EE445M/EE360L.6
Embedded and Real-Time Systems/ Real-Time Operating Systems

Lecture 7:

Digital Signal Processing, Digital Filters, FFT

Digital Filters

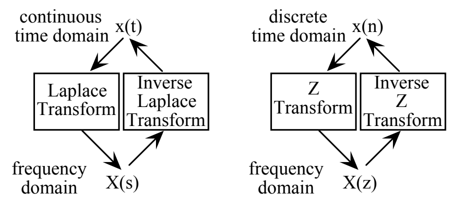
- Digital signal sampled from continuous analog signal $\mathbf{x_c(t)}$
  - $\mathbf{x(n) = x_c(nT)}$ with $-\infty < \mathbf{n} < +\infty$
  - finite precision, finite sampling frequency & frequency range
- Causal digital filter
  - calculates **y(n)** from **y(n-1)**, **y(n-2)**,... and **x(n)**, **x(n-1)**, **x(n-2)**,...
  - not future data (e.g., **y(n+1)**, **x(n+1)** etc.)
- Linear filter is constructed from a linear equation
- Nonlinear filter is constructed from a nonlinear equation
  - E.g. median filter
- Finite impulse response filter (FIR)
  - relates **y(n)** only in terms of **x(n)**, **x(n-1)**, **x(n 2)**,...
    - $y(n) = (x(n) + x(n\text{-}3))/2$
- Infinite impulse response filter (IIR)
  - relates **y(n)** in terms of both **x(n)**, **x(n 1)**,..., and **y(n–1)**, **y(n-2)**,...
    - $y(n) = (113 \bullet x(n) + 113 \bullet x(n\text{-}2)\ -\ 98 \bullet y(n\text{-}2))/128$

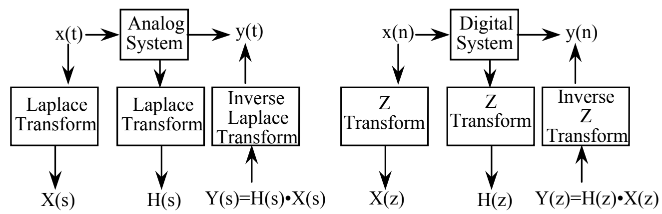
Transforms

- Time vs. frequency domain
  - Z-Transform

$$X(z) = Z[x(n)] \equiv \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

  - Laplace Transform





Gain and Phase Response

- Analog system
  - Gain ≡ **|H(s)|** at $s = j\,2\pi f$, for all frequencies, **f**
  - Phase ≡ **angle(H(s))** at $s = j\,2\pi f$
- Digital system
  - Transform, **H(z) = Y(z)/X(z)** from DC to ½ f_s
    - One can show that: $Z[x(n-m)] = z^{-m} Z[x(n)] = z^{-m} X(z)$
      - E.g., if $X(z) = Z[x(n)]$, $Z[x(n-2)] = z^{-2} \cdot X(z)$
  - Let
    - $z(f) \equiv e^{j2\pi f/fs} = \cos(2\pi f/f_s) + j \sin(2\pi f/f_s)$ for $0 \le f < \frac{1}{2} f_s$
    - $H(f) = H(z(f)) \equiv a + bj$, where **a** and **b** are real numbers
  - Gain ≡ $|H(f)| = \sqrt{a^2+b^2}$, as **f** varies from 0 to ½ f_s
  - Phase ≡ **angle(H(f))** $= \tan^{-1}(a/b)$, **f** from 0 to ½ f_s

Filter Example (1)

