A Fusion-based Approach for Tolerating Faults in Finite State Machines

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Abstract

Given a set of n different deterministic finite state machines (DFSMs) modeling a distributed system, we examine the problem of tolerating f crash or Byzantine faults in such a system. The traditional approach to this problem involves replication and requires $n \cdot f$ backup DFSMs for crash faults and $2 \cdot n \cdot f$ backup DFSMs for Byzantine faults. For example, to tolerate two crash faults in three DFSMs, a replication based technique needs two copies of each of the given DFSMs, resulting in a system with six backup DFSMs. In this paper, we question the optimality of such an approach and present a generic approach called (f, m)-fusion that permits lesser number of backups than the replication based approaches. Given n different DFSMs, we examine the problem of tolerating f faults using just m additional DFSMs. We introduce the theory of fusion machines and provide an algorithm to generate backup DFSMs for both crash and Byzantine faults. Further, we have implemented these algorithms and tested them for various examples.

1 Introduction

A distributed or parallel system can often be modeled as a collection of distinct and independent deterministic finite state machines or DFSMs (also referred to as *machines*). In this paper, we look at the problem of tolerating faults in these machines. Most commonly occurring faults can be categorized as crash (or fail stop) faults [11] and Byzantine faults [8]. In the case of crash faults, there is a loss in the execution state of the machine. In the case of Byzantine faults, the faulty machines can *lie* or reflect an incorrect execution state. In order to build reliable systems, it is important to detect these faults and recover the correct state of the system. As a motivating example, consider a small sensor network with three different sensors running DFSMs measuring

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the average heat, light and humidity in the environment over a fixed period of time, say a month. During execution, one of these sensors might fail, resulting either in the loss of its state (crash faults) or an inconsistency in the state (Byzantine faults). At the end of the month we need to determine the correct execution state of the system, i.e., the final states of each of the sensors.

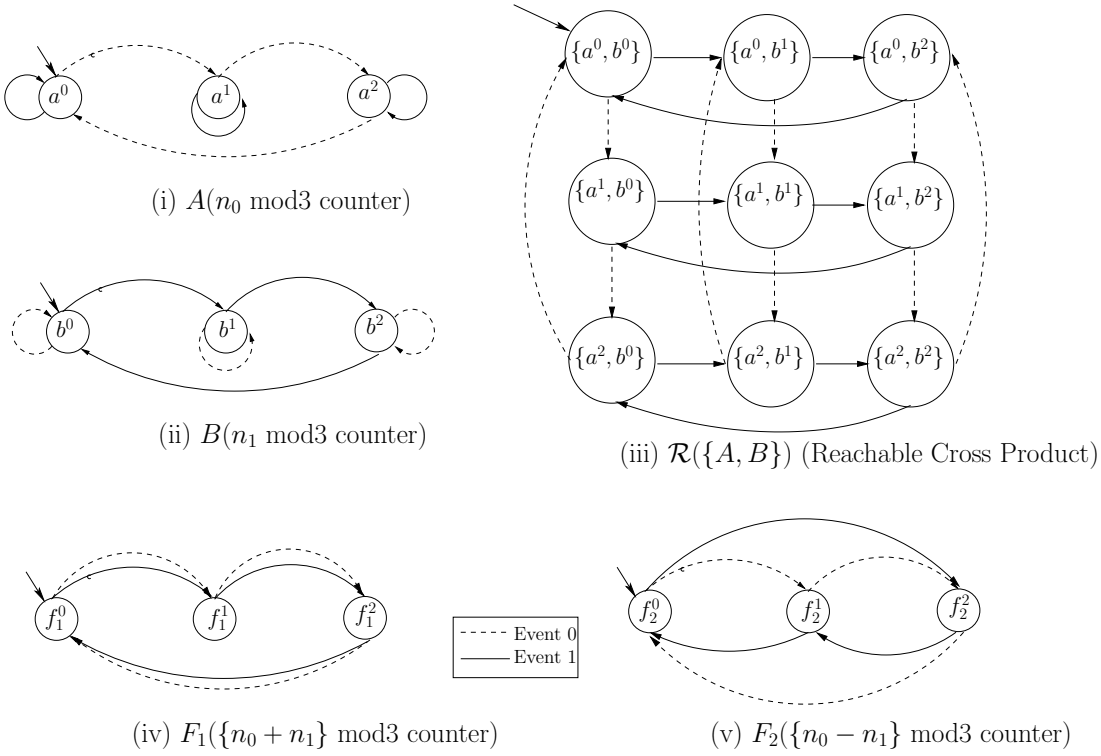
Traditional approaches to tolerating f faults in n different DFSMs require some form of replication. In the case of crash faults, we need to maintain f extra copies of each DFSM, resulting in a total of $n \cdot f$ backup DFSMs [7, 10, 14, 12]. In the case of Byzantine faults, since any f machines can lie about their current state, we need to obtain a majority on the current state of any failed DFSM. Hence, we need to maintain $2 \cdot f$ copies of each DFSM, resulting in a total of $2 \cdot n \cdot f$ backup DFSMs [12].

In this paper, we explore an alternate idea for fault tolerance that requires fewer backup machines than replication based approaches. Consider the DFSMs shown in Fig. 1(i) and 1(ii). These machines model mod-3 counters counting 0s and 1s respectively. We denote the number of 0s seen by the counters as n_0 and the number of 1s as n_1. A crash fault in one these machines will result in the loss of it's current state. In case of such a failure, we would like to recover the state of the failed machine.

Another way of looking at replication in DFSMs is by constructing a backup machine that is the *reachable cross product* (formally defined in section 2) of the original machines. As shown in Fig. 1 (iii), each state corresponding to this machine is a tuple, in which the first element corresponds to the state of A, and the second element corresponds to the state of B. We would need one such machine to tolerate a single fault. However, the reachable cross product could have a large number of states and would be equivalent to maintaining one copy each of the original DFSMs in terms of complexity. In the example shown in Fig. 1(i) and 1(ii), we can intuitively see that a machine which computes $\{n_0 + n_1\} \bmod 3$ (or $\{n_0 - n_1\} \bmod 3$) could be used to tolerate a single fault in the system. If machine A that counts $n_0 \bmod 3$ fails, then by using machine B ($n_1 \bmod 3$) and the machine F_1 ($\{n_0 + n_1\} \bmod 3$) we can compute the current state of the failed machine A. Note that, in this case F_1 is much smaller (number of states) than the reachable cross product (Fig. 1(iii)) with respect to the number of states.

In the previous example, it was easy to deduce the backup machine purely by observation. For any general set of DFSMs, it is not straightforward to generate such backup machines (for example, consider machines A and B in Fig. 2). The main objective of this paper is to automate the generation of efficient backup machines like F_1 for any given set of machines and formalize the underlying theory. Some of the questions that need to be answered are:

- Given a set of machines, are there backup machines with fewer states than the reachable cross product?
- What is the minimum number of backup machines required to tolerate f crash faults?
- Can these machines tolerate Byzantine faults? (For example, in Fig. 1, DFSMs A and B along with F_1 and F_2 can tolerate one Byzantine fault). What is the number of Byzantine faults that can be tolerated ?
- Is it possible to compute such backup machines efficiently?



(i) $A(n_0$ mod3 counter)

(ii) $B(n_1$ mod3 counter)

(iii) $\mathcal{R}(\{A, B\})$ (Reachable Cross Product)

(iv) $F_1(\{n_0 + n_1\}$ mod3 counter)

(v) $F_2(\{n_0 - n_1\}$ mod3 counter)

Figure 1: Mod 3 Counters

In this paper, we explore the idea of a fault graph and use that to define the minimum Hamming distance for a set of machines. Based on this, we introduce an approach called (f, m)-*fusion* that addresses the questions posed. Given n different DFSMs, we examine the problem of tolerating f faults using just m additional machines. For systems where two or more DFSMs need to be backed up, our approach always results in lesser number of backup machines than the replication based approaches.

