MANAGING CONGESTION IN DYNAMIC MATCHING MARKETS

NICK ARNOSTI, RAMESH JOHARI, AND YASH KANORIA

ABSTRACT. We construct a model of a decentralized two-sided matching market in which agents arrive and depart asynchronously. In our model, it is possible that an agent on one side of the market (an "employer") identifies an agent on the other side of the market (an "applicant") who is a suitable partner, only to find that the applicant is already matched. We find that equilibrium is generically inefficient for both employers and applicants. Most notably, for a wide range of parameter values, equilibrium employer welfare is driven to *zero* due to uncertainty about the availability of applicants.

We consider a simple intervention available to the platform: limiting the visibility of applicants. We find that this intervention can significantly improve the welfare of agents on both sides of the market; applicants pay lower application costs, while employers are less likely to find that the applicants they screen have already matched. Somewhat counterintuitively, the benefits of showing fewer applicants to each employer are greatest in markets in which there is a shortage of applicants.

Keywords: Matching, decentralized, market design, mean field

1. INTRODUCTION

Since the pioneering work of Diamond (1982a,b); Mortensen (1982a,b) and Pissarides (1984a,b) a large and still-growing body of work has studied the efficiency (or lack thereof) of labor markets in the presence of *search frictions*. [1] Search frictions are the costs that market participants incur in trying to find suitable trading partners. Informally, in the settings we consider, participants are divided into two sides of the market (e.g., "employers" and "applicants"), and the goal of each participant is to find a suitable match (if any exists) on the other side of the market. Two common forms of search friction are prevalent in such settings: *application* (or messaging) costs, which are paid each time a participant communicates interest to the other side of the market; and *screening* costs, which are paid each time a participant evaluates a prospective match on the other side of the market.

In both the traditional labor market and in a range of online platform marketplaces (e.g., dating markets, markets for lodging, online labor markets, etc.), information technology has progressively

N. Arnosti (`narnosti@stanford.edu`) and R. Johari (`ramesh.johari@stanford.edu`) are with the Department of Management Science and Engineering, Stanford University; Y. Kanoria (`ykanoria@gmail.com`) is with Columbia Business School.

[1]Other influential early work includes Hosios (1990) and Mortensen and Pissarides (1994). Later extensions have studied the role of wages (Shimer, 1996; Moen, 1997; Acemoglu and Shimer, 1999), incomplete information (Guerrieri, 2008), on-the-job search (Shi, 2009), and adverse selection (Guerrieri et al., 2010). Rogerson et al. (2005) conduct a helpful survey of the literature.

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lowered application costs. Intuitively, we might expect the reduction of search frictions to be associated with greater market efficiency: when it is easier to reach participants on the other side, in principle the set of potential matches is widened. The ideal outcome would be for this greater feasible set of matches to lead to higher welfare.

However, in practice, this desired outcome is far from guaranteed. In particular, reduction of search frictions in decentralized markets can lead to negative externalities that erase any potential efficiency gains, and indeed lead to worse performance overall. A key example of this phenomenon is *congestion* that arises as application costs are lowered; informally, in congested markets, participants send more applications than is desirable. Typically congestion is studied in the context of its effect on the *applicants*: as application costs are lowered, the increase in applications leads to the situation that *many applications that are sent are never even screened*. This welfare loss arises because each applicant imposes a negative externality on other applicants, and (in most markets) this externality is not internalized. Recent literature has studied variants of this phenomenon in a range of marketplaces (Halaburda and Piskorski, 2010; Coles et al., 2010; Lee et al., 2011).

By contrast, a key contribution of our paper is to study a novel source of inefficiency due to a negative externality on *employers* in congested matching markets. Specifically, the phenomenon on which we focus is that an agent on one side of the market (an "employer") may identify a suitable match on the opposite side (an "applicant"), only to later learn that this applicant is already matched and therefore *unavailable*. As a result, welfare losses are felt by employers, who incur excessive screening costs trying to match to applicants who are no longer available to match; this effect becomes more pronounced as application costs fall (so that applicants send more applications).