- Low-Q 60 Hz notch filter
  - $y(n) = (x(n)+x(n\text{-}3))/2$
- Z-Transform
  - $Y(z) = (X(z) + z^{-3}X(z))/2$
- Rewrite in the form **H(z)=Y(z)/X(z)**
  - $H(z) \equiv Y(z)/X(z) = ½ (1 + z^{-3})$
- Determine gain and phase response
  - $H(f) = ½ (1 + e^{-j6pf/fs}) = ½ (1 + \cos(6pf/f_s) - j \sin(6pf/f_s))$
  - **Gain** $\equiv |H(f)| = ½ \text{ sqrt}((1 + \cos(6pf/f_s))^2 + \sin(6pf/f_s)^2))$
  - **Phase** $\equiv \text{angle}(H(f)) = \tan^{-1}(-\sin(6pf/f_s)/(1 + \cos(6pf/f_s))$

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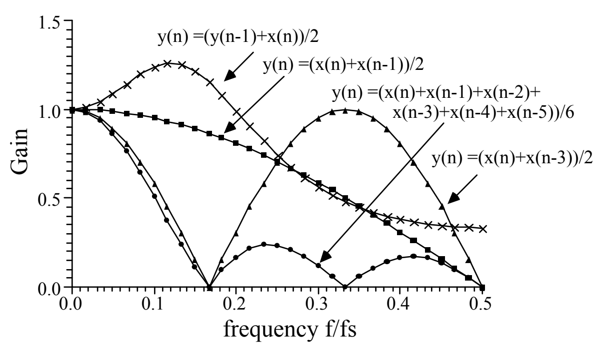
Filter Example (2)

- Gain vs. frequency response



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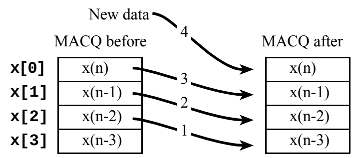
Filter Example (3)

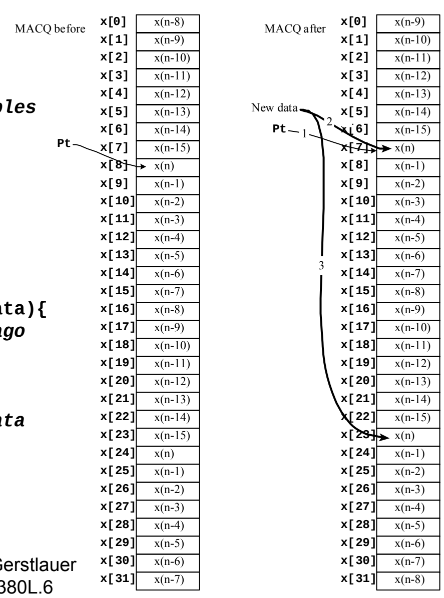
- Multiple Access Circular Queue (MACQ)




```
short x[4]; // MACQ
void ADC3_Handler(void){ short y;
  ADC_ISC_R = ADC_ISC_IN3; // ack ADC sequence 3 completion
  x[3] = x[2];  // shift data
  x[2] = x[1];  // units, ADC sample 0 to 4095
  x[1] = x[0];
  x[0] = ADC_SSFIFO3_R&ADC_SSFIFO3_DATA_M; // 0 to 4095
  y = (x[0]+x[3])/2;  // filter output
  Fifo_Put(y);        // pass to foreground
}
```


Pointer-Based MACQ

```
unsigned short x[32]; // two copies
unsigned short *Pt;   // pointer to current
unsigned short Sum;   // sum of last 16 samples
void LPF_Init(void){
  Pt = &x[0]; Sum = 0;
}
// calculate one filter output
// average previous 16 samples
// called at sampling rate
// Input: new ADC data
// Output: filter output, DAC data
unsigned short LPF_Calc(unsigned short newdata){
  Sum = Sum - *(Pt+16);  // sub 16 samples ago
  if(Pt == &x[0]){
    Pt = &x[16];         // wrap
  } else{
    Pt--;                // make room for data
  }
  *Pt = *(Pt+16) = newdata; // two copies
  return Sum/16;
}
```



