Rectangles are Better than Chains for Encoding Partially Ordered Sets

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Abstract

Partial orders have applications in various areas in computer science such as distributed systems, object oriented languages, knowledge representation systems and databases. In this paper we present a new technique to encode partially ordered sets or posets. Specifically, we use the family of two-dimensional posets, which we refer to as *rectangles*, to encode a given poset; an element x is less than another element y in the partial order if and only if x is less than y in at least one of the rectangular orders and y is not less than x in any of the rectangular orders. The least number of rectangular orders needed to represent a poset is referred to as its *rectangular dimension*. We establish upper bounds on rectangular dimension of posets in general and special families of posets in particular. We also provide examples of families of posets that can be encoded *optimally* using our technique but require much larger number of bits per element using other techniques such as adjacency matrix, adjacency list and dimension theory.

Key words partial order, poset, dimension, encoding graph, implicit representation

1 Introduction

The importance of managing the transitive closure of relationships has been acknowledged in several areas in computer science. Examples include the subsumption relation in knowledge representation systems [1, 5] and object oriented languages (OOL) [3], the causality relation in distributed systems [6, 14], and the transitive closure query in deductive database systems [1]. These hierarchies of objects are expected to grow larger in the future. Therefore it will become important to have techniques that enable such hierarchies to be stored in a compact manner while, at the same time,

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admitting fast response to queries such as *"is the object x related to the object y?"*. In distributed systems, for example, the order between any set of events is partial, and in many applications such as electronic commerce systems, the set of events and their order may be recorded for later queries such as "is bid x concurrent with bid y?"

A popular method to store a partially ordered set or *poset* is to view it as a directed acyclic graph and represent the graph using adjacency matrix or adjacency list. The adjacency matrix representation uses a matrix of size $n \times n$, where n is the number of elements in the set. Elements are assumed to be numbered from 1 to n. The $(i, j)^{th}$ entry of the matrix is 1 if and only if the i^{th} element is less than the j^{th} element in the partial order; otherwise it is 0. To compare two distinct elements, at most two entries of the matrix need to be examined. Thus the adjacency matrix representation uses $O(n)$ bits per element but allows $O(1)$ response to the *less-than* query ("is x less than y?"). In the adjacency list representation, a list is associated with each element; the list for element x contains all elements greater than x. The adjacency list representation is only useful for those posets which have low average degree.

Another method to represent a partial order on a set of elements is based on dimension theory introduced by Dushnik and Miller [4]. In this theory, a poset is represented as intersection of a collection of total orders on the set of elements, each of which extends the partial order. A collection of total orders whose intersection is the given partial order constitutes a *realizer* of the partial order. An element x is less than another element y in the partial order if and only if x is less than y in all the total orders in the realizer. A total order can be represented optimally using $\log n$ bits per element, where n is the number of elements, thereby furnishing a technique to represent a poset. If the realizer contains k total orders, then each element can be encoded using $O(k \log n)$ bits per element. Every element is simply represented by its rank in each total order. The *dimension* of a poset is the least number of total orders required to realize the corresponding partial order. Clearly, the dimension of a poset is one if and only if it is a total order. Yannakakis [18] established that it is in general NP-complete to test whether the dimension of a poset is at most k for any fixed $k \geqslant 3$. Poset dimension has been studied extensively; the interested reader can consult the book [16]. When the dimension is small, the dimension-theoretic representation is more concise than the adjacency matrix representation.

In our approach, we use posets with dimension at most two—called *two-dimensional* posets—as "building blocks" for realizing a given poset. For convenience, we refer to two-dimensional posets as *rectangles*. An element x is less than another element y in the given partial order if and only if x is less than y in at least one of the rectangular orders and y is not less than x is any of the rectangular orders. The set of rectangular orders that realizes a given partial order constitutes its *rectangular realizer*. Also, the *rectangular dimension* of a poset is the least number of rectangular orders needed to realize the corresponding partial order. Clearly, by definition, the rectangular dimension of a poset is one if and only if its dimension is at most two. Trivially, the rectangular dimension of a poset is upper bounded by its dimension.

It turns out that there are posets with arbitrarily high dimension but only constant rectangular dimension. As an illustration, consider the family of bipartite posets called *standard examples*. The standard example $\mathbf{S}_n$ for $n \geqslant 3$ is the poset induced by the 1-element and $(n-1)$-element subsets of n distinct elements when ordered by set containment. The graph representation of $\mathbf{S}_5$, for example, is shown in Figure 1. In the figure, all edges are directed upwards. It can be proved that a standard example has "large" dimension [16, Chapter 1]. Specifically, the dimension of $\mathbf{S}_n$ is given by n for each $n \geqslant 3$. This implies that $O(n \log n)$ bits per element are required to encode $\mathbf{S}_n$ using the dimension theory. On the other hand, we prove in this paper that the rectangular dimension of $\mathbf{S}_n$ is two for each $n \geqslant 3$. Each rectangle can be encoded using $O(\log n)$ bits. Therefore using rectangles leads to a much more efficient representation of $\mathbf{S}_n$. We further

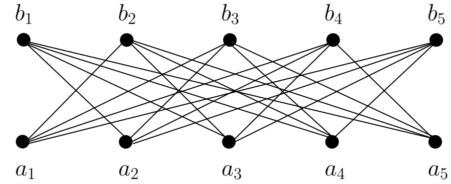

Figure 1: The standard example $\mathbf{S}_5$.

prove that the rectangular dimension of the generalized crown $\mathbf{S}_n^k$ for $n \geqslant 3$ and $k \geqslant 0$ [16, Chapter 2], which is a generalization of the standard example, is also two. Its dimension, however, is given by $\lceil 2(n+k)/(k+2) \rceil$. Note that encoding $\mathbf{S}_n$ and $\mathbf{S}_n^k$ requires a large number of bits per element (specifically, $O(n \log n)$ and $O(n \log(n+k))$, respectively) using adjacency list representation as well. A more detailed comparison with other techniques for representing a poset is given in Section 6.

We describe two methodologies to compute a rectangular realizer of a poset. The first methodology involves partitioning the ground set into one or more subsets; we refer to it as *point decomposition* (also known as *removal theorem* in the literature). The second methodology involves decomposing the partial order into one or more suborders; we refer to it as *order decomposition*. Using point decomposition method, we prove upper bounds of $n/3$ and $\lceil n/4 \rceil$ on rectangular dimension of general posets and bipartite posets, respectively, where n is the number of elements in the poset with $n \geqslant 3$. We also provide a bound on rectangular dimension of interval orders that depends on the number of intervals with *distinct* lengths present in an interval order. Using order decomposition method, we provide bounds on rectangular dimension of general posets based on degree of connectivity and of bipartite posets based on degree of adjacency. As a corollary, we derive an optimal encoding for generalized crown (and thereby for standard example) based on rectangles.

For encoding a rectangular order in a rectangular realizer, we use the *string representation* introduced in [8]. If a rectangle is represented using two chains, then each element requires $O(\log n)$ bits per element per rectangle. However, using string representation, the number of bits required can be as small as $O(1)$ per element per rectangle.

The paper is organized as follows. In Section 2, we provide the background on partially ordered sets and dimension theory and also describe the notation used in this paper. We formally define the notions of rectangle, rectangular realizer and rectangular dimension in Section 3. The two methodologies to compute rectangular realizers of posets are illustrated in Section 4 and Section 5, respectively. The related work is discussed in Section 6. In Section 7, we present our conclusions and outline some directions for future research.

2 Background and Notation

We use the terminology and notation given in [16]. Standard notation and definitions are repeated here for completeness. The reader already familiar with partially ordered sets and dimension theory can skip this section.