Lack of information about availability is a common problem in a wide range of online matching markets, and has significant welfare consequences. Fradkin (2013) notes that on AirBnB, employers are uncertain about applicant availability because transactions take time to complete, and applicants may not reliably update the calendars. On the online labor marketplace oDesk, potential employers can directly *invite* workers to apply, but in practice, the best workers may prove to be unavailable to the client. This fact typically has a strong negative effect on the employer's satisfaction on the site.[2] For example, Horton (2014) finds using an instrumental variables approach that a positive response to an invitation makes a client 40% more likely to fill their job opening.

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[2]Indeed, the oDesk site publicly states that "when clients send out invitations and freelancers don't reply, it's a frustrating experience that makes those clients less likely to hire anyone." See `https://support.odesk.com/entries/23127826-What-is-responsiveness-`.

A key reason that availability affects client satisfaction is that the screening effort they expend in determining whether to invite a worker is wasted if the invited worker turns out to be unavailable.

It is natural to ask whether the platform can do anything to improve the employer's experience. One obvious approach is to provide information about the likelihood that each individual will respond when contacted, and many online marketplaces such as oDesk, AirBnB, and OkCupid strive to do just that. However, these inferences are (at best) imperfect, as agent availability may depend on a variety of factors that are unobservable to the platform and to other market participants.[3]

We instead consider a simple yet powerful intervention that requires no special knowledge on the part of the platform: *limiting applications*. This approach is employed in many platform marketplaces; for example, on the dating site eHarmony, users are shown only 5 potential matches per day, while on Coffee Meets Bagel, users are shown exactly one profile per day in the app. Online labor markets such as oDesk and Elance both limit the number of applications workers are allowed to send in a fixed period of time.

In our analysis, we comprehensively evaluate the welfare impact of congestion in a dynamic matching market, and study the benefits that limiting applications can achieve. Our main findings are as follows. *First*, without intervention, both of the negative consequences of congestion discussed above are severe. In particular, the lack of availability information can drive equilibrium employer welfare to zero. *Second*, limiting applications can yield significant welfare benefits for both sides of the market. In particular, in cases where the employers obtain zero welfare without intervention, an appropriately chosen application limit raises their welfare to a constrained efficient benchmark. Further, we can deliver a large fraction of this surplus to employers at little or no cost to applicants.

We study the potential benefits of this intervention using a mathematical model of a dynamic matching market, presented in Section 2. In the game we consider, employers and applicants arrive over time, and live for (at most) a unit lifetime. Upon arrival, applicants apply to a subset of employers present in the system, and incur a fixed cost per application sent. Upon departure, employers screen the applications they receive for *compatibility*, i.e., fitness for a match; they pay a fixed cost per applicant screened. If a compatible applicant is found the employer can make her

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[3]As one example, on oDesk, Horton (2014) finds that workers have a wide distribution of hours worked per week (in contrast to the offline labor market, where hours worked per week is strongly peaked at 40 hours/week). Thus it is difficult to infer availability by simply examining the number of hours worked.

an offer, and applicants accept the first offer they receive (if any). If an employer makes an offer to an unavailable applicant, the offer fails and the employer continues to screen for a match.

Because there is a delay between when an application is sent (the "application time") and when the applicant is evaluated by the employer (the "screening time"), the state of the applicant can change by the time she is screened. In particular, although the applicant may have been unmatched at the application time, she may *already be matched* by the screening time. Unless the employer knows this, he may waste effort screening her. The combination of costly screening and uncertain availability is essential to our model.