Filter Design

Analog condition	Digital condition	Consequence
zero near s=j2πf line	zero near $z=e^{j2\pi f/fs}$	low gain near the zero
pole near s=j2πf line	pole near $z=e^{j2\pi f/fs}$	high gain near the pole
zeros in conjugate pairs	zeros in conjugate pairs	the output y(t) is real
poles in conjugate pairs	poles in conjugate pairs	the output y(t) is real
poles in left half plane	poles inside unit circle	stable system
poles in right half plane	poles outside unit circle	unstable system
pole near a zero	pole near a zero	high Q response

60Hz digital notch filter, $\mathbf{f_s}$ = 480 Hz

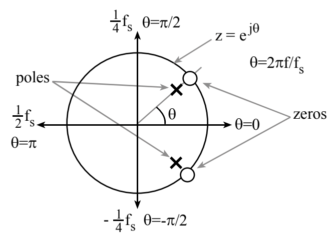
$$\theta = \pm 2\pi \cdot \frac{60}{f_s} = \pm \pi/4$$

Zeros on unit circle (gain=0 at 60 Hz):

$$z_1 = \cos(\theta) + j\sin(\theta),\ z_2 = \cos(\theta) - j\sin(\theta)$$

Poles next to the zeros, just inside the unit circle (flat pass band away from 60 Hz):

$p_1 = \alpha\, z1,\ p_2 = \alpha\, z_2$ where $0 < \alpha < 1$



IIR Filter (1)

- Transfer function

$$H(z) = \prod_{i=1}^{k} \frac{(z - z_i)}{(z - p_i)} = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)} = \frac{1 - 2\cos(\theta)z^{-1} + z^{-2}}{1 - 2\cos(\theta)z^{-1} + {}^{2}z^{-2}}$$

- With $\mathbf{f_s}$ = 480Hz and α = 7/8

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{49}{64} z^{-2}} \qquad y(n) = x(n) + x(n\text{-}2)\ \ \text{-}(49 * y(n\text{-}2))/64$$

- At **z** = 1, this reduces to

$$\text{DC Gain} = \frac{2}{1 + \frac{49}{64}} = \frac{128}{64 + 49} = \frac{128}{113}$$

For DC Gain of 1: $y(n) = (113 \cdot x(n) + 113 \cdot x(n\text{-}2)\ \text{-}\ 98 \cdot y(n\text{-}2))/128$

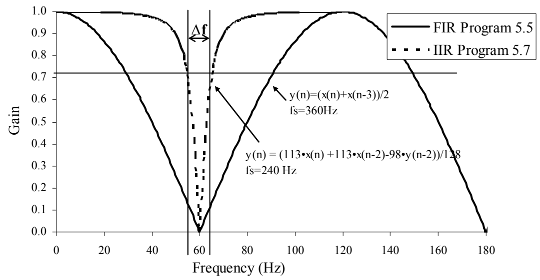
IIR Filter (2)

```
long x[3]; // MACQ for the ADC input data
long y[3]; // MACQ for the digital filter output
void ADC3_Handler(void){
  ADC_ISC_R = ADC_ISC_IN3;   // ack ADC completion
  x[2] = x[1]; x[1] = x[0];  // shift data
  y[2] = y[1]; y[1] = y[0];
  x[0] = ADC_SSFIFO3_R&ADC_SSFIFO3_DATA_M;
  y[0] = (113*(x[0]+x[2])-98*y[2])/128; // filter output
  Fifo_Put((short)y[0]);
}
```

Notch filter "Q":