We call the backup machines, *fusions* corresponding to the given set of machines. We assume a system model that has either crash faults or Byzantine faults. Note that, the technique discussed in this paper deals with determining the current *state* of the failed machines and not the entire DFSM (which is usually stored on some form of failure-resistant permanent storage medium).

Our work in [1] introduces the concept of the fusion of finite state machines. The main focus of the paper was to define the theory and algorithms for the special case where the number of backup machines is equal to the number of faults that need to be tolerated among a given a set of machines, i.e. (k, k)-fusion. The previous work outlined an exponential-time algorithm to generate a $(1, 1)$-fusion. The system model in that paper only allowed crash faults. In this paper, we generalize the notion of fusion to (f, m)-fusion. where f faults can be tolerated using m additional machines.

The work presented in [2] introduces the idea of fusible data structures. The authors have shown that commonly used data structures such as arrays, hash tables, stacks and queues can be fused into a single fusible structure, smaller than the combined size of the original structures. Our idea is similar to this approach in the sense that we generate a fused state machine that can

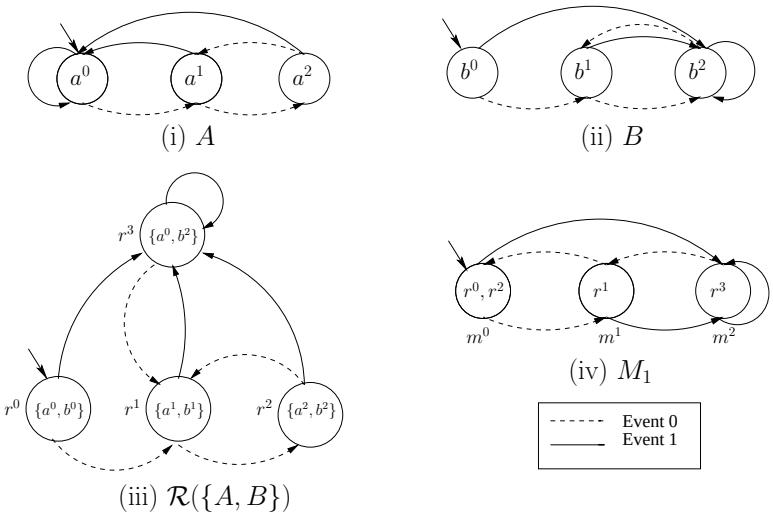


Figure 2: Reachable cross product, Order among DFSMs

enable recovery of any state machine that has crashed. The work presented in this paper effectively presents an algorithm to compute a fusion operation given a set of specific input machines.

Extensive work has been done [6, 5] on the minimization of completely specified DFSMs. In these approaches, the basic idea is to create equivalence classes of the state space of the DFSM and then combine them based on the transition functions. Even though our approach is also focussed on reducing the reachable cross product corresponding to a given set of machines, it is important to note that the machines we generate need not be equivalent to the combined DFSM. In fact, we implicitly assume that the input machines to our algorithm are reduced a priori using these techniques.

In the following sections, we look at the underlying theory behind this approach and also present an efficient algorithm for generating the minimum number of backup machines required to tolerate f faults. Note that, in some cases the smallest fusion could be the reachable cross product machine. However, our experiments suggest that there exist smaller fusions for many of the practical DFSMs in use. This can result in enormous savings in space, especially when a large number of machines need to be backed up. For example, consider a sensor network with 100 sensors, each running a mod-3 counter counting changes to different environmental parameters like temperature, pressure, humidity and so on. To tolerate a crash fault in such a system, replication based approaches would demand 100 new sensors for backup. Fusion, on the other hand, could possibly tolerate a fault by using only one new backup sensor with exactly three states. To summarize:

- We introduce the concept of (f, m)-fusion and explore the theory of such machines.
- Using this theory, we present separate algorithms to generate backup machines and recover from Byzantine or crash failures.
- We have implemented the algorithms in Java and have tested it with real world DFSMs.

2 Model and Notation

We now discuss the system model, followed by the notation used in the remainder of this paper. Our system consists of a set of independent servers, which include the original machines and the backups, each running a distinct DFSM with no shared state and no communication during a fault-free run. The events on the DFSMs originate from the environment and are applied to all the machines. For example, one or more client machines (the environment) could send ordered requests (events) which are applied on all the servers. If a received event does not belong to the event set of a server DFSM, then the event is ignored. Note that, synchronous operation is not essential to the underlying theory during normal conditions. The only requirement is that when there are failures, all DFSMs have acted on the same sequence of inputs before the state of the failed DFSM is recovered.

The machines in the system may undergo crash faults or Byzantine faults. We assume that the faults only impact the current state of the faulty machines, but the underlying DFSMs remains intact. When a fault occurs, no requests are sent by the clients till the execution state of the faulty machines have been recovered and the machines resume normal operation. In the case of crash faults, there is a loss of the execution state of the machines. In the case of Byzantine faults, the machines can enter an incorrect state on the application of an event.

Definition 1 *(DFSM) A DFSM, denoted by A, is a quadruple,* $(X_a, \Sigma_a, \alpha_a, a^0)$*, where,*

- X_a *is the finite set of states corresponding to A.*
- Σ_a *is the finite set of events corresponding to A.*
- $\alpha_a : X_a \times \Sigma_a \rightarrow X_a$*, is the transition function corresponding to A. If the current state of A is s, and an event* $\sigma \in \Sigma_a$ *is applied on it, the next state can be uniquely determined as* $\alpha_a(s, \sigma)$*.*
- a^0 *is the initial state corresponding to A.*

The *size* of a machine A, is the number of states in X_a, and is denoted by $|A|$.

A state, $s \in X_a$, is *reachable* iff there exists a sequence of events, which, when applied on the initial state a^0, takes the machine to state s. This is denoted by $s = \alpha^k(a^0)$, where α^k denotes a sequence of k operations, $\alpha^1, \ldots, \alpha^k$ applied to the initial state a^0. Our model assumes that all the states corresponding to the machines are reachable.

Consider any two machines, A (X_a, Σ_a, α_a, a^0) and B (X_b, Σ_b, α_b, b^0). Now construct another machine which consists of all the states in the product set of X_a and X_b with the transition function $\alpha'(\{a, b\}, \sigma) = \{\alpha_a(a, \sigma), \alpha_b(b, \sigma)\}$ for all $\{a, b\} \in X_a \times X_b$ and $\sigma \in \Sigma_a \cup \Sigma_b$. This machine ($X_a \times X_b$, $\Sigma_a \cup \Sigma_b$, α', $\{a^0, b^0\}$) may have states that are not reachable from the initial state $\{a^0, b^0\}$. If all such unreachable states are pruned, we get the *reachable cross product* of A and B, denoted $\mathcal{R}(\{A, B\})$. In the example shown in Fig. 2, $\mathcal{R}(\{A, B\})$ is the reachable cross product of A and B.

In the following subsection, we define a closed partition lattice corresponding to a given set of machines.

2.1 Closed Partition Lattice

A *partition* P, on the state set X_a of a DFSM, A $(X_a, \Sigma_a, \alpha_a, a^0)$ is the set $\{B_1, \ldots, B_k\}$, of disjoint subsets of the state set X_a, such that $\bigcup_{i=1}^{k} B_i = X_a$ and $B_i \cap B_j = \phi$ for $i \neq j$ [9]. An element B_i of a partition is called a *block*.

A partition, P, is said to be closed if each event, $\sigma \in \Sigma$, maps a block of P into another block. A closed partition P, corresponds to a distinct machine. Each state s of such a machine corresponds to a set of states in machine A. For example in Fig. 2, M_1 corresponds to a closed partition of the set of states of $\mathcal{R}(\{A, B\})$. M_1 has 3 states, $\{r^0, r^2\}$, $\{r^1\}$ and $\{r^3\}$, which we also refer to as the blocks of M_1. The closed partitions described here are also referred to as substitution property partitions or SP partitions in other literature [4].