In Section 3, we present a *mean field* analysis of our game. In particular, we assume that from the point of view of an employer, applicants are independently available with some fixed probability q; and from the point of view of an applicant, each application yields an offer independently with probability p. Under these two assumptions, solving for the optimal strategies of employers and applicants becomes straightforward. On the other hand, both p and q must satisfy consistency checks that ensure they arise from the optimal decisions of employers and applicants. Taken together, this pair of conditions—optimality and consistency—define a notion of equilibrium for our mean field model that we call *mean field equilibrium* (MFE); several recent papers have used a similar modeling approach.[4]

In Section 4, we prove that the mean field agent assumptions hold asymptotically in large markets. The proof is a significant technical contribution of our work, as much of the existing search literature relies on "large market" assumptions that are not formally stated or justified. Existing papers that provide such justification (see for example Galenianos and Kircher (2012)) consider a static setting; establishing similar results in our dynamic model presents a number of novel technical challenges.[5]

In Section 5, we study the welfare of employers and applicants in equilibrium, and the value of intervention by the platform. First, we study the market without intervention. As expected, we find a "tragedy of the commons" effect among applicants. Surprisingly, this effect remains severe *even* if application costs become very low: the increase in applications more than offsets any reduction in application costs.

One of our most striking conclusions is that without intervention, for a wide range of parameter values, employer welfare is *zero* in equilibrium. The intuition behind this result is that if applicant

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[4]See Weintraub et al. (2008); Iyer et al. (2013); Gummadi et al. (2013); Balseiro et al. (2013); Bodoh-Creed (2013).
[5]In addition, in our model each applicant will contact multiple firms, which Galenianos and Kircher acknowledge presents many difficulties even in the static setting.

applies to many positions, she might receive many offers, but can accept only one. Employers whose offers are rejected have wasted any effort they devoted to evaluating her. If each applicant expects many offers, this wasted effort can entirely offset the benefit that employers receive from consummated matches.

Finally, we study the value of intervention. We demonstrate that for each side of the market, for a wide range of parameter values, imposing an application limit can raise welfare to a constrained efficient benchmark. Importantly, we show that these benefits need not come at the expense of the other side of the market: we numerically demonstrate that using an application limit, we can *simultaneously* obtain for both employers *and* applicants at least 3/4 of the best surplus attainable when choosing the application limit optimally for each side of the market alone. We find that for many parameter values, this ratio is close to one, suggesting that the tension between the two sides of the market is minimal.

In Section 6, we interpret and discuss our results, and present several extensions and open directions. In particular, we contrast our insights with those obtained in a typical static model of a matching market. We observe that static models severely understate the impact of a lack of availability information on welfare; this is a major motivation for our analysis of a dynamic matching market.

2. The Model

In the market we consider, employers and applicants arrive over time, interact with each other, and eventually depart. Informally, we aim to capture the following behavior:

(1) Employers arrive to the market, and post an *opening*.

(2) When applicants arrive, they *apply* to a subset of the employers currently in the market.

(3) Upon exit from the system, employers may screen candidates to learn whether they are *compatible* with the job. They may make offers to compatible applicants.

(4) Whenever an applicant receives an offer, he chooses whether to accept it.

One could construct a static model in which the above stages happen sequentially, with every applicant applying simultaneously, and every employer making offers simultaneously. Instead, we model a dynamic and asynchronous market, the timing of which is described in more detail below. In Section 6 an Appendix B, we explain how a static model where each stage occurs once does not capture key features of such a market.

2.1. **Arrivals.** Our market starts empty at $t = 0$. Our dynamic markets are parameterized by $n > 0$, which describes the market size. Individual employers arrive at intervals of $1/n$, and applicants arrive at intervals of $1/(rn)$. Here $r > 0$ is a parameter that controls the relative magnitude of the two sides of the market. Employers remain in the system for a unit lifetime. Applicants depart the system according to a process that we describe below. We endow employers and applicants with unique IDs, which convey no other information about the agent.

Upon arrival, employers post an opening. They do not make any decisions at this time. Upon arrival, each applicant selects a value $m_a \in [0, n]$, and applies to each employer currently in the system with probability m_a/n. Note that for all $t \geq 1$, there are exactly n employers in the system, and thus the expected number of applications sent by an applicant who arrives after time $t \geq 1$ is m_a.[6]

2.2. **Applicant departure.** Applicants remain in the system for a maximum of one time unit. If they receive and accept an offer from any employer (as described below), they depart from the market at that time. Recall that (excluding application costs) an applicant earns a payoff of 1 as long as she is matched, and zero otherwise. Thus, *it is a dominant strategy for applicants to accept the first offer they receive.* We assume henceforth that applicants follow this strategy.