$$Q \equiv \frac{f_c}{\Delta f}$$

Δf: frequency range where gain is below 0.707 of the DC gain



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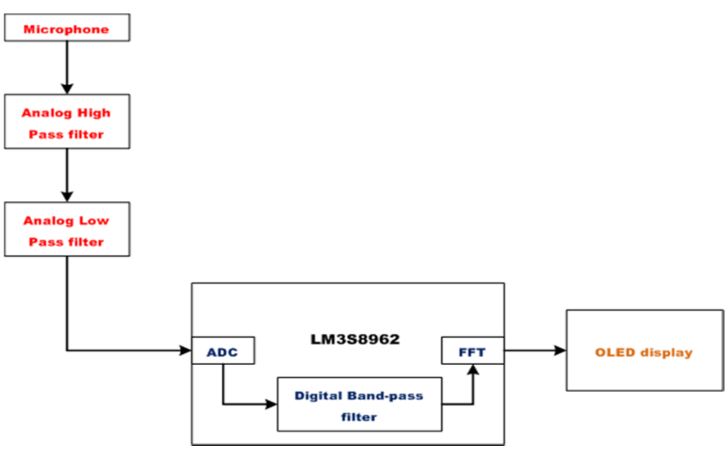
Lab4 Spectrum Analyzer



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Discrete Fourier Transform (DFT)

- Convert time to frequency domain

Input: N time samples Output: a set of N frequency bins

$\{a_n\} = \{a_0, a_1, a_2, \ldots, a_{N-1}\}$ $\{A_k\} = \{A_0, A_1, A_2, \ldots, A_{N-1}\}$

$$A_k = \sum_{n=0}^{N-1} a_n W_N^{kn}, \quad \text{where } W_N = e^{-j2\pi/N}, \ k=0,1,2,\ldots,N-1$$

- Inverse DFT

Input: a set of N frequency bins Ouptut: N time samples

$\{A_k\} = \{A_0, A_1, A_2, \ldots, A_{N-1}\}$ $\{a_n\} = \{a_0, a_1, a_2, \ldots, a_{N-1}\}$

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} A_k W_N^{-kn}, \quad \text{where } W_N = e^{-j2\pi/N}, \ n=0,1,2,\ldots,N-1$$

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DFT Properties

- Parameters
  - While the DFT deals only with samples and bins, assume data is ADC samples spaced at intervals $T=1/f_s$ (in sec)
  - Frequency bin k represents components at $k*f_S/N$ (in Hz)
  - The DFT resolution in Hz/bin is the reciprocal of the total time spent gathering time samples, i.e., 1/(NT)

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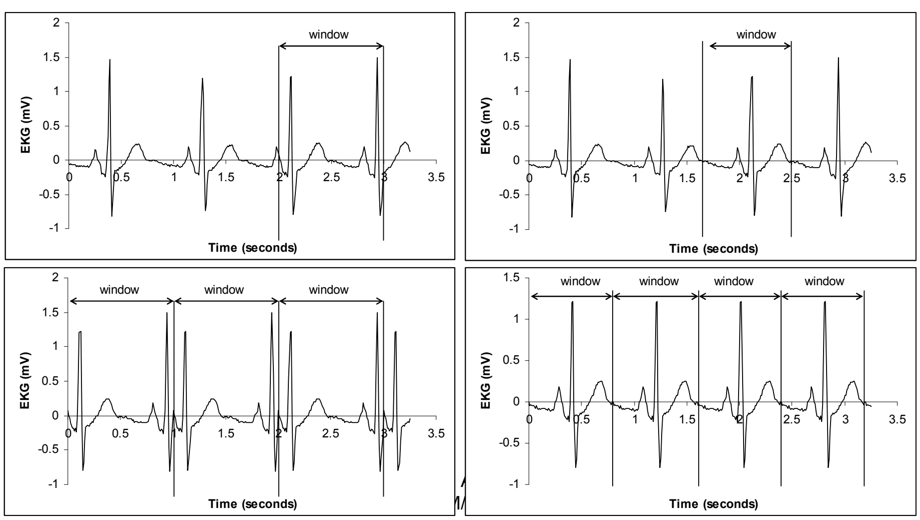
DFT Applications

- Applications
  - Measure S/N ratio
  - Identify noise
  - Filter design
- Four or five approximations
  - Finite min, max, range (max-min)
  - Finite precision & resolution (range/precision)
  - Sampling rate
  - Finite number of samples (spectral leakage)

Spectral Leakage

- Finite sequences, assumed to be periodic



Windowing (1)