2.1 Partially Ordered Sets

We consider *partially ordered set* or *poset* $\mathbf{P}$ to be a pair (X, P) where X is a set and P is a reflexive, antisymmetric, and transitive binary relation on X. We call X the *ground set* while P is a *partial order* on X. Elements of the ground set are also called *points*. We will only be concerned with *finite* posets in this paper. We write $x \leqslant y$ in P and $y \geqslant x$ in P when $(x,y) \in P$. The notations $x < y$ in P and $y > x$ in P mean $x \leqslant y$ in P and $x \neq y$. When the poset remains fixed throughout the discussion, we abbreviate $x < y$ in P by just writing $x < y$, etc. The *dual* of a partial order P on X is denoted by P^d and is defined by $\{(y,x) \mid (x,y) \in P\}$. The *dual* of a poset $\mathbf{P} = (X,P)$ is denoted by $\mathbf{P^d}$ and is defined by $\mathbf{P^d} = (X, P^d)$.

Let $\mathbf{P} = (X,P)$ be a poset and consider $x, y \in X$ with $x \neq y$. We say x and y are *comparable* in P, and write $x \perp y$ in P, when either $x < y$ in P or $y < x$ in P. On the other hand, x and y are *incomparable* in P, denoted $x \parallel y$ in P, if neither $x < y$ in P nor $y < x$ in P. Let Y and Z be disjoint subsets of X. We say Y and Z are *incomparable* in P and write $Y \parallel Z$ in P if $y \parallel z$ in P, for every $y \in Y$ and $z \in Z$.

When $\mathbf{P} = (X,P)$ is a poset and Y is a nonempty subset of X, the *restriction* of P to Y, denoted by $P(Y)$, is a partial order on Y and we call $(Y, P(Y))$ (also denoted $\mathbf{P(Y)}$) a *subposet* of (X,P). A poset $\mathbf{P} = (X,P)$ is called a *chain* if every distinct pair of points from X is comparable in P. Similarly, we call a poset an *antichain* if every distinct pair of points from X is incomparable in P. When (X,P) is a chain, we call P a *linear order* (also, *total order*). A nonempty subset $Y \subseteq X$ is called a *chain* (respectively, *antichain*) if the subposet $(Y, P(Y))$ is a chain (respectively, antichain).

A chain C of a poset is a *maximum chain* if no other chain of the poset contains more points than C. A *maximum antichain* can be dually defined. The *height* of a poset $\mathbf{P} = (X,P)$ is the number of points in a maximum chain and is denoted by $\text{height}(X,P)$ (or $\text{height}(\mathbf{P})$). The *width* of a poset $\mathbf{P} = (X,P)$, denoted by $\text{width}(X,P)$ (or $\text{width}(\mathbf{P})$), is the number of points in a maximum antichain.

A poset $\mathbf{P} = (X,P)$ is *connected* if for every $x, y \in X$ with $x \neq y$ there is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ of points from X so that $x_i \perp x_{i+1}$ in P for $i = 0, 1, 2, \ldots, n-1$. We call an element a *loose point* if it is not comparable to any other element in the set.

We call a poset $\mathbf{P} = (X,P)$ *bipartite* if there is no chain in the poset involving more than two elements. In that case, the ground set X can be partitioned into two disjoint subsets L and U so that $P \subseteq L \times U$ and we write $\mathbf{P} = (X,P) = (L,U,P)$. Note that the partition of X into L and U may not be unique.

We depict a poset pictorially by its *Hasse diagram* or *covering graph*. Elements are denoted by solid circles. There is an edge from x to y if $x < y$ and there is no z so that $x < z < y$. Unless otherwise stated, the edges are directed from bottom to top.

Let Y and Z be two disjoint subsets of the set X, and let L and M be the linear orders on Y and Z, respectively. We use $L < M$ to represent the linear order on $Y \cup Z$ in which the elements of Y occur in the order given by L followed by the elements of Z as ordered by M.

2.2 Chain Realizer and Chain Dimension

Let $\mathbf{P} = (X,P)$ be a poset. A family $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ of total orders on X is called a *(chain) realizer* of P on X (also, $\mathcal{C}$ *realizes* $\mathbf{P}$) if $P = \bigcap \mathcal{C} = \bigcap_{i=1}^{t} C_i$. The *(chain) dimension* of a poset $\mathbf{P} = (X,P)$, denoted by $\dim(X,P)$ (or $\dim(\mathbf{P})$), is the least positive integer t for which there exists a family $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ of t total orders on X so that $\mathcal{C}$ realizes $\mathbf{P}$ [16].

For example, the chain dimension of the poset in Figure 2 is three. The three total orders that realize it are given by $a_3 < a_1 < b_1 < a_2 < b_2 < b_3$, $a_1 < a_2 < b_2 < a_3 < b_3 < b_1$ and $a_2 < a_3 < b_3 < a_1 < b_1 < b_2$.

Remark 1 *We use the term "chain realizer" instead of the term "realizer" to emphasize the difference between a realizer of a poset and a rectangular realizer of a poset defined in the next section.*

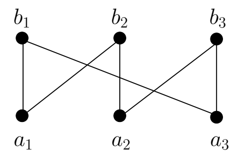

Figure 2: An example of a poset.

The same remark also applies to "chain dimension".

3 Rectangles

In this section, we formally define the notions of rectangle, rectangular realizer and rectangular dimension of a poset.

Definition 1 (rectangle) *A* rectangle *is a two-dimensional poset.*

When a poset $\mathbf{P} = (X, P)$ is a rectangle, we call P as a *rectangular order*. For a rectangular order R, we use $R.1$ and $R.2$ to refer to two total orders that realize R. In case the dimension of R is one, both $R.1$ and $R.2$ refer to the same total order. Clearly, $R = R.1 \cap R.2$. A chain as well as an antichain is a rectangle.

Definition 2 (rectangular realizer) *Let $\mathbf{P} = (X, P)$ be a poset. A family $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ of rectangular orders on X is called a* realizer *of P on X (also, $\mathcal{R}$* realizes *$\mathbf{P}$) if for every $x, y \in X$, $x < y$ in P if and only if $y \not< x$ in R_i for each $i \in [1, t]$ and $x < y$ in R_j for some $j \in [1, t]$.*

If $x \parallel y$ in P, then in a rectangular realizer $\mathcal{R}$ of P two cases are possible. Either $x \parallel y$ in all rectangular orders in $\mathcal{R}$, or $x < y$ in some rectangular order in $\mathcal{R}$ and $y < x$ in some other rectangular order in $\mathcal{R}$. On the other hand, if $x < y$ in P, then $x < y$ in some rectangular order in $\mathcal{R}$ and $x < y$ or $x \parallel y$ in all other rectangular orders in $\mathcal{R}$. The notion of rectangular dimension can now be defined as follows:

Definition 3 (rectangular dimension) *The* rectangular dimension *of a poset $\mathbf{P} = (X, P)$, denoted by* $\mathrm{rdim}(X, P)$ *(or* $\mathrm{rdim}(\mathbf{P})$*), is the least positive integer t for which there exists a family $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ of t rectangular orders on X so that $\mathcal{R}$ realizes $\mathbf{P}$.*

As an example, the rectangular dimension of the poset depicted in Figure 2 is two. The two rectangular orders realizing it are given by $\{(a_1 < a_2 < a_3 < b_1 < b_2 < b_3), (a_3 < a_2 < a_1 < b_3 < b_2 < b_1)\}$ and $\{(b_1 < a_2 < b_2 < a_3 < b_3 < a_1), (b_3 < a_1 < b_2 < a_3 < b_1 < a_2)\}$.

The rectangular dimension of a poset and its dual are identical because the dual of a two-dimensional poset is again a two-dimensional poset. The notions of rectangular realizer and rectangular dimension defined for a poset can be generalized to any (acyclic) relation on a ground set. For a collection of rectangular orders $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$, let $\mathrm{rel}(\mathcal{R})$ denote the relation realized by $\mathcal{R}$. For example, consider the rectangular orders $\{(a < b < c), (c < a < b)\}$ and $\{(a < b < c), (b < c < a)\}$. The relation realized by the two orders collectively is given by $\{(a, b), (b, c)\}$. Note that the relation is not transitive because it does not contain the ordered pair (a, c). Next, we define two concepts that we use when deriving rectangular realizers for posets using the point and order decomposition methods. Let $X = Y \cup Z$ be a partition of X, and let Q and R be rectangular orders on Y and Z, respectively. Then $P = Q \cup R$ is also a rectangular order on X. Evidently, two chains realizing P are given by $P.1 = Q.1 < R.1$ and $P.2 = R.2 < Q.2$. Moreover, P satisfies the following properties:

1. $P(Y) = Q$ and $P(Z) = R$, and
2. $y \parallel z$ in P for all $y \in Y$ and $z \in Z$.