2.3. **Employer departure.** Each employer stays in the system for one time unit, and then departs. Upon departure, employers see the set of applicants to their opening. Initially, they do not know which of these applicants are compatible for their job, nor do they know which would accept the job if offered it (i.e. which applicants remain in the system). The employer takes a sequence of "screening" and "offer" actions, instantaneously learning the result of each, until they choose to exit the market.

At each stage of the employer's sequential decision process, she may screen any unscreened applicant, thereby learning whether this applicant is compatible for the job. If the employer has yet to match, she may also make an offer to any applicant whom she has found to be compatible (the applicant responds immediately to the offer; as mentioned above, we assume that they accept it if and only if it is the first offer that they receive). Employers are also allowed to exit the market at any point.

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[6]An alternate model might be that applicants directly choose the *number* of applications they send. We choose a probabilistic specification primarily for technical convenience, to ease our later mean field analysis: in a setting where applicants deterministically send a fixed number of applications, an additional dependency is introduced across the employers.

Note that before making an offer to any applicant, employers must screen this applicant and find him compatible. (We provide microfoundations for this mechanical restriction in Appendix A.) We assume that that each employer-applicant pair is compatible with probability β (independently across all such pairs), and that this is common knowledge.[7]

2.4. **Utility.** If a compatible pair matches to each other, the employer earn v and the applicant earns w. Without loss of generality, we normalize $v = w = 1$.[8] Applicants pay a cost c_a for each application that they send, and employers pay a cost c_s for each applicant that they screen. The net utility to an agent will be the difference between value obtained from any match, and costs incurred. Our agents are risk neutral; that is, they seek to maximize their expected utility.

Assumption 1. $\max(c_a, c_s) < \beta$.

This assumption rules out the uninteresting case where costs are so high that no activity occurs in the marketplace. If $c_s \geq \beta$, then regardless of applicant behavior, it would be optimal for employers to exit the market rather than screening. Similarly, if $c_a \geq \beta$, then because employers hire only compatible applicants, no applicant strategy can earn positive surplus.

For later reference, it will be useful to consider *normalized* versions of the screening and application costs, given by

$$(1) \qquad\qquad c_s' = c_s/\beta, \qquad\qquad c_a' = c_a/\beta.$$

3. The large market: A stationary mean field model

In principle, the strategic choices facing an agent in the model described above may be quite complicated. Consider the case of an employer who knows that he has only one competitor. If he finds that his first applicant has already accepted another offer, he learns that every other applicant is still looking for a job. Similar logic suggests that in thin markets, information revealed during screening may induce significant shifts in the employer's beliefs. This could conceivably cause optimal employer behavior to be quite complex.

As the market thickens, however, one might expect that the correlations between agents on the same side of the market become weak. In particular, employers screening a pool of applicants

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[8]The assumption that $v = w$ is without loss of generality because we never compare absolute welfare of agents on opposite sides of the market.

might reasonably assume that learning that one applicant has already accepted an offer does not inform them about the availability of other applicants. Further, if the employer cannot distinguish individual applicants, each one should appear to be available with equal probability. Similarly, applicants who know nothing about individual employers may be justified in assuming that each of their applications convert to offers independently and with equal probability p.

In this section we develop a formal stationary *mean field* model for our dynamic matching market, and introduce a notion of game-theoretic equilibrium for this model; in particular, we study a model that arises from a limiting regime where the market thickens.

In our formal model, agents make the following assumptions.