- Spectral leakage can be virtually eliminated by "windowing" time samples prior to the DFT
  - Windows taper smoothly down to zero at the beginning and the end of the observation window
  - Time samples are multiplied by window coefficients on a sample-by-sample basis
- Windowing sinewaves places the window spectrum at the sinewave frequency
  - Convolution in frequency
- Window coefficients w(k)
  - Normalized so that the RMS value of the time samples is the same before and after windowing

$$\frac{1}{N}\sum_{n=0}^{N-1}\left|w(k)\right|^2 = 1$$

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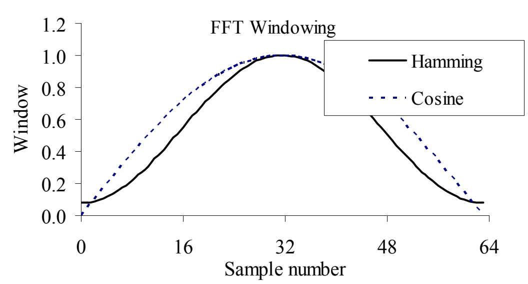
Windowing (2)

- Various windowing functions
  - Hamming w(k) = 0.54-0.46*cos(2pk/(N-1))
  - Hann w(k) = $(\sin(pk/(N-1)))^2$
  - Cosine w(k) = sin(pk/(N-1))
  - Triangle w(k) = (2/N)(N/2 - |k – (N-1)/2|)



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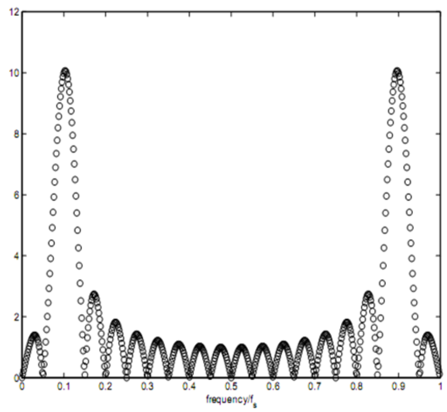
Fast Fourier Transform (FFT)

- Faster version of the Discrete Fourier Transform (DFT)
  - FFT spectrum of a cosine with a frequency of $0.1f_s$
    - f_s = 10kHz
    - Cosine freq = 1kHz
  - Interested region from 0 to $f_s/2$
    - Symmetric around $f_s/2$



FFT Library (1)

```
for(t = 0; t < 64; t++){ // collect 64 ADC samples
      data = OS_Fifo_Get();  // get from producer
      x[t] = data & 0xFFFF;  // real 0 to 1023, imaginary 0
}

cr4_fft_64_stm32(y,x,64);  // complex 64-point FFT

for(t = 0; t < 32; t++){   // first half
      real = y[t] & 0xFFFF;       // bottom 16 bits
      imag = y[t] >> 16;          // top 16 bits
      mag[t] = sqrt(real*real+imag*imag);
      LCD_Plot(mag);
}
```

http://www.ece.utexas.edu/~valvano/EE345M/sqrt.c

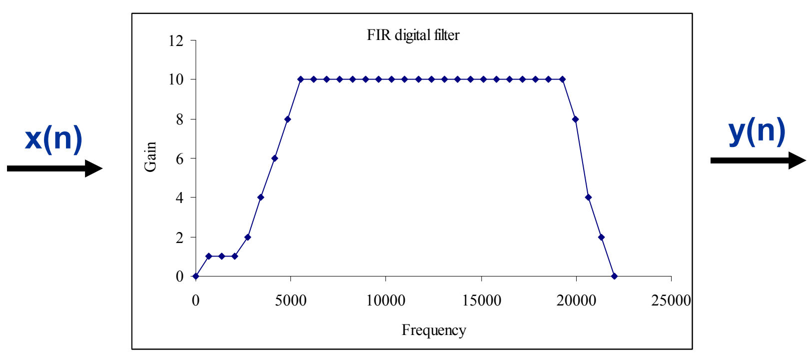
FFT Library (2)

```
for(t = 0; t < 1024; t++){ // collect 1024 ADC samples
    data = OS_Fifo_Get();  // get from producer
    x[t] = data;           // real 0 to 1023, imaginary 0
}
cr4_fft_1024_stm32(y,x,1024);  // complex FFT
for(t = 0; t < 512; t++){   // first half
    real = y[t]&0xFFFF;       // bottom 16 bits
    imag = y[t]>>16;          // top 16 bits
    data = sqrt(real*real+imag*imag);
    ST7735_PlotdBfs(data);
    if((t%4)==3){
      ST7735_PlotNext();    // 4 pixel per tick
    }
}
ST7735_PlotNext();   // 128 ticks across screen
```