Definition 4 (disjoint composition) *Let $X = Y \cup Z$ be a partition of X, and let Q and R be rectangular orders on Y and Z, respectively. Then we say that the rectangular order $P = Q \cup R$ on X is obtained by* disjoint composition *of the rectangular orders Q on Y and R on Z.*

Definition 5 (non-interference) *A rectangular order R is said to be* non-interfering *with a partial order P if $R \subseteq P$.*

Also, a rectangular realizer is *non-interfering* with a partial order if every rectangular order in the realizer is non-interfering with the partial order. In other words, if two elements are incomparable in the partial order, they are also incomparable in all rectangular orders in the realizer. In that case, the partial order is given by the union of all rectangular orders in the realizer. Trivially, every partial order (even a relation) has a non-interfering rectangular realizer.

4 Point Decomposition Method and its Applications

4.1 The Main Idea

Given a poset $\mathbf{P} = (X, P)$, we partition the ground set X into two subsets: Y and $X \setminus Y$. We first compute rectangular realizers, say $\mathcal{Q}$ and $\mathcal{R}$, of the two induced subposets $(Y, P(Y))$ and $(X \setminus Y, P(X \setminus Y))$, respectively. A rectangular realizer $\mathcal{S}$ of $P(Y) \cup P(X \setminus Y)$ on X can then be obtained by disjoint composition of $\mathcal{Q}$ and $\mathcal{R}$; each rectangular order of $\mathcal{S}$ is obtained by disjoint composition of corresponding rectangular orders of $\mathcal{Q}$ and $\mathcal{R}$ (padding can be done if necessary). Finally, we compute a rectangular realizer, say $\mathcal{T}$, of the relation $P \setminus (P(Y) \cup P(X \setminus Y))$ on X. To guarantee that $\mathcal{S} \cup \mathcal{T}$ constitutes a rectangular realizer of P on X, it suffices to ensure that $\mathcal{T}$ is non-interfering with P. Note that it is not necessary that the relation realized by $\mathcal{T}$, given by $\mathrm{rel}(\mathcal{T})$, be exactly $P \setminus (P(Y) \cup P(X \setminus Y))$. It is sufficient that $P \setminus (P(Y) \cup P(X \setminus Y)) \subseteq \mathrm{rel}(\mathcal{T}) \subseteq P$. We often choose Y such that $\mathcal{T}$ consists of a single rectangular order.

Our results in this section are based on the notion of *indistinguishable elements*. Let $\mathbf{P} = (X, P)$ be a poset and consider an element $x \in X$ and a subset $Y \subseteq X$. We denote the subset of elements in Y that are less than x in P, that is, $\{y \in Y \mid y < x \text{ in } P\}$ by $D(x, Y)$ (called the *down set* of x in Y). Similarly, the subset of elements in Y that are greater than x in P, that is, $\{y \in Y \mid x < y \text{ in } P\}$ is denoted by $U(x, Y)$ (called the *up set* of x in Y). For convenience, we abbreviate $D(x, X)$ by $D(x)$ and $U(x, X)$ by $U(x)$.

Definition 6 (indistinguishable elements) *Two elements $x, y \in X$ are said to be* indistinguishable *with respect to Y in P if $D(x, Y) = D(y, Y)$ and $U(x, Y) = U(y, Y)$.*

The elements of $X \setminus Y$ can be partitioned into equivalence classes such that elements in the same class are mutually indistinguishable with respect to Y in P. These equivalence classes are referred to as *Y–indistinguishable classes* of $X \setminus Y$ in P. The class for which $D(x, Y) = U(x, Y) = \emptyset$ for each element x in the class is called *Y–disconnected class*. (No element in the Y–disconnected-class is comparable to any element in Y.) When deriving the rectangular realizer $\mathcal{T}$ to represent $P \setminus (P(Y) \cup P(X \setminus Y))$ described earlier, it suffices to consider at most one representative element from every Y–indistinguishable class.

For example, for the poset shown in Figure 2, $D(b_1, \{a_3, b_3\}) = \{a_3\}$. Furthermore, there are three $\{a_3, b_3\}$–indistinguishable classes of $\{a_1, a_2, b_1, b_2\}$, namely $\{b_1\}$, $\{a_1, b_2\}$ and $\{a_2\}$. The class $\{a_1, b_2\}$ is the $\{a_3, b_3\}$–disconnected class.

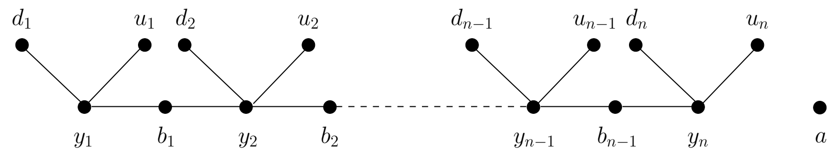

Figure 3: The Y–indistinguishable classes of $X \setminus Y$ in P where $(Y, P(Y))$ is a critical subposet of the poset (X, P) (note that in the Hasse diagram, the line segments between elements are directed from left to right instead of the usual bottom to top).

4.2 Applications

In this section, we present some removal theorems and later use them to provide bounds on rectangular dimension of various posets.

4.2.1 Removal Theorems

Out first theorem is based on the notion of *critical subposet*. A pair (x, y) from X with $x \parallel y$ in P forms a *critical pair* in the subposet $(Y, P(Y))$ if $D(x, Y) \subseteq D(y, Y)$ and $U(x, Y) \supseteq U(y, Y)$. A subposet $(Y, P(Y))$ is called a *critical subposet* of the poset (X, P) if for every incomparable pair (x, y) from Y either (x, y) or (y, x) forms a critical pair in the subposet $(X \setminus Y, P(X \setminus Y))$.

Theorem 1 (critical subposet removal theorem) *Let $(Y, P(Y))$ be a critical subposet of the poset $\mathbf{P} = (X, P)$ where $Y \subsetneq X$. Then,*

$$\mathrm{rdim}(X, P) \leqslant 1 + \max\{\mathrm{rdim}(X \setminus Y, P(X \setminus Y)),\ \dim(Y, P(Y))\}$$

Proof: First, we compute a rectangular order R that contains all ordered pairs in the relation $P \setminus (P(Y) \cup P(X \setminus Y))$. However, R may interfere with P. But, R is such that the ordered pairs in R that do not belong to P only involve the elements of Y. Such ordered pairs are reversed later. Now, to compute R, we claim that the elements of Y can be "viewed" as a chain with respect to the elements of $X \setminus Y$. More precisely, it is possible to linearize the partial order $P(Y)$ to obtain a total order T on Y that satisfies the following property: for all elements $x, y \in Y$ if $x < y$ in T then $D(x, X \setminus Y) \subseteq D(y, X \setminus Y)$ and $U(x, X \setminus Y) \supseteq U(y, X \setminus Y)$. Let Q_D and Q_U be the relations as defined:

$$Q_D \;=\; \{(x, y) \mid x, y \in Y,\ x \parallel y \text{ in } P(Y) \text{ and } D(x, X \setminus Y) \subsetneq D(y, X \setminus Y)\}$$

$$Q_U \;=\; \{(x, y) \mid x, y \in Y,\ x \parallel y \text{ in } P(Y) \text{ and } U(x, X \setminus Y) \supsetneq U(y, X \setminus Y)\}$$

We establish that the relation $P(Y) \cup Q_D$ is acyclic. The main idea is that any cycle in $P(Y) \cup Q_D$, if it exists, must involve a pair from Q_D because $P(Y)$ is acyclic. Note that for every pair $(x, y) \in P(Y)$, $D(x, X \setminus Y) \subseteq D(y, X \setminus Y)$, and for every pair $(x, y) \in Q_D$, $D(x, X \setminus Y) \subsetneq D(y, X \setminus Y)$. Hence if an element p is involved in a cycle then $D(p, X \setminus Y) \subsetneq D(p, X \setminus Y)$—a contradiction. Similarly, it can be proved that the relation $P(Y) \cup Q_D \cup Q_U$ is acyclic as well. All incomparable pairs that remain after taking the transitive closure of $P(Y) \cup Q_D \cup Q_U$ are $(X \setminus Y)$–indistinguishable in P and therefore can be ordered either way in T. The required total order T on Y is given by any linearization of the relation $P(Y) \cup Q_D \cup Q_U$.