A partition P_1 is less than or equal to another partition P_2 ($P_1 \leq P_2$) if each block of P_2 is contained in a block of P_1. If DFSMs X_1 and X_2 correspond to partitions P_1 and P_2 respectively, then machine X_1 is less than or equal to machine X_2 ($X_1 \leq X_2$) iff $P_1 \leq P_2$. In Fig 2, each block of $\mathcal{R}(\{A, B\})$ is contained in a block of M_1 and hence, $M_1 \leq \mathcal{R}(\{A, B\})$. Two machines X_1 and X_2 are said to be incomparable iff $X_1 \not\leq X_2$ and $X_2 \not\leq X_1$.

Consider two machines X_1 and X_2 such that $X_1 \leq X_2$. It is clear that given the state of X_2 we can determine the state of X_1. For example, in Fig 2, $M_1 \leq \mathcal{R}(\{A, B\})$. When $\mathcal{R}(\{A, B\})$ is in state r^1, M_1 is in state m^1.

Given a set of n machines, $\mathcal{A} = \{A_1, \ldots, A_n\}$, their reachable cross product is denoted by $\mathcal{R}(\mathcal{A})$. Every machine in $\mathcal{A}$ is less than or equal to $\mathcal{R}(\mathcal{A})$. Hence, given the state of $\mathcal{R}(\mathcal{A})$, we can determine the state of any of the machines in $\mathcal{A}$.

It can be seen that the set of all closed partitions corresponding to a machine, form a lattice under the $\leq$ relation[4]. In this paper, we consider the lattice of all closed partitions corresponding to $\mathcal{R}(\mathcal{A})$. Fig. 3 shows the closed partition lattice corresponding to $\mathcal{R}(\{A, B\})$ (denoted $\top$), shown in Fig. 2(iii). An arrow from one machine to another indicates that the former is less than the latter. Both A (Fig. 2(i)) and B (Fig. 2(ii)) are contained in the lattice. The bottom element (denoted $\perp$) is always a single block partition containing all the states of $\top$. Henceforth, we use $\top(X_\top, \Sigma, \alpha_\top, t^0)$ or *top* to denote the reachable cross product of the given set of machines in our system. It is important to note that we never have to create the entire closed partition lattice for a given set of machines during implementation.

We now define the lower cover of a machine, a concept used later in section 5.

Definition 2 *(Lower Cover) The lower cover of any machine $A(X_a,\ \Sigma_a,\ \alpha_a,\ a^0)$, is the set of machines corresponding to the maximal partitions of X_a that are less than A.*

For example, in Fig. 3, the lower cover of machine A consist of machines M_3 and M_4.

The lower cover of machine $A(X_a,\ \Sigma_a,\ \alpha_a,\ a^0)$ consists of all the incomparable machines obtained by combining two states of X_a into a block and computing the new largest closed partition which is less than this new (possibly not closed) partition.

In our closed partition lattice, the lower cover of $\top$ is called the *basis* of the lattice. In the lattice shown in Fig. 3, the machines A, B, M_1 and M_2 constitute the basis.

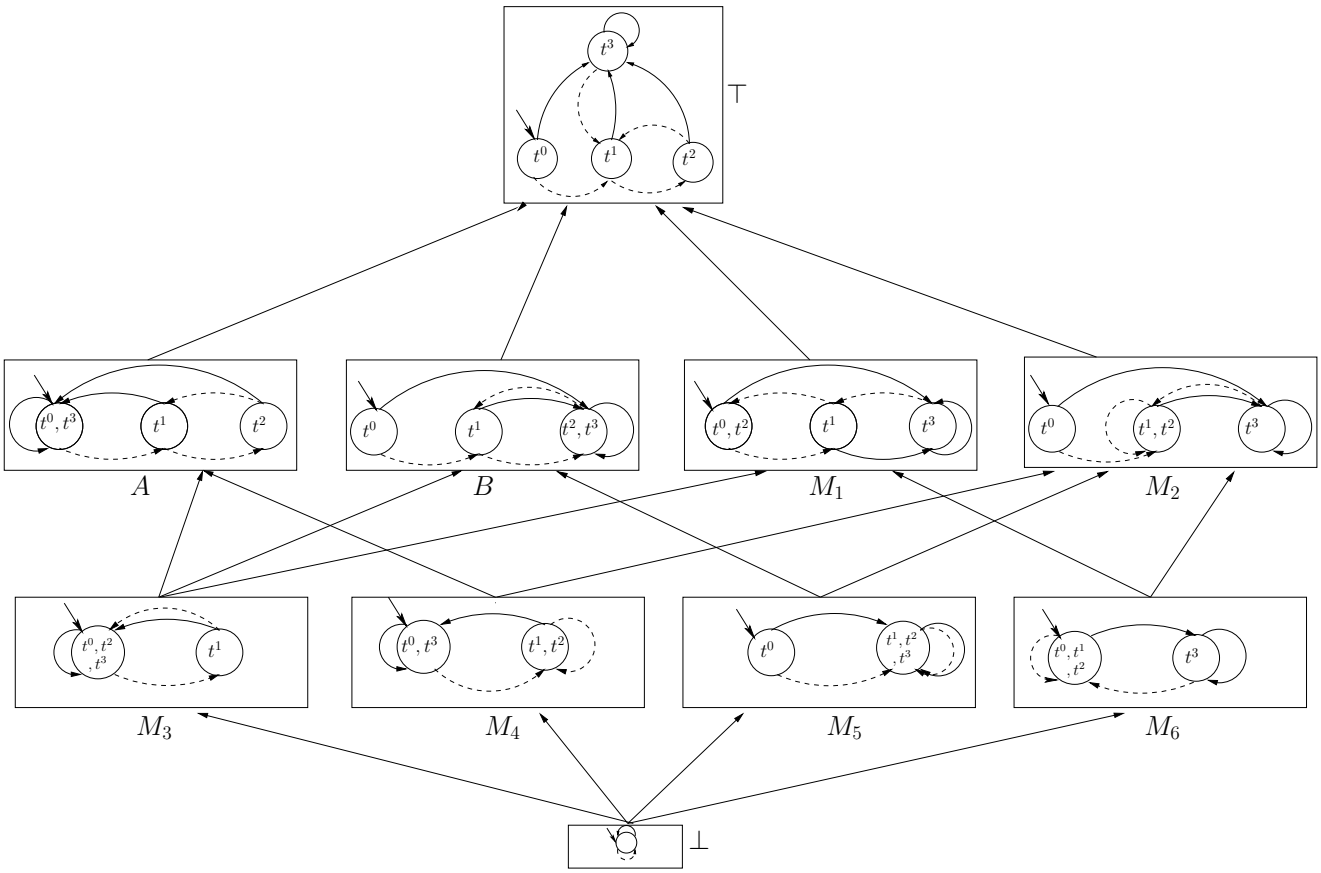


Figure 3: Closed Partition Lattice For Fig. 2

3 Fault Tolerance of Machines

In this section, we introduce concepts that enable us to answer fundamental questions about the fault tolerance in a given set of machines.

We begin with the idea of a *fault graph* of a set of machines $\mathcal{M}$, for a machine T, where all machines in $\mathcal{M}$ are less than or equal to T. This is a weighted graph and is denoted by $G(T, \mathcal{M})$.

The fault graph is an indicator of the capability of the set of machines in $\mathcal{M}$ to correctly identify the current state of T. As described in the previous section, since all the machines in $\mathcal{M}$ are less than or equal to T, the set of states of any machine in $\mathcal{M}$ corresponds to a closed partition of the set of states of T. Considering the lattice shown in Fig. 3, we construct the fault graph $G(\top, \{A\})$. The machine A has three states, $\{t^0, t^3\}$, $\{t^1\}$ and $\{t^2\}$. Given just the current state of machine A, it is possible to determine if $\top$ is in state t^1(exact) or t^2 (exact) or one of t^0 and t^3(ambiguity). Hence, A distinguishes between all pairs of states of $\top$ except (t^0, t^3). This information is captured by the fault graph.

