors, then so is $G \sqcap H$.
- If G and H are market clearing prices, then so is $G \sqcap H$.

Dual of Lattice-Linearity

reverse-forbidden$(G, i, B) \equiv \forall H \in L : H \leq G : (G[i] = H[i]) \Rightarrow \neg B(H)$.
B is *dual-lattice-linear* iff:
$\forall G \in L : \neg B(G) \;\Rightarrow\; \exists i :$ reverse-forbidden(G, i, B).
$B_{stableMarriage}$ and $B_{marketClearing}$ are also dual-lattice-linear.
$\Rightarrow$

- the set of stable marriages and market clearing prices are also closed under joins
- one can traverse the lattice backwards to find the woman-optimal stable marriage or the greatest market clearing prices.

Note: $B_{shortestPath}$ is not dual-lattice-linear.

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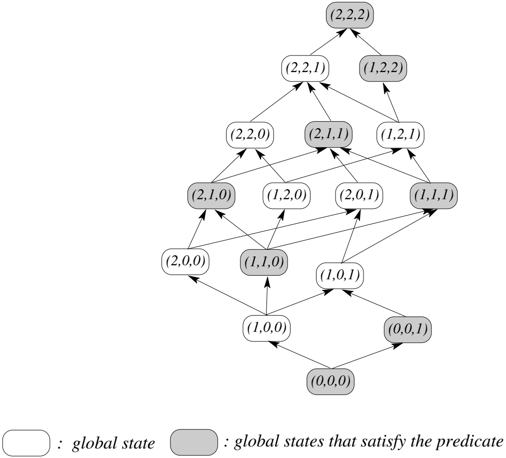
Enumerating All Feasible Solutions

Assume that B is lattice-linear as well as dual-lattice-linear.

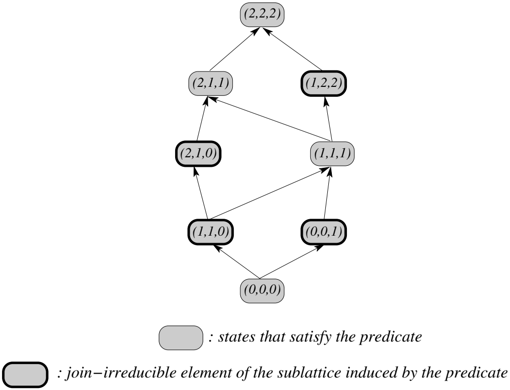


- L_B (the subset of elements in L that satisfy B) forms a sublattice of L
- L_B is a distributive lattice.

Slicing: Can we represent L_B concisely? [Mittal and Garg 01]

Join-irreducible Elements

join-irreducible element: cannot be represented as join of two other elements



Theorem

[Birkhoff's Representation Theorem] A distributive lattice can be recovered exactly *from the set of its join-irreducible elements.*

Algorithm to find All Join-Irreducible Elements

for all $e \in E$:
 compute $J(B, e)$

$J(B, e)$: the minimum global state of $(E, \leq)$ that

- satisfies B, and
- contains e

Feasible predicate: $B_e(G) \equiv B(G) \wedge (e \in G)$
Observation: B_e is a conjunction of two lattice-linear predicates.
We can use LLP algorithm to find the least global state satisfying $J(B, e)$

Applications of Slicing

- Constrained Stable Marriages: We get a generalization of rotation poset [Irving and Gusfield].
- Constrained Market Clearing Prices: A poset that captures all integral market clearing prices.

Conclusions

How to Solve Many Combinatorial Optimization Problems
Find the least feasible element

- View State space as the set of consistent global states
- Each process starts with the most desirable choice and moves to less desirable
- Define a "feasibility" predicate B
- Check if B satisfies the lattice-linearity condition

Other algorithms as special cases of the LLP Algorithm:

- Gale's Top Trading Cycle Algorithm,
- Prim's MST Algorithm,
- Horn's satisfiability algorithm,
- Johnson's algorithm to transform graphs with negative cost edges

Future Work

- Techniques when the feasibility predicate is not lattice-linear.