We use y_i to refer to the i^{th} element in the total order T for $i = 1, 2, \ldots, n$ where $n = |Y|$. We now compute the Y–indistinguishable classes of $X \setminus Y$ in P. Figure 3 shows various Y–indistinguishable classes of $X \setminus Y$ in P and how they relate to Y. In the figure we depict each equivalence class by a single representative element, namely a for A, and d_i, u_i and b_i for D_i, U_i and B_i, respectively, for each i. The classes other than the Y–disconnected class A can be partitioned into three categories. The first family of classes, denoted by $\mathcal{D}$, consists of classes D_i for $i = 1, 2, \ldots, n$ so that an element $x \in X \setminus Y$ belongs to D_i if $x < y_i$ in P but $y_{i-1} \parallel x$ in P. Note that when $i = 1$ only the first condition is applicable, that is, $x \in D_1$ if $x < y_1$ in P. The second family of classes, denoted by $\mathcal{U}$, consists of classes U_i for $i = 1, 2, \ldots, n$ so that an element $x \in X \setminus Y$ belongs to U_i if $y_i < x$ in P but $x \parallel y_{i+1}$ in P. Again, note that when $i = n$ only the first condition applies, that is, $x \in U_n$ if $y_n < x$ in P. The third family of classes, denoted by $\mathcal{B}$, contains classes B_i for $i = 1, 2, \ldots, n-1$ such that an element $x \in X \setminus Y$ is contained in B_i if $y_i < x < y_{i+1}$. Clearly, all Y–indistinguishable classes of $X \setminus Y$ in P are covered by $\{A\} \cup \mathcal{D} \cup \mathcal{U} \cup \mathcal{B}$. The required rectangular order R is given by:

$$
\begin{aligned}
R.1 \quad &= \quad a < d_n < d_{n-1} < \cdots < d_1 < y_1 < b_1 < y_2 < b_2 < \cdots < y_n < \\
&\quad\quad u_n < u_{n-1} < \cdots < u_1 \\
R.2 \quad &= \quad d_1 < y_1 < u_1 < b_1 < d_2 < y_2 < u_2 < b_2 < \cdots < b_{n-1} < d_n < \\
&\quad\quad y_n < u_n < a
\end{aligned}
$$

Now, we independently compute representations for the subposets $(Y, P(Y))$ and $(X \setminus Y, P(X \setminus Y))$. To represent the former, we use a chain realizer, and, to represent the latter, we use a rectangular realizer. Set $t = \max\{\dim(Y, P(Y)), \mathrm{rdim}(X \setminus Y, P(X \setminus Y))\}$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ be a chain realizer of $P(Y)$ on Y, and let $\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$ be a rectangular realizer of $P(X \setminus Y)$ on $X \setminus Y$. We construct a family of t rectangular orders $\mathcal{T} = \{T_1, T_2, \ldots, T_t\}$ where the rectangular order T_i on X is given by disjoint composition of the order C_i on Y and the order S_i on $X \setminus Y$ for $i = 1, 2, \ldots, t$. We choose a chain realizer and not a rectangular realizer for representing $(Y, P(Y))$ because when we linearize $P(Y)$ we may introduce ordered pairs from $Y \times Y$ that are not present in P. These pairs are reversed by the chain realizer. Finally, $\{R\} \cup \mathcal{T}$ constitutes a rectangular realizer of P on X. $\square$

A chain trivially constitutes a critical subposet of any poset because it does not contain any incomparable pair. Also, the dimension of a chain is one. Thus, from critical subposet removal theorem, it follows that:

Theorem 2 (chain removal theorem) *Let C be a chain in the poset $\mathbf{P} = (X, P)$. Then,*

$$\mathrm{rdim}(X, P) \leqslant 1 + \mathrm{rdim}(X \setminus C, P(X \setminus C))$$

Does a similar theorem exist for an antichain? The answer is in general no. But in case the antichain consists of only two elements, a removal theorem can indeed be provided.

Theorem 3 (incomparable pair removal theorem) *Let $\mathbf{P} = (X, P)$ be a poset and (x, y) be an incomparable pair P. Then,*

$$\mathrm{rdim}(X, P) \leqslant 1 + \mathrm{rdim}(X \setminus \{x, y\}, P)$$

Proof: For convenience, set $Y = \{x, y\}$. We give a rectangular order R containing the relation $P \setminus P(X \setminus Y)$ (in this case, $P(Y) = \emptyset$) that is non-interfering with P. Figure 4 depicts various

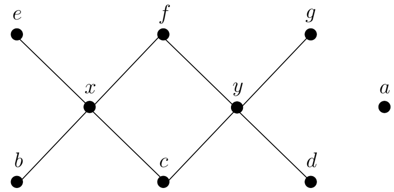

Figure 4: The Y–indistinguishable classes of $X \setminus Y$ in P where $Y = \{x, y\} \subseteq X$ with $x \parallel y$ in P.

Y–indistinguishable classes and how they relate to Y. As shown, there are seven such classes represented by a, b, c, d, e, f and g. The required rectangular order R is given by:

$$\begin{aligned} R.1 &= a < b < c < x < e < d < y < f < g \\ R.2 &= d < c < y < g < b < x < f < e < a \end{aligned}$$

It can be verified that R contains $P \setminus P(X \setminus Y)$ and does not interfere with P. Let $\mathcal{S}$ be a rectangular realizer of $P(X \setminus Y)$ on $X \setminus Y$. We can obtain a rectangular realizer $\mathcal{T}$ of $P(X \setminus Y)$ on X by disjointly composing each rectangular order of $\mathcal{S}$ with the empty order on Y. Then $\{R\} \cup \mathcal{T}$ constitutes a rectangular realizer of P on X. $\Box$

4.2.2 Establishing Upper Bounds on Rectangular Dimension

Since $\mathrm{rdim}(X, P) \leqslant \dim(X, P)$ and $\dim(X, P) \leqslant |X|/2$ when $|X| \geqslant 4$, trivially, $\mathrm{rdim}(X, P) \leqslant |X|/2$ when $|X| \geqslant 4$. The bound is not tight for rectangular dimension as shown in this section. Before we give a bound on the rectangular dimension of a general poset, we provide a bound on the rectangular dimension of a bipartite poset. Recall that a poset $\mathbf{P} = (X, P)$ is *bipartite* if it does not contain any chain involving more than two elements. In that case, the ground set X can be partitioned into two disjoint subsets L and U so that $P \subseteq L \times U$ and we write $\mathbf{P} = (X, P) = (L, U, P)$.

Theorem 4 *Let $\mathbf{P} = (X, P) = (L, U, P)$ be a bipartite poset with nonempty L and U. Then,*

$$\mathrm{rdim}((X, P)) \leqslant \min\{\lceil |L|/2 \rceil, \lceil |U|/2 \rceil\}$$

Proof: Without loss of generality, assume that $\lceil |L|/2 \rceil \leqslant \lceil |U|/2 \rceil$ and further that $|L|$ is even. In case $|L|$ is odd, we add an element to L that is not connected to any other element. Set $t = \lceil |L|/2 \rceil$. Using incomparable pair removal theorem repeatedly, we successively remove two elements from L which, by definition of L, are incomparable in P until we have exhausted all elements in L. Clearly, t pairs of elements are removed. Thus,

$$\mathrm{rdim}(X, P) \leqslant t + \mathrm{rdim}(U, P(U)) = t + \mathrm{rdim}(U, \emptyset) = t + 1$$

Note, however, that we do not need a separate rectangle to represent the poset $(U, \emptyset)$. This is because the rectangular order constructed in the proof of the incomparable pair removal theorem is non-interfering with P. This implies that for all $x, y \in U$, $x \parallel y$ in each of the other t rectangular orders. Therefore the fact that the elements of U form an antichain is already captured in the other t rectangular orders. As a result, $\mathrm{rdim}(X, P) \leqslant t = \lceil |L|/2 \rceil$. $\Box$

Clearly, either $|L| \leqslant |X|/2$ or $|U| \leqslant |X|/2$. Thus, from Theorem 4, it follows that:

Corollary 5 *Let* $\mathbf{P} = (X, P)$ *be a bipartite poset. Then,*

$$\mathrm{rdim}(X, P) \leqslant \lceil |X|/4 \rceil$$

For a general poset, a slightly weaker upper bound can be given.