Every state of T corresponds to a node of the fault graph $G(T, \mathcal{M})$ and the graph is totally connected. The weight of the edge between nodes corresponding to states t^i and t^j of the fault graph is the number of machines in $\mathcal{M}$ that have states t^i and t^j in distinct blocks. Hence, in the

fault graph $G(\top, \{A\})$, shown in Fig. 4(i), the edge (t^0, t^3) has weight 0 and all other edges have weight 1.

Definition 3 *(Fault Graph) Given a set of machines $\mathcal{M}$ and a machine $T = (X_T, \Sigma, \alpha, t^0)$ such that $\forall M \in \mathcal{M} : M \leq T$, the fault graph $G(T, \mathcal{M})$ is a weighted graph with $|X_T|$ nodes such that*

- *Every node of the graph corresponds to a state in X_T*
- *The graph is completely connected*
- *The weight of the edge between two nodes (corresponding to any two states t^i and t^j in X_T) of the fault graph is the number of machines in $\mathcal{M}$ that have states t^i and t^j in distinct blocks*

Given the states of $|\mathcal{M}| - x$ machines in $|\mathcal{M}|$, it is always possible to determine if T is in state t^i or t^j iff the weight of the edge (t^i, t^j) is greater than x. Consider the graph shown in Fig. 4(ii). Given the state of just any one machine in $\{A, B\}$, we can determine if $\top$ is in state t^0 or t^1, since the weight of that edge is greater than 1, but cannot do the same for the edge (t^0, t^3), since the weight of the edge is only 1.

To understand the idea of fault graphs and their significance to tolerating faults in state machines, we draw an analogy between fault tolerance in DFSMs and fault tolerance in a block of bits using erasure codes [13]. Consider the fault graph $G(T, \mathcal{M})$, where $T = \mathcal{R}(\mathcal{M})$ is the reachable cross product of $\mathcal{M}$. The state of all the machines in $\mathcal{M}$ can be represented by exactly one of the states in T i.e., the machine T lists all the valid states of the system $\mathcal{M}$. The weights of the edges in the graph $G(T, \mathcal{M})$ are in indicator of the how easy it is to distinguish between those states.

The set of states of $T = \mathcal{R}(\mathcal{M})$ is equivalent to the set of all valid code words in an erasure code. The weight of the edge separating the states is the *Hamming distance* [3] between the valid code words. In this case, instead of conventional bit errors, the states of the machines in $\mathcal{M}$ are either incorrect (Byzantine) or unavailable (crash). To tolerate bit errors, erasure coding involves adding redundant bits to the data bits. Similarly, in this case we add more machines to the system. These new machines are less than T, the cross product of the original machines. When faults occur, the cross product the states of machines in the system may be an incorrect state of the T machine. However, if the system is fault-tolerant, this recovered cross product state will be closest in distance to the correct state in T.

There is one important difference between erasure codes involving bits and the state machines problem. In erasure codes, the value of the redundant bits depend on the data bits. In case of state machines, it is not feasible to transmit the state of all machines after each event has happened to calculate the state of the backup machines. Hence, the backup machines have to be designed to act on the same inputs as the original machines and independently transition to suitable states. This implies that instead of designing a erasure code once and reusing it for on different inputs, for state machines we need to *re-design* the set of backup machines for every distinct set of original machines.

The distance between any two states (or nodes in the fault graph) is the weight of the edge between those nodes in the fault graph.

Definition 4 *(distance) Given a set of machines $\mathcal{M}$ and their reachable cross product T $(X_T, \Sigma, \alpha, t^0)$, the distance between any two states $t_i, t_j \in X_T$, denoted $d(t_i, t_j)$ is the weight of the edge between the nodes corresponding to t_i and t_j in the fault graph $G(T, \mathcal{M})$.*

Given a fault graph, $G(T, \mathcal{M})$, the smallest distance between the nodes in the fault graph gives us an idea of the fault tolerance capability of the set $\mathcal{M}$. Consider the graph, $G(\top, \{A, B, M_1, M_2\})$, shown in Fig. 4 (iii). Since the smallest distance in the graph is 3, we can remove any two machines from $\{A, B, M_1, M_2\}$ and still regenerate the current state of $\top$. As seen before, given the state of $\top$, we can determine the state of any machine less than $\top$. Therefore, the set of machines $\{A, B, M_1, M_2\}$ can tolerate two crash faults.

The least distance in $G(T, \mathcal{M})$ is denoted $d_{min}(T, \mathcal{M})$.

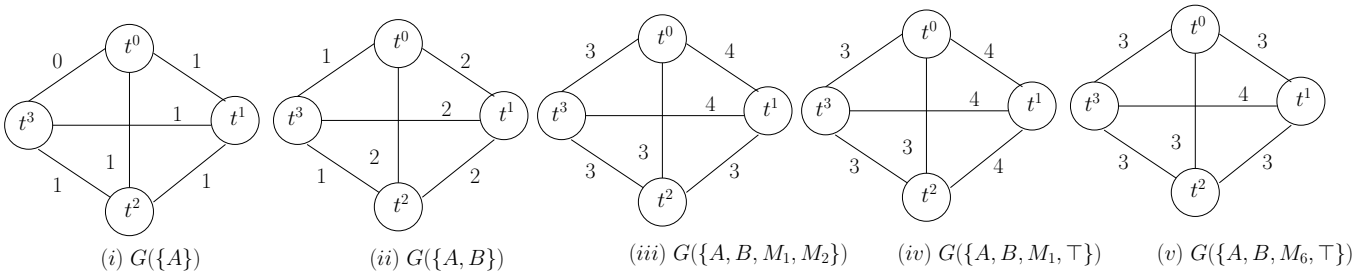


Figure 4: Fault Graphs, $G(\top, \mathcal{M})$, for sets of machines shown in Fig. 3

Theorem 1 *A set of machines $\mathcal{M}$, can tolerate up to f crash faults iff $d_{min}(T, \mathcal{M}) > f$, where T is the reachable cross-product of all machines in $\mathcal{M}$.*

Proof: $\Rightarrow$ Given that $d_{min}(T, \mathcal{M}) > f$, we show that any $\mathcal{M}-f$ machines from $\mathcal{M}$ can accurately determine the current state of T. It is obvious that the current state of any DFSM in $\mathcal{M}$ can be determined if the state of T is known. The distance between any two nodes in fault graph $G(T, \mathcal{M})$ is greater than f since $d_{min}(T, \mathcal{M}) > f$. i.e., $f + 1$ or more machines separate any two states in the fault graph. Hence, for any pair of states t_i, t_j in T, after f crash failures in $\mathcal{M}$, there will always be at least one machine remaining that can distinguish between t_i and t_j. This implies that it is possible to accurately determine the current state of T by using any $\mathcal{M} - f$ machines from $\mathcal{M}$.

$\Leftarrow$ We now show that the system cannot tolerate f crash faults when $d_{min}(T, \mathcal{M}) \leq f$. $d_{min}(T, \mathcal{M}) \leq f$ implies that there exists states t_i and t_j in $G(T, \mathcal{M})$ separated by distance k, where $k \leq f$. Hence there exist exactly k machines (say the set $\mathcal{M}' \subset$ of $\mathcal{M}$ in $\mathcal{M}$) that can distinguish between states (t_i, t_j) in T. Assume that all these k machines crash (since $k \leq f$) when T is in either t_i or t_j. Using the states of the remaining machines in $\mathcal{M}$, it is not possible to determine whether T was in state t_i or t_j. Therefore, it is not possible to exactly regenerate the state of any machine in $\mathcal{M}$ using the remaining machines.