if V is the FFT output magnitude in volts
$dB_{FS} = 20 \log_{10}(V/3)$; *// full scale is 3.0 volts*

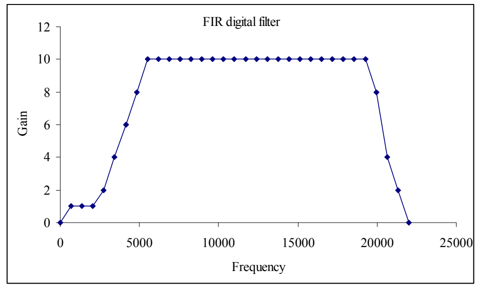
Alternative FIR Filter Design (1)

- Specify gain versus frequency
- Specify phase versus frequency



Alternative FIR Filter Design (2)

- Y(z) = H(z) X(z)
- h(n) = IFFT {H(z)}
- Convolution
  - y(n) = h(n)*x(n)



- Constants $h_0, h_1, \ldots h_{N-1}$
- $y(n) = h_0 \cdot x(n) + h_1 \cdot x(n-1) + \ldots + h_{N-1} \cdot x(n-(N-1))$
- N multiplies, N-1 additions per sample

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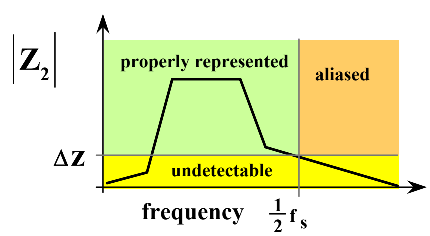
How to Choose Sampling Rate



- Nyquist Rate
- Limitation of display
- Limitation of processor
- Limitation of RAM
- Limitation of human eyes and ears
- Limitation of communication channel

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How to Choose Number of Samples

- Frequency resolution = f_s/N
  - Increase in N results in better frequency resolution
  - However, increase in N leads to a bigger MACQ buffer and more multiplies and additions
- Does not need to be a power of 2
  - DFT calculated once, off line

Design Process (1)

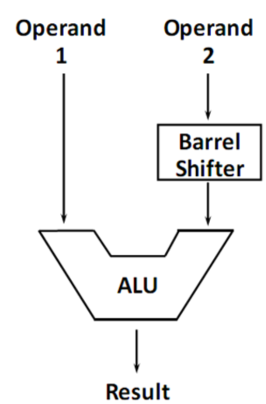
- Specify desired gain and phase, 0 to ½ f_s
  - k goes from 0 to N/2 ($f = k f_s/N$)
  - H(k) is complex
  - |H(k)| is gain
  - angle(H(k)) is phase
- For ½ f_s to f_s
  - H(N-k) is complex conjugate of H(k)
  - Poles and zeros are in complex conjugate pairs

Design Process (2)

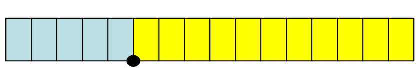
- Take IDFT of H(k) to yield h(n)
  - n goes from 0 to N-1
  - h(n) will be real, because H(k) symmetric
- The digital filter is
  - $y(n) = h_0 \cdot x(n) + h_1 \cdot x(n-1) + \ldots + h_{N-1} \cdot x(n-(N-1))$

Binary Fixed Point Notation

- Binary fixed-point is faster than decimal fixed-point
- Qn number (16 bit)
  - n: specifies the resolution = 2^{-n}
  - 16-n: specifies range
- Eg: 10.450 (unsigned number) with n=11?
- How is this number stored as an integer?