Theorem 6 *Let* $\mathbf{P} = (X, P)$ *be a poset with* $|X| \geqslant 3$*. Then,*

$$\mathrm{rdim}(X, P) \leqslant |X|/3$$

Proof for Theorem 6: The proof is by induction on the number of elements in X.

Base Case ($|\mathbf{X}| \leqslant \mathbf{5}$): It can be verified by doing case analysis that whenever $|X| \leqslant 5$, $\dim(X, P) \leqslant 2$ implying that $\mathrm{rdim}(X, P) \leqslant 1$ [16, Page 23].

Induction Step: Suppose that $\mathrm{rdim}(X, P) \leqslant |X|/3$ whenever $|X| \leqslant k$ where $k \geqslant 5$. Now consider a poset (X, P) with $k+1$ elements. In case the poset (X, P) contains a chain C involving three elements, from Theorem 2, $\mathrm{rdim}(X, P) \leqslant 1 + \mathrm{rdim}(X \setminus C, P(X \setminus C)) \leqslant 1 + |X \setminus C|/3$ $= 1 + (|X| - 3)/3 = |X|/3$. Thus assume that the poset (X, P) does not contain any chain involving more than two elements, or, in other words, it is a bipartite poset. From Corollary 5, $\mathrm{rdim}(X, P) \leqslant \lceil |X|/4 \rceil \leqslant |X|/3$. □

4.2.3 Bounding Rectangular Dimension of Interval Orders

Recall that a poset $\mathbf{P} = (X, P)$ is an *interval order* if there exists an *interval representation* function F assigning to each element $x \in X$ a non-degenerate closed interval $F(x) = [a_x, b_x]$ of the real line $\mathbf{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbf{R}$ [16, Page 190]. We define the *range* of an interval order (X, P) with respect to an interval representation function F, denoted by $\mathrm{range}(X, P, F)$, as the cardinality of the set $\{b_x - a_x \mid x \in X \text{ with } F(x) = [a_x, b_x]\}$. In case $\mathrm{range}(X, P, F) = 1$, the interval order (X, P) is called a *semi-order* [16, Page 192]. We show that the rectangular dimension of an interval order is at most two more than its range. This is an extension of the result in the dimension theory that if $\mathrm{range}(X, P, F) = 1$ then $\dim(X, P) \leqslant 3$ [16, Page 196].

Theorem 7 *Let* $\mathbf{P} = (X, P)$ *be an interval order with interval representation* F*. Then,*

$$\mathrm{rdim}(X, P) \leqslant \mathrm{range}(X, P, F) + 2$$

Proof: Set $t = \mathrm{range}(X, P, F)$. Clearly, the ground set X can be partitioned into t subsets X_i for $i = 1, 2, \ldots, t$ such that each induced subposet $(X_i, P(X_i))$ is a semi-order. We claim that each subposet $(X_i, P(X_i))$ is also a critical subposet of the poset (X, P).

Consider a subposet $(X_i, P(X_i))$ and elements $x, y \in X_i$ with $x \parallel y$ in $P(X_i)$. Let $b_x - a_x = b_y - a_y = d$ (say) where $F(x) = [a_x, b_x]$ and $F(y) = [a_y, b_y]$. Either $a_x \leqslant a_y$ or $a_y \leqslant a_x$. Without loss of generality, assume that $a_x \leqslant a_y$. We establish that $D(x) \subseteq D(y)$ and $U(x) \supseteq U(y)$ which in turn implies that (x, y) is a critical pair in (X, P) and hence a critical pair in $(X \setminus Y, P(X \setminus Y))$. Consider an element $z \in X$ with $F(z) = [a_z, b_z]$. If $z \in D(x)$ then $b_z < a_x \leqslant a_y$ implying that $z \in D(y)$. Similarly, if $z \in U(y)$ then $a_z > b_y = a_y + d \geqslant a_x + d = b_x$ or, equivalently, $a_z > b_x$ implying that $z \in U(x)$.

The dimension of a semi-order is at most three. Thus, by repeatedly apply the critical subposet removal theorem, we obtain:

$$\mathrm{rdim}(X, P) \leqslant \mathrm{range}(X, P, F) - 1 + \max\{\mathrm{rdim}(X_t, P(X_t)), 3\}$$

Since the rectangular dimension of a semi-order is also at most three, $\mathrm{rdim}(X, P) \leqslant \mathrm{range}(X, P, F) - 1 + 3 = \mathrm{range}(X, P, F) + 2$. □

5 Order Decomposition Method and its Applications

5.1 The Main Idea

Given a poset $\mathbf{P} = (X, P)$, we first decompose the partial order P into t suborders P_i for $i = 1, 2, \ldots, t$. It is not necessary for the suborders to be disjoint. We next compute a rectangular realizer $\mathcal{R}_i$ for each subposet (P, X_i) such that $\mathcal{R}_i$ is non-interfering with P. Then the collection of rectangular orders $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_t$ constitutes a rectangular realizer of P on X. When we specify a decomposition of a partial order into suborders, we do not enumerate the reflexive pairs which can always be added later.

5.2 Applications

In this section, we provide upper bounds on rectangular dimension of posets based on two measures: "degree of connectivity" and "degree of adjacency".

5.2.1 Bounding Rectangular Dimension of Posets based on Degree of Connectivity

Suppose the Hasse diagram (or covering graph) of a poset $\mathbf{P} = (X, P)$ is such that every element in the graph has at most one outgoing edge. In this case the covering graph resembles a forest of trees. In particular, for every element $x \in X$, $U(x)$ forms a chain in P. Such a poset belongs to the class of *series-parallel* posets [10, 12, 17]. The dimension of a series-parallel poset is at most two and hence its rectangular dimension is at most one [10, 12, 17]. Similarly, a poset whose Hasse diagram is such that every element has at most one incoming edge also has rectangular dimension of one.

A natural question to ask is: what other posets have "small" rectangular dimension? In this section, we show that posets with "low" indegree have "small" rectangular dimension. Furthermore, we show that posets in which the indegree of every element is either "low" or "high" (but not "medium") also have "small" rectangular dimension.

For a poset $\mathbf{P} = (X, P)$ and an element $x \in X$, the *indegree* of x in P, denoted $\deg_{\mathrm{D}}(x)$, is defined as the number of elements less than x in P [16, Page 165]. Let $\Delta_{\mathrm{D}}(X, P)$ denote $\max\{\deg_{\mathrm{D}}(x) \mid x \in X\}$. The *outdegree* of an element can be dually defined.

Theorem 8 *Let $\mathbf{P} = (X, P)$ be a poset with $\Delta_{\mathrm{D}}(X, P) \leqslant k$ where $k \geqslant 1$. Then,*

$$\mathrm{rdim}(X, P) \leqslant k$$

Proof: The central idea is to decompose the partial order into at most k suborders such that the subposet induced by each suborder is a rectangle.