Mean Field Assumption 1 (Employer Mean Field Assumption)**.** *Each applicant in an employer's applicant set is available with probability q, and the availability of applicants in the applicant set is independent.*

Mean Field Assumption 2 (Applicant Mean Field Assumption)**.** *Each application yields an offer with probability p, independently across applications to different openings.*

Mean Field Assumption 3 (Large Market Assumption)**.** *The number of applications sent by an applicant who chooses $m_a = m$ is Poisson distributed with mean m. If all applicants select $m_a = m$, the number of applications received by each employer is Poisson distributed with mean rm.*

Under these assumptions, optimal agent behavior simplifies greatly. We describe optimal employer and applicant responses in Section 3.1. For applicants, we show that there exists a unique optimal choice of m, given p; and for employers, we show that given q, their optimal response is either to employ a simple sequential screening strategy or to exit immediately (they may also randomize when indifferent between these options).

Of course p and q are not given exogenously, but rather arise endogenously from the choices made by agents. In Section 3.2, we derive "consistency checks" that p and q should satisfy, if they indeed arise from the conjectured employer and applicant strategies.

The work in Sections 3.1 and 3.2 allows us to define a *mean field equilibrium* (MFE) in Section 3.3. Informally, a mean field equilibrium consists of strategies for employers and applicants that are best responses to the stationary market dynamics that they induce. We prove that there exists a *unique* MFE. We conclude with Section 3.4, which discusses a simple intervention available to

the market operator: placing a limit ℓ on the value m_a chosen by each applicant. We show that unique MFE continue to exist in this setting.

We emphasize that a key result of our paper is that our mean field model is in fact the correct limit of our dynamic market as the thickness n grows. In particular, Theorems 3 and 4 in Section 4 justify our study of MFE: they state that the mean field assumptions hold as n approaches infinity, and that as a consequence any MFE is an approximate equilibrium in the game with finite but sufficiently large n.

3.1. **Optimal decision rules.** We first study how agents respond when confronted with a world where the mean-field assumptions hold.

3.1.1. *Applicants.* As discussed in Section 2, it is a dominant strategy for applicants to accept the first offer (if any) that they receive, and we assume applicants follow this rule. Therefore the only decision an applicant a needs to make on arrival is her choice of m_a, the expected number of applications sent.

If an applicant a chooses $m_a = m$, they incur an expected cost of $c_a \cdot m$. If the applicant applies to Poisson(m) employers, and each application independently yields an offer with probability p, then at least one offer is received—i.e., the applicant matches to an employer—with probability $1 - e^{-mp}$. Thus, the expected payout of an applicant in the mean-field environment who selects $m_a = m$ is $1 - e^{-mp} - c_a m$. Applicants choose $m \geq 0$ to maximize this payoff. Because their objective is strictly concave and decays to $-\infty$ as $m \to \infty$, this problem possesses a unique optimal solution identified by first-order conditions. If $p \leq c_a$, the optimal choice is $m = 0$. Otherwise, applicants select $m = \frac{1}{p} \log \left(\frac{p}{c_a} \right)$. We define $\mathcal{M}$ to be the function that maps p to the unique optimal value of m:

$$(2) \qquad \mathcal{M}(p) = \begin{cases} 0, & \text{if } p \leq c_a; \\ \frac{1}{p} \log \left(\frac{p}{c_a} \right), & \text{if } p > c_a. \end{cases}$$

3.1.2. *Employers.* Next, we consider the optimal strategy for employers, when Mean Field Assumption 1 holds. We consider a simple strategy, which we denote ϕ^1. A employer playing ϕ^1 sequentially screens candidates in her applicant list. When she finds a compatible applicant, she makes an offer to this candidate; otherwise, she considers the next candidate. This process repeats until one applicant accepts or no more applicants remain.

The optimal strategy for the employers is straightforward to characterize. First suppose an employer has exactly one applicant. The employer will prefer to screen the applicant if $\beta q - c_s > 0$, i.e., if $q > c'_s$; exit if $q < c'_s$; and is indifferent if $q = c'_s$. Now it is clear that if an employer has more than one applicant in her list, since all applicants are *ex ante* homogeneous from the perspective of the employer, the same reasoning holds: the employer will screen or exit immediately according to whether q is larger or smaller than c'_s, respectively. (Note the essential use of Mean Field Assumption 1: if there is correlation in the availability of successive applicants in the employer's list, the preceding reasoning no longer holds.) The following proposition summarizes the preceding discussion.