□

Byzantine faults may include machines which lie about their state. Similar to erasure coding theory, the number of Byzantine faults that can be tolerated by the system of DFSMs is $d_{min}/2$. Consider the machines $\{A, B, M_1, M_2\}$ shown in Fig. 3. As shown before, these machines can tolerate two crash faults. Let us consider the case where $\top$ is in state t^3 and two of the machines, say B and M_1, lie about their state. Let the states of the machines A, B, M_1 and M_2 be $\{t^0, t^3\}$, $\{t^0\}$, $\{t^0, t^2\}$ and $\{t^3\}$ respectively. Since, we do not know which machines are lying, in this case

we cannot determine the state of $\top$ correctly. If we pick the state which appears the most number of times among these sets, we will determine the state of $\top$ as t^0, which we know is incorrect. Hence, these set of machines cannot tolerate two Byzantine faults. Assuming that only one of the machines, say B, lies about it's state, let the states of the machines A, B, M_1 and M_2 are $\{t^0, t^3\}$, $\{t^0\}$, $\{t^3\}$ and $\{t^3\}$ respectively. Here, we can determine correctly, that the state of $\top$ is t^3, since the majority of machines distinguish between all pairs of states. Hence, since $d_{min}(\top, \{A, B, M_1, M_2\})$ is three, these set of machines can tolerate only one Byzantine fault.

Theorem 2 *A set of machines $\mathcal{M}$, can tolerate up to f Byzantine faults iff $d_{min}(T, \mathcal{M}) > 2f$, where T is the reachable cross-product of all machines in $\mathcal{M}$.*

Proof: $\Rightarrow$ Given that $d_{min}(T, \mathcal{M}) > 2f$, we show that any $\mathcal{M} - f$ correct machines from $\mathcal{M}$ can accurately determine the current state of T in spite of $f$incorrect inputs. It is obvious that the current state of any DFSM in $\mathcal{M}$ can be determined if the state of T is known. The distance between any two nodes in fault graph $G(T, \mathcal{M})$ is greater than $2f$ since $d_{min}(T, \mathcal{M}) > 2f$. i.e., $2f + 1$ or more machines separate any two states in the fault graph. Hence, for any pair of states t_i, t_j in T, after f Byzantine failures in $\mathcal{M}$, there will always be at least $f + 1$ correct machine remaining that can distinguish between t_i and t_j. This implies that it is possible to accurately determine the current state of T by simply taking a majority vote.

$\Leftarrow$ We now show that the system cannot tolerate f crash faults when $d_{min}(T, \mathcal{M}) \leq 2ff$. $d_{min}(T, \mathcal{M}) \leq 2f$ implies that there exists states t_i and t_j in $G(T, \mathcal{M})$ separated by distance k, where $k \leq 2f$. If all f machines from this k machines lie about their correct state, we have only $k - f$ correct machines remaining. Hence it is not possible to discount the current states of the f lying machines (since $k - f \leq f$). Therefore, it is not possible to exactly regenerate the state of any machine in $\mathcal{M}$ using the remaining machines.

□

Henceforth, we only consider machines less than or equal to the top element of the closed partition lattice ($\top$) corresponding to the input set of machines $\mathcal{A}$. So, for notational convenience, we use $G(\mathcal{M})$ instead of $G(\top, \mathcal{M})$ and $d_{min}(\mathcal{M})$ instead of $d_{min}(\top, \mathcal{M})$. From theorems 1 and 2, it is clear that we can determine the inherent fault tolerance in a given set of machines $\mathcal{A}$, simply by finding $d_{min}(\mathcal{A})$.

Observation 1 *Given a set of n machines $\mathcal{A}$, the system can tolerate up to $d_{min}(\mathcal{A}) - 1$ crash faults and $(d_{min}(\mathcal{A}) - 1)/2$ Byzantine faults.*

4 Theory of Fusion Machines

To tolerate faults in a given set of machines, we need to add backup machines so that the fault tolerance of the system (original set of machines along with the backups) increases to the desired value. To simplify the discussion, in the remainder of this paper, unless specified otherwise, we mean crash faults when we simply say faults. As seen in theorem 2, the discussion for crash faults also applies to Byzantine faults (where $f/2$ Byzantine faults can be tolerated instead of f crash faults).

Given a set of n machines $\mathcal{A}$, we add m backup machines $\mathcal{F}$, each less than or equal to the top, such that the set of machines in $\mathcal{A} \cup \mathcal{F}$ can tolerate f faults. We call the set of m machines in $\mathcal{F}$, an (f, m)-fusion of $\mathcal{A}$. From theorem 1, we know that, $d_{min}(\mathcal{A} \cup \mathcal{F}) > f$.

Definition 5 *(Fusion) Given a set of n machines $\mathcal{A}$, we call the set of m machines $\mathcal{F}$, an (f, m)-fusion of $\mathcal{A}$, if $d_{min}(\mathcal{A} \cup \mathcal{F}) > f$.*

Any machine belonging to m is referred to as a *fusion* machine or just a *fusion*. Note that, the top is also a fusion. Consider the set of machines, $\mathcal{A} = \{A, B\}$, shown in Fig. 3. From Fig. 4(ii), $d_{min}(\{A, B\}) = 1$. Hence the set of machines, $\{A, B\}$, cannot tolerate even a single fault.

Let us assume that we want to generate a set of machines $\mathcal{F}$, such that, $\mathcal{A} \cup \mathcal{F}$ can tolerate 2 faults. It can been seen from Fig. 4(iii) that $d_{min}(\{A, B, M_1, M_2\}) = 3$, and hence the set of machines $\{A, B, M_1, M_2\}$ can tolerate up to 2 faults. In this case, the set $\{M_1, M_2\}$ forms a $(2, 2)$-fusion of $\{A, B\}$.

Based on the values of f and m, we discuss three cases of (f, m)-fusion:

- $f = m$: In this case, the number of fusion machines equals the number of faults. The set of machines in $\{M_1, M_2\}$, shown in Fig. 3, form a $(2, 2)$-fusion corresponding to $\{A, B\}$.
- $f < m$: The traditional approach of replication is the simplest example for this case. To tolerate two faults in any two machines $\{A, B\}$, replication will require two additional copies each of A and B. Hence, $\{A, A, B, B\}$ is a $(2, 4)$-fusion of $\{A, B\}$.
- $f > m$: From observation 1, if a system is inherently fault tolerant, then no additional machines may be needed to tolerate faults. In the example shown in Fig. 3, let us assume that the original set of machines are $\{A, B, M_1\}$. Since, $d_{min}(\{A, B, M_1\}) = 2$, these machines can tolerate one fault without any additional machine.

As we have seen before, $d_{min}(\{A, B, M_1, M_2\}) > 2$. Any machine in the set $\{A, B, M_1, M_2\}$ can at most contribute 1 to the weight of any edge in the graph $G(\{A, B, M_1, M_2\})$. Hence, even if we remove one of the machines, say M_2, from this set, $d_{min}(\{A, B, M_1\})$ will still be greater than 1. This implies that $\{M_1\}$ is a $(1, 1)$-fusion of $\{A, B\}$. Similarly, $\{M_2\}$ is also a $(1, 1)$-fusion of $\{A, B\}$. This property is generalized in the following theorem.

Theorem 3 *(Subset of a Fusion) Given a set of n machines $\mathcal{A}$, and an (f, m)-fusion $\mathcal{F}$, corresponding to it, any subset $\mathcal{F}' \subseteq \mathcal{F}$ such that $|\mathcal{F}'| = m - t$ is a $(f - t, m - t)$-fusion when $t \leq min(f, m)$.*

Proof: Since, $\mathcal{F}$ is an (f, m)-fusion of $\mathcal{A}$, according to the definition of (f, m)-fusion, $d_{min}(\mathcal{A} \cup \mathcal{F}) > f$.

Any machine, $F \in \mathcal{F}$, can at most contribute a value of 1 to the weight of any edge of the graph, $G(\mathcal{A} \cup \mathcal{F})$. Similarly, t machines in the set $\mathcal{F}$ can contribute a value of at most t to the weight of any edge of the graph, $G(\mathcal{A} \cup \mathcal{F})$. Therefore, even if we remove t machines from the set of machines in $\mathcal{F}$, $d_{min}(\mathcal{A} \cup \mathcal{F}) > f - t$.