For each element $x \in X$, number all elements in $D(x)$ from 1 to $|D(x)|$. The i^{th} suborder P_i for $i = 1, 2, \ldots, k$ is given by the reflexive transitive closure of the set $\{(x, y) | x \in X \text{ and } y \text{ is the } i^{th} \text{ element in } D(x), \text{ if it exists}\}$. Clearly, each element has at most one incoming edge in the Hasse diagram of (X, P_i). Hence (X, P_i) is a series-parallel poset, which implies that P_i is a rectangular order. Since $P_i \subseteq P$, P_i is non-interfering with P. Therefore it follows that the set $\{P_1, P_2, \ldots, P_k\}$ constitutes a rectangular realizer of P on X. □

We now show that if the indegree of every element in a poset is either at most k or at least $|X| - k$, then the rectangular dimension of the poset is at most $\lceil 3k/2 \rceil + 1$.

Theorem 9 *Let* $\mathbf{P} = (X, P)$ *be a poset such that for every element* $x \in X$*, either* $\deg_{\mathrm{D}}(x) \leqslant k$ *or* $\deg_{\mathrm{D}}(x) \geqslant |X| - k$ *where* $k \geqslant 1$*. Then,*

$$\mathrm{rdim}(X, P) \leqslant \lceil 3k/2 \rceil + 1$$

Proof: It suffices to prove that $\mathrm{rdim}(X, P) \leqslant \lceil 3k/2 \rceil + 1$ when $k \leqslant \lfloor |X|/2 \rfloor$. Our approach is to partition the ground set X into two disjoint subsets L and U such that (1) for every element $x \in L$, $|D(x)| \leqslant k$, and (2) for every element $x \in U$, $|D(x)| \geqslant |X| - k$. Clearly, there is no element in U that is less than some element in L; otherwise $D(x) \supset D(y)$ with $x \in L$, $y \in U$, and $y < x$ in P implying that $|D(x)| > |D(y)| \geqslant |X| - k \geqslant \lceil |X|/2 \rceil$—a contradiction. We now bound the size of U in case it is non-empty. Consider a minimal element x of $(U, P(U))$. By definition, $D(x) \subseteq L$. Therefore $|L| \geqslant |D(x)| \geqslant |X| - k$ which in turn implies that $|U| \leqslant k$.

Set $t = \lceil 3k/2 \rceil + 1$. Let T be some total order on $L \cup U$. We construct a rectangular order R on X as follows:

$$\begin{aligned} R.1 &= T(L) < T(U) \\ R.2 &= T^d(L) < T^d(U) \end{aligned}$$

Note that $R(L) = \emptyset$, $R(U) = \emptyset$ and every element of L is less than every element of U in R. Therefore some ordered pairs in R need to be reversed. However, since every element in U has high indegree, the number of such ordered pairs is small. Let $Q_1 = P(L) \cup P(U)$ and $Q_2 = \{(y, x) \mid x \in L,\ y \in U \text{ and } x \parallel y \text{ in } P\}$. Informally, Q_1 captures those ordered pairs in P that do not belong to R, and Q_2 reverses those ordered pairs in R that are not present in P.

To represent (X, Q_1), we observe that the indegree of every element in (X, Q_1) is at most k; for elements in L, it follows from the definition of L, and for elements in U, it follows from the fact that $|U| \leqslant k$. As a result, we can use the construction in the proof of Theorem 8 to compute a non-interfering rectangular realizer of Q_1 on X consisting of at most k rectangular orders, say $\mathcal{S}$.

To represent (X, Q_2), we observe that (X, Q_2) is actually a bipartite poset, say (L', U', Q_2), where $L' = U$ and $U' = L$. Further, $|L'| = |U| \leqslant k$. As a result, we can use the construction in the proof of Theorem 4 to compute a non-interfering rectangular realizer of Q_2 on X consisting of at most $\lceil k/2 \rceil$ rectangular orders, say $\mathcal{T}$.

Finally, $\{R\} \cup \mathcal{S} \cup \mathcal{T}$ constitutes a rectangular realizer of P on X consisting of at most $1 + k + \lceil k/2 \rceil = \lceil 3k/2 \rceil + 1$ rectangular orders. □

5.2.2 Bounding Rectangular Dimension of Bipartite Posets Based on Degree of Adjacency

In this section, we prove that bipartite posets with high degree of adjacency have small rectangular dimension. We then prove that a generalized crown has rectangular dimension of two by showing that it has high degree of adjacency. To prove our results, we use two properties defined by Spinrad, Brandstädt and Stewart in [15].

Definition 7 (adjacency property [15]) *A bipartite poset* $\mathbf{P} = (X, P) = (L, U, P)$ *satisfies the* adjacency property *if there is an ordering of elements of* U *such that for each element* $x \in L$*, elements of* $U(x) \subseteq U$ *occur consecutively in the ordering.*

Definition 8 (enclosure property [15]) *A bipartite poset* $\mathbf{P} = (X, P) = (L, U, P)$ *that satisfies the adjacency property also satisfies the* enclosure property *if for all elements* $x, y \in L$ *with* $U(x) \subseteq U(y)$*, elements in* $U(y) \setminus U(x) \subseteq U$ *occur consecutively in the ordering dictated by the adjacency property.*

Spinrad, Branstädt and Stewart in [15] prove that a bipartite poset satisfying both adjacency and enclosure properties has dimension of at most two. This implies that the rectangular dimension of such a poset is at most one. We generalize their result to arbitrary bipartite posets.

Theorem 10 *Let* $\mathbf{P} = (X, P) = (L, U, P)$ *be a bipartite poset. Suppose* P *is decomposed into* t *suborders* P_i *for* $i = 1, 2, \ldots, t$ *such that each induced bipartite subposet* (X, P_i) *satisfies the adjacency property. Then,*

$$\mathrm{rdim}(X, P) \leqslant t + 2$$

If each bipartite subposet (X, P_i) *for* $i = 1, 2, \ldots, n$ *also satisfies the enclosure property, then,*

$$\mathrm{rdim}(X, P) \leqslant t$$

Proof: For each bipartite subposet (X, P_i) for $i \in [1, t]$, we construct a rectangular order R_i that contains P. However, R_i may interfere with P. But R_i is such that the ordered pairs in R_i that do not belong to P only involve the elements of L. Such ordered pairs are reversed later.

Let T be some total order on $L \cup U$ that is consistent with the order on U as dictated by the adjacency property. We denote the i^{th} element in the total order $T(U)$ by u_i for $i = 1, 2, \ldots, n$ where $n = |U|$. The two total orders $R_i.1$ and $R_i.2$ on X that realize the rectangular order R_i are both constructed recursively. To construct $R_i.1$, we start with the empty order G_0. At the i^{th} step for $i = 1, 2, \ldots, n$, we "append" to G_{i-1} the element u_i and those elements of $D(u_i)$ that are not already in G_{i-1} as follows:

$$G_i = G_{i-1} < T(D(u_i) \setminus G_{i-1}) < u_i$$

Finally, to obtain $R_i.1$, we "append" to G_n the remaining elements of L (that are not already in there) as follows: $R_i.1 = G_n < T(L \setminus G_n)$. The total order $R_i.2$ is constructed in a similar fashion except that the elements of U are considered in the reverse order. More precisely, starting from the empty order H_0, H_i for $i = 1, 2, \ldots, n$ is constructed recursively as given:

$$H_i = H_{i-1} < T(D(u_{n-i+1}) \setminus H_{i-1}) < u_{n-i+1}$$

Finally, we "prepend" to H_n the remaining elements of L (that are not already in there) to obtain $R_i.2$ as follows: $R_i.2 = T(L \setminus H_n) < H_n$. We show that the rectangular order R_i satisfies two important properties. First, it contains all ordered pairs in P. Second, all "other" ordered pairs in R, given by $R \setminus P$, only involve the elements of L.

Since $R_i.1(U) = T(U)$ and $R_i.2(U) = T^d(U)$, $R_i(U) = \emptyset$ implying that the elements of U form an antichain in R_i. Note that $L \setminus G_n = L \setminus H_n$. Let $A = L \setminus G_n$. Informally, an element of A is a "loose" point not connected to any other element in X. Clearly, by construction, for all $a \in A$ and $u \in U$, $a \| u$ in R_i. It now remains to be shown that $R_i \cap (U \times (L \setminus A)) = \emptyset$ and $R_i \cap ((L \setminus A) \times U) = P$.