EE382N: Advanced Embedded Systems Architecture (lecture 5)



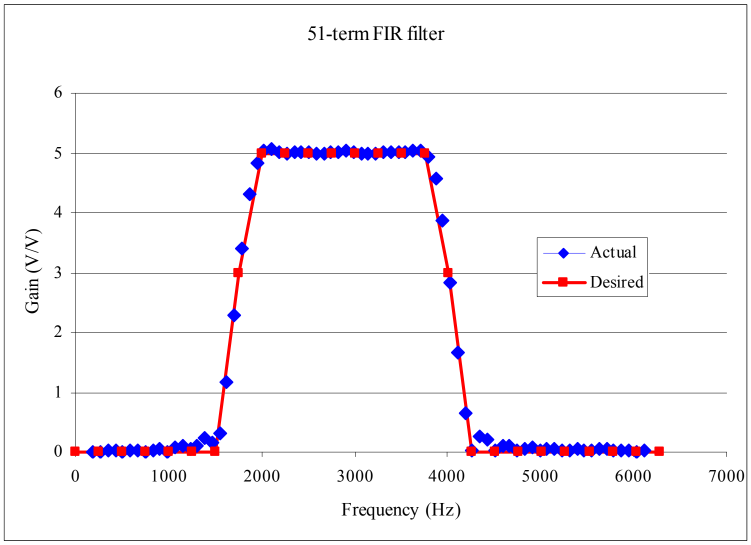
Value = Integer/2048

10.450 ≈ 01010.1110011010

Example

- Open FIRdesign51.xls
- Change sampling rate to 10,000 Hz
- Adjust red desired gain to make BPF
  - Pass 2 to 4 kHz
  - Look at sharp corner versus round corner
- Notice linear phase
- Copy 51 coefficients into software

2kHz to 4kHz BPF



FIR Filter SW Design

```
const long h[51]={-3,-9,4,5,0,17,5,-20,-5,-7,-22,
     24,41,-8,2,1,-74,-31,71,20,33,125,-119,-350,67,
     462,67,-350,-119,125,33,20,71,-31,-74,1,2,-8,41,
     24,-22,-7,-5,-20,5,17,0,5,4,-9,-3};
static unsigned int n=50;   // 51,52,… 101
short Filter(short data){unsigned int k;
  static long x[102]; // this MACQ needs twice
  long y;
  n++;
  if(n==102) n=51;
  x[n] = x[n-51] = data;  // two copies of new data
  y = 0;
  for(k=0;k<51;k++){
    y = y + h[k]*x[n-k];  // convolution
  }
  y = y/256;  // fixed point
  return y;
}
```


Circular Buffering

Array Index	Filter Coefficient Array h[]	Circular Buffer Array xcirc[]
0	$h[0]$	$x[n - newest]$
1	$h[1]$	$x[n - newest + 1]$
$\vdots$	$\vdots$	$\vdots$
		$x[n-1]$
$newest$		$x[n]$
$oldest$		$x[n-N+1]$
		$x[n-N+2]$
$\vdots$	$\vdots$	$\vdots$
$N-2$	$h[N-2]$	$x[n - newest - 2]$
$N-1$	$h[N-1]$	$x[n - newest - 1]$

"Communication system design using DSP algorithms" by Steven A. Tretter (Chapter 3, page 73)

Optimization

- Pointer implementations of MACQ faster
- Do not try and shift the data
- Convolution x[n]*h[n] takes N multiplies, N-1 additions per sample
  - Can be optimized to N/2 multiplies
  - Coefficients are symmetric
- Assembly optimization with MLA
  - Multiply with accumulate