Proposition 1. *Let ϕ^1 be the strategy of sequentially screening applicants, offering them the job if and only if they are qualified, until either an applicant is hired or no more applicants remain. Then ϕ^1 is uniquely optimal if and only if $q > c'_s$, exiting immediately is uniquely optimal if and only if $q < c'_s$, and any mixture of these strategies is optimal if $q = c'_s$.*

Motivated by this proposition, we define ϕ^α to be the strategy that plays ϕ^1 with probability α and exits immediately otherwise. Define the correspondence $\mathcal{A}(q)$ by:

(3)
$$
\mathcal{A}(q) = \begin{cases} \{0\} & \text{if } q < c'_s \\ [0,1] & \text{if } q = c'_s \\ \{1\} & \text{if } q > c'_s. \end{cases}
$$

This correspondence captures the optimal employer response, as described in Proposition 1, so that $\mathcal{A}(q) = \{\alpha \in [0,1] : \phi^\alpha \text{ is optimal for the employer, given } q\}$.

3.2. **Consistency.** In the previous section, we discussed the best responses available to employers and applicants when the mean field assumptions hold; that is, given p and q, we found the strategies that agents would adopt. However, p and q are clearly *determined* by agent strategies. In this section we identify consistency conditions that p and q must satisfy, given specified agent strategies.

We focus on strategies that could conceivably be optimal, as identified in the preceding section. We assume that all applicants choose the same $m \geq 0$, and that all employers play ϕ^α, i.e., they play ϕ^1 with probability α and exit immediately otherwise. From any m and α, we derive a unique prediction for the pair (p, q).

We emphasize at the outset that our analysis aims only to derive the correct consistency conditions under the mean field assumptions. We provide rigorous justification for these assumptions via the theorems in Section 4. As a consequence, those theorems also justify the consistency conditions described below.

We start by deriving a consistency condition for q, given p and the strategy adopted by applicants. Intuitively, q should be equal to the long-run fraction of offers that are accepted. Fix the value of m chosen by applicants, and let X be the number of offers received by a single applicant. This applicant will accept an offer if and only if $X > 0$, so the expected fraction of offers that are accepted is $P(X > 0)/E[X]$. If Mean Field Assumptions 2 and 3 hold, then X is Poisson with mean mp, so we should have

$$(4) \qquad q = \frac{1 - e^{-mp}}{mp}.$$

To derive a consistency condition for p, note that when employers follow ϕ^α, only compatible applicants receive offers. Thus, p should equal β times the long-run fraction of qualified applications that result in offers. Because an applicant's availability does not influence whether they receive an offer (as it is unobserved by prospective employers), this should be equal to β times the fraction of applications by qualified *available* applicants that result in offers. Fix an employer playing ϕ^α, and let Y be the number of qualified, available applicants received by this employer. This employer successfully hires if and only if $Y > 0$ and she decides to screen. Thus, the fraction of qualified available applicants who receive offers should be $\alpha P(Y > 0)/E[Y]$. We conclude that

$$(5) \qquad p = \alpha\beta\frac{P(Y > 0)}{E[Y]} = \alpha\beta\frac{1 - e^{-rm\beta q}}{rm\beta q},$$

where the final step comes from Mean Field Assumptions 1 and 3, which jointly imply that Y is distributed as a Poisson random variable with mean $rm\beta q$.

The equations (4) and (5) are a system for p and q, given the values of m and α (as well as the parameters r and β). The following Theorem states that the pair of consistency equations (4) and (5) have a unique solution.

Theorem 1. *For fixed m, α, r, and β, there exists a unique solution (p, q) to (4) and (5).*

We refer to the unique pair (p, q) that solve (4) and (5) as a *mean field steady state* (MFSS). This pair provides a prediction of how a large market should behave, given specific strategic choices of the

agents. For later reference, given strategies m and α (and parameters r and β), let $\mathcal{P}(m, \alpha; r, \beta)$ and $\mathcal{Q}(m, \alpha; r, \beta)$ denote the unique values of p and q guaranteed by Theorem 1, respectively. Because our analysis is conducted with r and β fixed, we will omit the dependence on r and β in favor of the more concise $\mathcal{P}(m, \alpha)$ and $\mathcal{Q}(m, \alpha)$.