Consider elements $a \in L \setminus A$ and $b \in U$. In case $b < a$ in $R_i.1$, there exists an element $c \in U$ so that $b < c$ in T and $a \in D(c)$. However, by construction, $c < b$ in $R_i.2$ implying that $a < b$ in $R_i.2$. As a result, $a \parallel b$ in R_i. Thus no element of U is less than an element of $L \setminus A$ in R_i or $R_i \cap (U \times (L \setminus A)) = \emptyset$.

Now, consider the other case when $a < b$ in $R_i.1$. Suppose $a \in D(b)$. By construction, $a < b$ in $R_i.2$ which implies that $a < b$ in R_i. Thus every ordered pair in P is also an ordered pair in R_i or $P \subseteq R_i$. Finally, suppose $a \notin D(b)$. Then there is an element $c \in U$ with $c < b$ in T such that $a \in D(c)$. We claim that $b < a$ in $R_i.2$. Assume the contrary, that is, $a < b$ in $R_i.2$. Since $a \notin D(b)$, there is an element $d \in U$ such that $d < b$ in T^d with $a \in D(d)$. Therefore (1) $a \in D(c) \cap D(d)$, (2) $c < b < d$ in T, and (3) $a \notin D(b)$. Combining the three, we can infer that elements in $U(a)$ do not occur consecutively in T, which violates the adjacency property—a contradiction. Thus

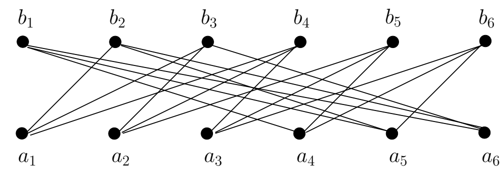

Figure 5: The crown $\mathbf{S}_4^2$.

if an element of $L \setminus A$ is incomparable with an element of U in P then so is the case in R_i or $R_i \cap ((L \setminus A) \times U) \subseteq P$.

It follows that the family of rectangular orders $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ "almost" realizes P on X but for some ordered pairs involving the elements of L, which need to be reversed. To accomplish that, we need two more rectangles. The first of the two rectangular orders, say R_{t+1}, is obtained by disjoint composition of $T(L)$ on L with the empty order on U. The second of the two rectangular orders, say R_{t+2}, is obtained by disjoint composition of $T^d(L)$ on L with the empty order on U. Clearly, $\mathcal{R} \cup \{R_{t+1}, R_{t+2}\}$ realizes P on X.

We now prove the second part of the theorem. Suppose each bipartite subposet (X, P_i) for $i = 1, 2, \ldots, n$ satisfies the enclosure property as well. This implies that $\mathrm{rdim}(X, P_i) \leqslant 1$ for each $i \in [1, t]$. Thus each suborder P_i is a rectangular order on X and, evidently, $\{P_1, P_2, \ldots, P_t\}$ constitutes a rectangular realizer of P on X. $\square$

We now show a generalized crown can be represented using only a small number of rectangles. Recall that the *generalized crown* $\mathbf{S}_n^k$ [16, Chapter 2], for integers $n \geqslant 3$ and $k \geqslant 0$, is a bipartite poset with $L = \{a_1, a_2, \ldots, a_{n+k}\}$ and $U = \{b_1, b_2, \ldots, b_{n+k}\}$. For each $i = 1, 2, \ldots, n+k$, $b_i \parallel \{a_i, a_{i+1}, \ldots, a_{i+k}\}$ in $\mathbf{S}_n^k$, and $b_i > a_j$ in $\mathbf{S}_n^k$ for each $j = i+k+1,\ i+k+2, \ldots, i+k+n-1$. In the definition, the subscripts are to be interpreted cyclically. Figure 5 depicts $\mathbf{S}_4^2$.

Theorem 11 *The rectangular dimension of a generalized crown is at most two.*

Proof: We decompose the partial order of the generalized crown $\mathbf{S}_n^k$ into two suborders such that both satisfy the adjacency and enclosure properties. To accomplish this, it helps to redefine the generalized crown $\mathbf{S}_n^k$ from the point of view of the elements in L. For each $i = 1, 2, \ldots, n+k$, $a_i \parallel \{b_{i-k}, b_{i-k+1}, \ldots, b_i\}$ in $\mathbf{S}_n^k$, and $a_i < b_j$ in $\mathbf{S}_n^k$ for each $j = i+1, i+2, \ldots, i+n-1$. Of course, the subscripts are to be interpreted cyclically.

For each element $a_i \in L$, we partition the set $U(a_i) = \{b_{i+1}, b_{i+2},\ \ldots,\ b_{i+n-1}\}$ into two subsets denoted by $U_1(a_i)$ and $U_2(a_i)$. The first subset $U_1(a_i)$ consists of elements b_j for $j = i+1, i+2, \ldots, \min\{i+n-1, n+k\}$. In case $i+n-1 > n+k$, the second subset $U_2(a_i)$ consists of elements b_j for $j = 1, 2, \ldots, i-k-1$; otherwise it is empty. For an illustration, refer to Figure 6.

Now, the first suborder P_1 is given by $\{(a, b) \mid a \in L \text{ and } b \in U_1(a)\}$, and the second suborder P_2 is given by $\{(a, b) \mid a \in L \text{ and } b \in U_2(a)\}$. Clearly, both subposets $(L \cup U, P_1)$ and $(L \cup U, P_2)$ satisfy the adjacency and enclosure properties. $\square$

In contrast, the dimension of a generalized crown can be "large". Specifically, $\dim(\mathbf{S}_n^k) = \lceil 2(n+k)/(k+2) \rceil$ for each $n \geqslant 3$ and $k \geqslant 0$ [16, Chapter 2]. Note that the class of generalized crowns includes the class of standard examples. Therefore the rectangular dimension of a standard example is also at most two.

Using Theorem 10, a simple heuristic to compute a rectangular realizer of a bipartite poset can be derived based on the following observation: *any bipartite poset with at most two minimal*

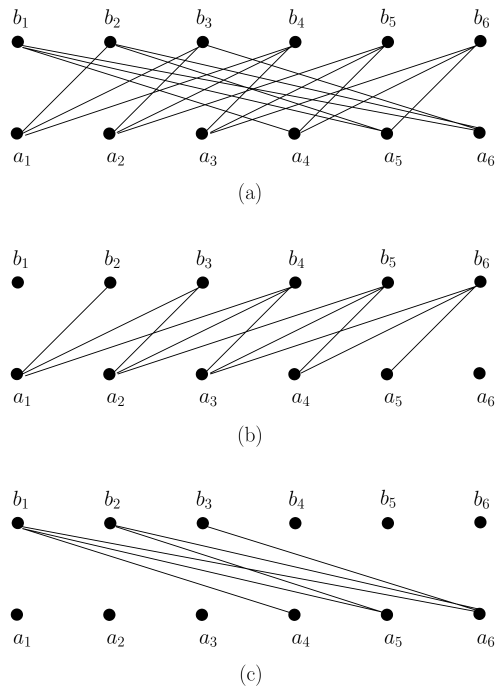

Figure 6: (a) The crown $\mathbf{S}_4^2$, (b) and (c) its two subposets that satisfy the adjacency and enclosure properties.

elements satisfies both adjacency and enclosure properties. To see why, consider a bipartite poset $\mathbf{P} = (X, P) = (L, U, P)$ with $|L| = 2$. Suppose the two minimal elements are a and b, and let S be some total order on U. We construct another total order T on U as follows:

$$\begin{aligned} T \;&=\; S\Big(U(a) \setminus U(b)\Big) < S\Big(U(a) \cap U(b)\Big) < S\Big(U(b) \setminus U(a)\Big) < \\ &\;\; S\Big(U \setminus (U(a) \cup U(b))\Big) \end{aligned}$$

Clearly, all elements of $U(a)$ as well as $U(b)$ occur consecutively in T. Now, assume that $|L| > 2$. A suborder $P_1 \subseteq P$ satisfying the adjacency property can now be obtained as follows. Pick two elements from L with maximum outdegrees. They serve as elements a and b. Add all their outgoing edges to P_1. It turns out that we can add more edges to P_1 and still ensure that P_1 satisfies the adjacency property. Specifically, for every element $x \in L \setminus \{a, b\}$, we can compute the largest subset of $U(x)$ whose elements occur consecutively in T and add the corresponding edges to P_1. Finally, we compute $P \setminus P_1$ and repeat the above steps to compute P_2 and so on.