Hence, for any subset $\mathcal{F}' \subseteq \mathcal{F}$, of size $m - t$, $d_{min}(\mathcal{A} \cup \mathcal{F}') > f - t$. This implies that $\mathcal{F}'$ is an $(f - t, m - t)$-fusion of $\mathcal{A}$.

□

It is important to note that the converse of this theorem is not true. For example, consider the machines M_1 and M_6 shown in Fig. 3. Even though both $\{M_1\}$ and $\{M_6\}$ are $(1,1)$-fusions of $\{A, B\}$, since $d_{min}(\{A, B, M_1, M_6\}) = 2$, $\{M_1, M_6\}$ is not a $(2,2)$-fusion of $\{A, B\}$.

We now consider the existence of an (f, m)-fusion for a given set of machines $\mathcal{A}$. The top machine distinguishes between all the states of $X_\top$. So the basic intuition is that, if the union of m top machines along with $\mathcal{A}$ cannot tolerate f faults, then there cannot exist an (f, m)-fusion for $\mathcal{A}$. Let us consider the existence of a $(2, 1)$-fusion for the set of machines $\{A, B\}$, shown in Fig. 3. From Fig. 4(ii), $d_{min}(\{A, B\}) = 1$. We need exactly one machine F, such that, $d_{min}(\{A, B, F\}) > 2$. Even if F was the top machine, $d_{min}(\{A, B, \top\}) = 2$. Hence, there cannot exist a $(2, 1)$-fusion for $\{A, B\}$. We formalize this in the following theorem.

Theorem 4 *(Existence of an (f, m)-fusion) Given a set of n machines $\mathcal{A}$, there exists an (f, m)-fusion of $\mathcal{A}$, iff, $m + d_{min}(\mathcal{A}) > f$.*

Proof: $\Rightarrow$

Assume that there exists an (f, m)-fusion $\mathcal{F}$ for the given set of machines $\mathcal{A}$. We will show that $m + d_{min}(\mathcal{A}) > f$.

Since, $\mathcal{F}$ is an (f, m)-fusion fusion of $\mathcal{A}$, $d_{min}(\mathcal{A} \cup \mathcal{F}) > f$. The m machines in $\mathcal{F}$, can at most contribute a value of m to the weight of each edge in $G(\mathcal{A} \cup \mathcal{F})$. Hence, $m + d_{min}(\mathcal{A})$ has to be greater than f.

$\Leftarrow$

Assume that $m + d_{min}(\mathcal{A}) > f$. We will show that there exists an (f, m)-fusion for the set of machines $\mathcal{A}$.

Consider a set of m machines $\mathcal{F}$, containing m replicas of the top. These m top machines, will contribute exactly m to the weight of each edge in $G(\mathcal{A} \cup \mathcal{F})$. Since, $m + d_{min}(\mathcal{A}) > f$, $d_{min}(\mathcal{A} \cup \mathcal{F}) > f$. Hence, $\mathcal{F}$ is an (f, m)-fusion of $\mathcal{A}$.

□

From this theorem, given a set of n machines $\mathcal{A}$ and an (f, m)-fusion $\mathcal{F}$, corresponding to it, $|\mathcal{F}| \geq f - d_{min}(\mathcal{A})$. Given a set of machines, we now define an order among (f, m)-fusions corresponding to them.

Definition 6 *(Order among (f, m)-fusions) Given a set of n machines $\mathcal{A}$, an (f, m)-fusion $\mathcal{F} = \{F_1, ..F_m\}$, is less than another (f, m)-fusion $\mathcal{G}$ ($\mathcal{F} < \mathcal{G}$) iff the machines in $\mathcal{G}$ can be ordered as $\{G_1, G_2, ..G_m\}$ such that $\forall 1 \leq i \leq m : (F_i \leq G_i) \wedge (\exists j : F_j < G_j)$.*

An (f, m)-fusion $\mathcal{F}$ is *minimal*, if there exists no (f, m)-fusion $\mathcal{F}'$, such that, $\mathcal{F}' < \mathcal{F}$. From Fig. 4(iv), $d_{min}(\{A, B, M_1, \top\}) = 3$, and hence, $\mathcal{F}' = \{M_1, \top\}$ is a $(2, 2)$-fusion of $\{A, B\}$. We have seen that $\mathcal{F} = \{M_1, M_2\}$, is a $(2, 2)$-fusion of $\{A, B\}$. Since $\mathcal{F} < \mathcal{F}'$, $\mathcal{F}'$ is not a minimal $(2, 2)$-fusion. In the lattice shown in Fig. 3, $\{M_3, M_4, M_5, M_6\}$ are a set of minimal machines. It can be seen that $d_{min}(\{A, B, M_3, M_4, M_5, M_6\}) > 2$ and $\{M_3, M_4, M_5, M_6\}$ is a minimal $(2, 4)$-fusion of $\{A, B\}$.

Using the analogy of Hamming distances presented in section 3, it is easy to see that the idea of fusions and crash fault tolerance can be extended to Byzantine faults. An (f, m)-fusion can tolerate f crash faults and $f/2$ Byzantine faults.

5 Algorithms

In this section, we present algorithms to generate backup machines, detect faults and recover from them. We first briefly discuss the notation used for representing machines in the algorithms. We are given a set $\mathcal{A}$ of n machines. The following algorithms assume that all state machines are expressed with respect to the reachable cross product of $\mathcal{A}$, denoted $\top$.

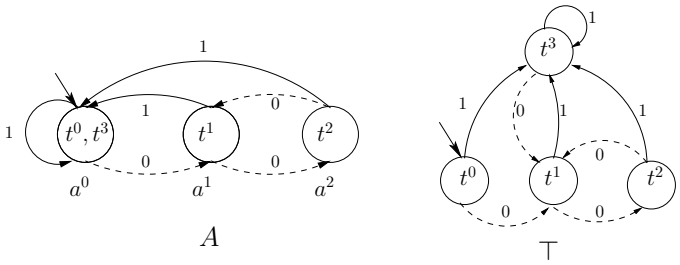


Figure 5: Set Representation of States

All the states in machines less than or equal to $\top$ can be represented as a set consisting of states belonging to $\top$. For example in Fig. 3, the states in all the machines are expressed in this notation.

Algorithm 1 `Generate set representation`

Input: $T = (X_t, \Sigma, \alpha_t, t^0), A = (X_a, \Sigma_a, \alpha_a, a^0) : A \leq T$

Output: $S = \{s_0, \ldots, s_{|X_a|}\}$: the set corresponding to states in X_a where s_i corresponds to the state a^i in X_a

1: **for all** $\sigma \in \Sigma$ **do**
2: $X \leftarrow X_t$
3: $i, j \leftarrow 0$
4: **while** $t_j \in X$ **do**
5: $s_i \leftarrow s_i \bigcup \{t_j\}$
6: $X \leftarrow X/\{t_j\}$ // remove t_j from X
7: $a_i \leftarrow \alpha_a(\sigma, a_i)$, $t_j \leftarrow \alpha_t(\sigma, t_j)$ // determine next transitions to update i, j
8: **return** S

Given machines T and A, algorithm 1 outlines the procedure to map the elements of the bigger machine to the smaller machine and hence, generate the set representation for states in A. Every state in machine T is a set containing exactly one element.

For example, in Fig. 5, consider the machines $\top$ and A. Since $A \leq \top$, the states of A can be represented as a set of states of $\top$. Firstly, the initial state of $\top$ is mapped to the initial state of A. On the application of event 0 to the initial states of both machines, $\top$ transitions from $t^0 \rightarrow t^1$ and $a^0 \rightarrow a^1$. Hence t^1 is mapped to a^1. Similarly, on the application of event 1 to t^0 and a^0 respectively, $\top$ transitions from $t^0 \rightarrow t^3$ and $a^0 \rightarrow a^0$. Hence t^3 is mapped to a^0. Continuing this procedure for all the states and events corresponding to the machines, we get the set representation for the states in A. States a^0, a^1 and a^2 can be represented by the sets $\{t^0, t^3\}$, $\{t^1\}$ and $\{t_2\}$ respectively.