6 Related Work

The *boolean dimension* of a poset $\mathbf{P} = (X, P)$, defined by Gambosi, Nešetřil and Talamo [7], is the least positive integer t so that there exist a boolean formula $F(b_1, b_2, \ldots, b_t)$ on boolean variables $b_1, b_2, \ldots, b_t$ and total orders $\{C_1, C_2, \ldots C_t\}$ satisfying the following: for every $x, y \in X$, $x \leqslant y$ in P if and only if $F(b_1, b_2, \ldots, b_t)$ evaluates to true where b_i is true if and only if $x \leqslant y$ in C_i. The notions of dimension and rectangular dimension can be derived from the notion of boolean dimension by appropriately defining the boolean formula. For example, to derive the notion of dimension, the boolean formula can be set to $b_1 \wedge b_2 \wedge \cdots \wedge b_t$. On the other hand, to derive the notion of rectangular dimension, the boolean formula can be set to $\Big((b_1 \wedge b_2) \vee (b_3 \wedge b_4) \vee \cdots \vee (b_{t-1} \wedge b_t)\Big) \bigwedge \Big((b_1 \vee b_2) \wedge (b_3 \vee b_4) \wedge \cdots \wedge (b_{t-1} \vee b_t)\Big)$. Clearly, the boolean dimension of a poset is less than or equal to its rectangular dimension. However, using boolean dimension entails the added complexity of representing the boolean formula F.

The *encoding dimension* of a poset $\mathbf{P} = (X, P)$ is defined as the least positive integer t with $t = \sum_{i=1}^{m} \lceil \log_2(k_i) \rceil$ such that P can be embedded into $K_1 \times K_2 \times \cdots \times K_m$ where K_i denotes a chain of length k_i [9]. The encoding dimension minimizes the number of bits as opposed to the chain dimension which minimizes the number of chains in a chain realizer. Clearly, if the objective is to minimize the number of bits, the code assigned to each element when rectangles are used is at most two times the encoding dimension of a poset. However, posets with large encoding dimension that can be encoded optimally using rectangles include standard examples and generalized crowns.

Garg and Skawratananond introduce the notion of *string* in [8]. A poset is said to form a string if there is a function f assigning integer values to each element in the set so that an element x is less than another element y in the partial order if $f(x) < f(y)$. In case $f(x) \leqslant f(y)$, we say that x is less than or equal to y in the string. Trivially, a chain is a string and the dimension of a string is at most two. A set of strings realizes a partial order if an element x is less than another element y in the partial order if and only if x is less than or equal to y in all the strings and x is less than y in at least one of the strings. The *string dimension* of a poset is the least number of strings required to realize the corresponding partial order. Garg and Skawratananond [8] prove that the string dimension of a poset, which is itself not a string, is equal to its dimension. However, encoding posets using strings could potentially use much fewer bits than chains. For example, encoding an antichain using strings requires only 1 bit per element; every element is assigned bit 0. It can be proved that every poset can be encoded using strings with at most n bits per element, where n is the number of elements. Algorithms by Fidge [6] and Mattern [14] for encoding the causality relation between events in a distributed computation using vector clocks can be viewed as special cases of encoding using strings.

It is possible to expand the class of extensions available for use in realizers in dimension theory. In particular, interval orders can be used instead of chains. In that case, the *interval dimension* of a poset is defined as the least number of interval orders whose intersection is same as the given partial order [16]. Posets with arbitrarily large interval dimension but constant rectangular dimension include standard examples and generalized crowns.

Capelle [2] propose a technique in which the given partial order is decomposed into a set of interval orders for which optimal encoding is already known. The scheme is similar to our order decomposition method but for the "building block" which is an interval order instead of a two-dimensional poset. While the focus in [2] is on developing a heuristic for decomposition and evaluating its performance experimentally, we primarily concern ourselves with establishing theoretical upper bounds on the rectangular dimension of a poset in general and special families of posets in particular.

In *PQ-encoding*, proposed by Zibin and Gil [19], the set of elements is partitioned into *slices*; each element belongs to exactly one slice. Each slice satisfies the following property: there exists an ordering of elements such that for every element x in the slice, the elements in $x \cup D(x)$ appear consecutively in the ordering. An element is assigned a code which consists of (number of slices + 3) integers. Using PQ-encoding, a *less-than* query can be answered in $O(1)$ time. However, PQ-encoding may not produce an optimal encoding for a poset. For a poset $\mathbf{P} = (X, P)$, let $\text{numslices}(X, P)$ denote the least number of such slices into which X can be partitioned. For the standard example $\mathbf{S}_n$, no slice can contain more than two maximal elements implying that numslices($\mathbf{S}_n$) = $\lceil n/2 \rceil$. In this case, therefore, PQ-encoding assigns a code consisting of $O(n \log n)$ bits to each element which requires much more space than the encoding obtained using rectangles. Similarly, for the generalized crown $\mathbf{S}_n^k$, no slice can contain more than $k+1$ maximal elements. Therefore numslices($\mathbf{S}_n^k$) = $\lceil (n+k)/(k+1) \rceil$, which implies that the PQ-encoding for the generalized crown is suboptimal.

Madej and West [13] give the notion of *interval inclusion number* of a poset. An *interval inclusion representation* of a poset is a set-valued function f that assigns to every element in the set a union of intervals from the real line $\mathbf{R}$ so that an element x is less than another element y in the partial order if $f(x) \subsetneq f(y)$. The interval inclusion number of a poset is the least positive integer t for which there is an interval inclusion representation of the poset that maps to every element a union of at most t intervals from the real line $\mathbf{R}$. The algorithm proposed by Agrawal, Borgida and Jagdish [1] for managing transitive relationships in a directed acyclic graph can be viewed as computing an interval inclusion representation of the poset. Many results on the rectangular dimension of a poset in this paper also hold for its interval inclusion number. However, for the boolean lattice B_n, which contains 2^n elements and whose interval inclusion number is equal to $\lceil n/2 \rceil$, to our knowledge, $O(n \log n)$ bits per element are required if interval inclusion representation is used. Our method, in contrast, assigns optimal $O(n)$ bits to each element (by utilizing string representation).

A graph with n vertices has an *implicit representation* if it is possible to assign a label of size $O(\log n)$ to every vertex so that the adjacency relation between every pair of vertices can be ascertained by simply examining the labels of the two vertices only [11]. Due to the limit on the size of the label, not every graph (or poset) has an implicit representation. Kannan, Naor and Rudich study in [11] which graphs permit implicit representation; the coding scheme may vary with graphs. We, on the other hand, investigate the behaviour of a specific coding scheme, based on rectangles, for different classes of posets.

7 Conclusion and Future Work

In this paper, we present a novel technique for encoding posets using rectangles. We establish upper bounds on rectangular dimension of posets in general and special families of posets in particular. We also provide examples of posets that can be encoded optimally using our technique but require may more bits per element element using other techniques such as adjacency matrix, adjacency list, dimension, interval dimension and PQ-encoding [16, 9, 19, 8].

Many interesting questions still remain. For instance, given n distinct elements, it can be proved that as many as $2^{\theta(n^2)}$ different posets can be constructed. Using information theory, therefore, a lower bound of $\theta(n/\log n)$ can be obtained on the rectangular dimension of a poset (even a bipartite poset) in the worst case. In this paper, we have only established upper bounds of $n/3$ and $\lceil n/4 \rceil$ on the rectangular dimension of general posets and bipartite posets, respectively, when $n \geqslant 3$. Can we tighten the upper bound to match the lower bound? What classes of posets have large rectangular dimension? Is it possible to compute the rectangular dimension of a poset efficiently or is the problem NP-hard?

We also plan to investigate in detail the relationship between the rectangular dimension of a poset and other measures of dimension such as interval dimension and interval inclusion number.

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