5.1 Generation of Backup Machines

In this section, we present an algorithm to generate the minimum set of machines required to tolerate f crash faults among a given set of n machines, with a time complexity polynomial in the size of the top machine. From theorem 2, it is clear that this set of machines can tolerate $f/2$ Byzantine faults.

Given a set of n machines $\mathcal{A}$, algorithm 2 generates the smallest set of machines $\mathcal{F}$, such that, the set of machines in $\mathcal{A} \cup \mathcal{F}$ can tolerate f faults. The outer while loop terminates when the required set of machines is generated, i.e., when the $d_{min}(\mathcal{A} \cup \mathcal{F})$ is equal to (or exceeds) f.

The $\top$ machine is always a valid fusion. Hence, we start with the $\top$, which clearly increases d_{min} by 1. We then try to find such a machine in the lower cover of the $\top$, and continue traversing down the lattice, until we encounter the bottom machine or till the lower cover does not contain such a machine. In the inner loop, the algorithm successively iterates to find out other DFSMs that are less than the previous guess. The inner loop terminates when there is no machine in the lower cover whose addition results in the required fault tolerance i.e., the loop continues as long as it finds at least one machine in the lower cover which increases the minimum weight of the system. (line 6 of the algorithm).

Consider the example shown in Fig. 3, with $\mathcal{A} = \{A, B\}$, and $f = 2$. We need to find a set of machines $\mathcal{F}$ such that $d_{min}(\mathcal{A} \cup \mathcal{F}) > 2$. Since $\mathcal{F}$ is empty to begin with, $G(\mathcal{A} \cup \mathcal{F}) = G(\{A, B\})$, shown in Fig. 4(ii). The addition of machine M_1, belonging to the lower cover of $\top$, increases d_{min} of the system by 1. (i.e. $d_{min}(\mathcal{A}) < d_{min}(\mathcal{A} \bigcup \{M_1\})$). Similarly, machine M_6 in the lower cover of M_1 satisfies this property, while machine M_3 does not. Hence, M_6 is chosen by the algorithm for its next iteration. Since no machine less than M_6 increases d_{min}, M_6 is added to the fusion set.

Algorithm 2 `Generate Fusion`

Input: $\mathcal{A}$: given set of machines, f : number of crash faults to be tolerated
Output: $\mathcal{F}$: set of fusion machines

```
1:  F ← φ ;
2:  while d_min(A ∪ F) ≤ f do
3:      M ← ⊤
4:      while M ≠ ⊥ do
5:          C ← lower_cover(M)
6:          if ∃F ∈ C : d_min(F ⋃ A ⋃ F) > d_min(A ⋃ F) then
7:              M ← F
8:          else
9:              break
        F ← {M} ⋃ F
10: return F
```

Given a set of n machines $\mathcal{A}$, algorithm 2 returns the smallest set of machines $\mathcal{F}$, such that $\mathcal{F}$ is a minimal $(f, |\mathcal{F}|)$-fusion of $\mathcal{A}$. There maybe other solutions which have more machines, where each of the machines is less than the machines returned by this algorithm.

We now proceed to prove the correctness of the algorithm 2. If a machine M has the states t^i

and t^j in distinct blocks, we say that M *covers* the edge (t^i, t^j). Also the least distance in the fault graph $G(T, \mathcal{M})$ is denoted $d_{min}(T, \mathcal{M})$ and the edges with this value are called the *weakest* edges of $G(T, \mathcal{M})$. It can be seen from the algorithm that every machine added, covers a set of edges. This set of edges (the input to algorithm 2) is called the edge set of that machine. The edge set of F_j is denoted by E_j.

Lemma 1 *Given a set of n machines $\mathcal{A}$, and the set $\mathcal{F}$ returned by algorithm 2, let $F_i \in \mathcal{F}$ be the machine returned in the i^{th} iteration. Then, $\forall F_i, F_j \in \mathcal{F} : i < j \Rightarrow E_i \subseteq E_j$.*

Proof: If $\mathcal{F}' \subseteq \mathcal{F}$ is the current fusion set during the execution of algorithm 2, then the edge set for the next iteration consists of the minimal edges of the fault graph $G(\mathcal{A} \bigcup \mathcal{F}')$. Every time a machine is added to $\mathcal{F}'$, the weights of the edges in $G(\mathcal{A} \bigcup \mathcal{F}')$ can increase by at most one and the weight of every minimal edge is incremented by exactly one. Hence, after every iteration the edge set for the next iteration can not decrease in size. This implies $\forall F_i, F_j \in \mathcal{F} : i < j \Rightarrow E_i \subseteq E_j$. □

Theorem 5 *Given a set of n machines $\mathcal{A}$, algorithm 2 returns the smallest set of machines $\mathcal{F}$, such that $\mathcal{F}$ is a minimal $(f, |\mathcal{F}|)$-fusion of $\mathcal{A}$.*

Proof:

1. $\mathcal{F}$ is a fusion with the minimum number of elements.

   We show that $\mathcal{F}$ is an $(f, |\mathcal{F}|)$-fusion of $\mathcal{A}$ where, $|\mathcal{F}| = f - d_{min}(\mathcal{A})$. As seen in the previous section, this is the minimum number of machines in any (f, m)-fusion of $\mathcal{A}$.

   The addition of any machine, $F \leq \top$, to the set of machines in $\mathcal{A} \cup \mathcal{F}$, can increase $d_{min}(\mathcal{A} \cup \mathcal{F})$ by at most 1. In each iteration of the loop in algorithm 2, we find a machine covering the weakest edges in $G(\mathcal{A} \cup \mathcal{F})$ and add it to $\mathcal{F}$. Hence, in each iteration of the while loop we increase $d_{min}(\mathcal{A} \cup \mathcal{F})$ exactly by 1 adding exactly one extra machine.

   Initially, since $\mathcal{F} = \phi$, $d_{min}(\mathcal{A} \cup \mathcal{F}) = d_{min}(\mathcal{A})$, and finally, $d_{min}(\mathcal{A} \cup \mathcal{F}) = f$. Therefore, the number of machines added to $\mathcal{F}$ is $f - d_{min}(\mathcal{A})$. Since, $d_{min}(\mathcal{A} \cup \mathcal{F}) > f$, $\mathcal{F}$ is a $(f, |\mathcal{F}|)$-fusion of $\mathcal{A}$.

2. $\mathcal{F}$ is a minimal fusion.

   Lemma 1 implies that if an edge e occurs in the edge set of any machine in $\mathcal{F}$ and there are k machines in $\mathcal{F}$ that cover e, then in any valid $(f, |\mathcal{F}|)$-fusion there are at least k machines that cover edge e.

   Let there be an (f, m)-fusion $\mathcal{G} = \{G_1, ..G_m\}$, such that $\mathcal{G}$ is less than (f, m)-fusion $\mathcal{F}$ ($\mathcal{F} = \{F_1, F_2, ..., F_m\}$). Hence $\forall j : G_j \leq F_j$.

   Let $G_i < F_i$ and let E_i be the set of edges that needed to be covered by F_i. It follows from algorithm 2, that G_i does not cover at least one edge say e in E_i (otherwise algorithm 2 would have returned G_i instead of F_i). If e is covered by k DFSMs in $\mathcal{F}$, then e has to be covered by k machines in G.

We know that there is a pair of machines F_i, G_i such that F_i covers e and G_i does not cover e. For all other pairs F_j, G_j if G_j covers e then F_j covers e (since $G_j \leq F_j$). Hence e can be covered by no more than $k-1$ in $\mathcal{G}$. This implies that $\mathcal{G}$ is not a valid fusion.

□

We now consider the time complexity for algorithm 2. Let us assume that $|\top| = N$. The time complexity to generate the lower cover of any machine R, less than the top, is $O(N^2 \cdot |\Sigma|)$ [9]. While generating a machine in the lower cover, we can determine if adding that machine increases d_{min} without any additional time. Since the number of machines in the lower cover is $O(N^2)$, each iteration of the inner while loop requires $O(N^2 \cdot |\Sigma|) + O(N^2) = O(N^2 \cdot |\Sigma|)$ time. As we traverse down the lattice, we combine at least two blocks of F. Thus the inner while loop in algorithm 2 is executed at most N times. Hence, the time complexity of inner loop of algorithm 2 is $O(N^2 \cdot |\Sigma|) * O(N) = O(N^3 \cdot |\Sigma|)$.

In each iteration of the outer loop, we add a new machine to the fusion set. Hence there can be at most f f iterations of the loop. Since the time complexity for each iteration of the outer loop is $O(N^3 \cdot |\Sigma|) + O(N^2) = O(N^3 \cdot |\Sigma|)$ the time complexity of algorithm is $O(N^3 \cdot |\Sigma| \cdot f)$. Therefore, algorithm 2 generates the set of fusion machines with a time complexity polynomial in the size of the top machine.

5.2 Recovering from Faults

Given a set of n machines $\mathcal{A}$, and a corresponding (f, m)-fusion $\mathcal{F}$, we present an algorithm to recover from f crash faults or $f/2$ Byzantine faults.

As mentioned earlier, we use the set representation for all states of the machines in our system.

Given the current state of all the machines in our system, algorithm 3 returns the correct state of the top machine from which the state of all the individual machines can be determined. The algorithm iterates through the current states of all the machines in the system and picks the state, $t_c \in X_\top$, which appears the most number of times in these states.

Algorithm 3 `Recover`

Input: S : set of current states of the machines in $\mathcal{A} \cup \mathcal{F}$,
`count` : vector of size N initialized to 0
Output: t_c : the correct state of the top machine.

```
1: for all s ∈ S do
2:     for all t_i ∈ s do
3:         ++count[i]
4: return t_c : 1 ≤ c ≤ N and count[c] is the maximal element in count
```

Consider the machines A, B, M_1 and M_2 shown in Fig. 3. As shown before, this system of machines can tolerate two crash faults and one Byzantine fault. Let us consider the case of crash faults, where both machines B and M_1 have crashed and the machines A and M_2 are in states $\{t^0, t^3\}$ and $\{t^3\}$ respectively. Algorithm 3 will return t^3 since the count corresponding to t^3, `count`[3] = 2, which is greater than the count corresponding to t^0 (`count`[0] = 1), t^1 (`count`[1] = 0) and t^2 (`count`[2] = 0).

Now let us assume that machines A, B and M_2 are in states in $\{t^0, t^3\}$, $\{t^0\}$, $\{t^0\}$. Assume that M_1 has a Byzantine failure reflecting an incorrect state $\{t^1, t^2, t^3\}$. Algorithm 3 will return t^0 since the count corresponding to t^0, `count`$[0] = 3$, which is greater than the count corresponding to t^1 (`count`$[1] = 1$), t^2 (`count`$[2] = 1$) and t^3 (`count`$[3] = 2$).

We now proceed to prove the correctness of algorithm 3.

Theorem 6 *Given a set of n machines $\mathcal{A}$ and a corresponding (f, m)-fusion $\mathcal{F}$, the algorithm shown in Fig. 3 can recover the correct state of all the machines in the system in the event of f crash faults or $f/2$ Byzantine faults.*

Proof: We first consider the case of crash faults. Since $\mathcal{F}$ is an (f, m)-fusion of $\mathcal{A}$, the set of machines in $\mathcal{A} \cup \mathcal{F}$ can tolerate f crash faults. Let us denoted the correct state of the top machine as t_c. Given the current state of any $n + m - f$ machines in this set (in the tuple set representation), we can uniquely determine the state of the top machine as t_c. Hence, t_c will be present in each and every state of these $n + m - f$ machines and no other $t_i \in X_\top$ will be present in all of these states. Therefore, `count`$[c]$ will be the maximum and algorithm 3 will return t_c.

We now consider the case of Byzantine faults. Let any $f/2$ machine lie about their state. Since t_c is the correct state of the top machine, it will be present in the current states of all of the non-faulty machines. Hence `count`$[c] \geq n + m - f/2$. If an adversary wants to put the system in an incorrect state, say t_z, then the count associated with t_{nc} has to be greater than the count associated with t_c. Hence, `count`$[z]$ has to be greater than $n + m - f/2$. In a worst case scenario, all the Byzantine machines contribute to the count of t_z. Even then, there must be at least $n+m-f/2-f/2 = n+m-f$ machines which have t_z present in the them. Now if any f machines not in this set undergo a crash fault, the algorithm will not be able to determine the unique state of the top, since `count`$[z]$ will be equal to `count`$[c]$ which is $n + m - f$. This is in contradiction to the definition of the system as f- crash fault tolerant. Hence algorithm 3 can recover the state of the system from $f/2$ Byzantine faults.

□

Let us consider the time complexity of algorithm 3. If the size of the top is N, the number of elements in the current state s of any machine is $O(N)$. The time complexity for the inner loop which updates the `count` vector is $O(N)$. Since there are $n + m$ machines in the system, the outer loop executes $O(n + m)$ times. The time complexity of algorithm 3 is $O((n + m).N)$.

6 Implementation and Results

We have implemented the algorithm specified in section 5 in Java (JDK 6.0) on a machine with an Intel Core Duo processor with 1.83 GHz clock frequency and 1 GB RAM. We tested the algorithms for many practical DFSMs including TCP and the MESI cache coherency protocol along with the examples shown in Fig. 2 (denoted A and B in the results table). In this section we present our results for crash faults.

In the results table, along with the original machines, we have tabulated the number of crash faults tolerated (f), the size of the top ($|\top|$) and the sizes of the backup fusion machines generated by algorithm 2 (|Backup Machines|), the state space required for a replication based solution (|Replication|) and the state space required for our fusion based solution (|Fusion|).

Given a set of n machines, $\mathcal{M} = \{M_1, M_2, \ldots M_n\}$, in order to tolerate f crash faults among them, replication will require f copies of each machine. Hence the state space for replication is calculated as $(\prod_{i=1}^{i=n} |M_i|)^f$. If the set of backup machines generated by algorithm 2 is, $\mathcal{F} = \{F_1, F_2, \ldots F_m\}$, the state space for fusion is simply calculated as $\prod_{i=1}^{i=m} |F_i|$.

Original Machines	f	$\|\top\|$	|Backup Machines|	|Replication|	|Fusion|
MESI, 1-Counter, 0-Counter, Shift Register	2	87	[39 39]	82944	1521
Even Parity , Odd Parity Checker, Toggle Switch, Pattern Generator, MESI	3	64	[32 32 32]	2097152	32768
1-Counter, 0-Counter, Divider, A, B	2	82	[18 28]	59049	504
MESI, TCP, A, B	1	131	[85]	396	85
Pattern Generator, TCP, A, B	2	56	[44 56]	156816	2464

The results indicate that there are many practical examples for which our algorithm yields huge savings in state space compared to replication based approaches. Since the largest running time for the execution of our program was only 13.2 minutes, our algorithm can be used to generate backup machines for larger, more complex machines within a feasible time frame.

7 Conclusion and Future Work

In this paper, we develop the theory for (f, m)-fusion and present a polynomial time algorithm to generate the minimum set of machines required to tolerate both crash and Byzantine faults among a given set of machines. We have also implemented and tested this algorithm for real world DFSM models such as TCP and MESI.

The concept of (f, m)-fusion gives us a wide spectrum of choices for fault-tolerance. Replication is just a special case of (f, m)-fusion. Our approach shows that there are many cases for which we can do better. Hence, if we want to tolerate 5 crash faults among 1000 machines, replication will require 5000 extra machines. Using our algorithm we may achieve this with just 5 extra machines.

In this paper, we only consider machines belonging to the closed partition lattice of $\top$. It is possible that machines outside the lattice may provide more efficient solutions. Also, our algorithm returns the minimum number of backup machines required to tolerate faults in a given set of machines. We may be able to generate smaller machines if the system under consideration permits a larger number of backup machines.

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