3.3. **Mean field equilibrium.** In this section we define *mean field equilibrium* (MFE), a notion of game theoretic equilibrium for our stationary mean field model. Informally, a MFE should be a pair of strategies such that (1) agents play optimally given their beliefs about the marketplace, i.e., the values p and q in the mean field assumptions; and (2) agent beliefs are consistent with the strategies being played, i.e., (p, q) is an MFSS corresponding to the agents' strategies. Section 3.1 addressed the first point; and Section 3.2 addressed the second. We define a mean field equilibrium by composing the maps defined in those sections.

Definition 1. *A mean field equilibrium (MFE) is a pair (m^*, α^*) such that $m^* = \mathcal{M}(\mathcal{P}(m^*, \alpha^*))$ and $\alpha^* \in \mathcal{A}(\mathcal{Q}(m^*, \alpha^*))$.*

In an MFE, m^* and α^* are optimal responses (under the mean field assumptions) to the steady-state (p, q) that they induce. For future reference, we define $p^* = \mathcal{P}(m^*, \alpha^*)$ and $q^* = \mathcal{Q}(m^*, \alpha^*)$. Our main theorem in this section establishes existence and uniqueness of MFE.

Theorem 2. *Fix r, β, c_a, c_s, and suppose Assumption 1 holds. Then there exists a unique mean field equilibrium (m^*, α^*).*

3.4. **A market intervention: application limits.** As noted in the Introduction, we are interested in comparing the outcome of the market described above to the outcome when the platform operator intervenes to try to improve the welfare of employers and/or applicants. We consider a particular type of intervention: a limit on the number of applications that can be sent by any individual.

In our model with application limits, agent payoffs are identical to before, as are the strategies available to employers. Applicants, however, are restricted to selecting $m_a \leq \ell$. In the corresponding mean field model, given p, applicants choose m_a to maximize $1 - e^{-m_a p} - c_a m_a$ (their expected payoff), subject to $m_a \in [0, \ell]$. The applicant objective is concave in m_a, so this problem has a unique solution given by

$$(6) \qquad \qquad \mathcal{M}_\ell(p) = \min(\ell, \mathcal{M}(p)).$$

The consistency conditions are identical to those in Section 3.2. We define a *mean field equilibrium* of the market with application limit ℓ as a pair $(m_\ell^*, \alpha_\ell^*)$ solving the following pair of equations:

$$(7) \qquad m_\ell^* = \mathcal{M}_\ell(\mathcal{P}(m_\ell^*, \alpha_\ell^*)), \quad \alpha_\ell^* \in \mathcal{A}(\mathcal{Q}(m_\ell^*, \alpha_\ell^*)).$$

The following proposition is an analog of Theorem 2 for the market with an application limit.

Proposition 2. *Fix r, β, c_a, c_s such that Assumption 1 holds, and let (m^*, α^*) be the corresponding MFE in the market with no application limit. Then for any $\ell \geq 0$ there exists a unique mean field equilibrium in the market with application limit ℓ. If $m^* \leq \ell$, then $(m_\ell^*, \alpha_\ell^*) = (m^*, \alpha^*)$. Otherwise, $m_\ell^* = \ell$ and α_ℓ^* is the unique solution to $\alpha_\ell^* \in \mathcal{A}(\mathcal{Q}(\ell, \alpha_\ell^*))$.*

For future reference, we define $p_\ell^* = \mathcal{P}(m_\ell^*, \alpha_\ell^*)$, $q_\ell^* = \mathcal{Q}(m_\ell^*, \alpha_\ell^*)$.

4. MEAN FIELD APPROXIMATION

In this section, we show that our mean field model is (in an appropriate sense) a reasonable approximation to our finite system when the market grows large (i.e., when $n \to \infty$). Formally, we show two results. First, we show that the mean field assumptions hold as $n \to \infty$, as long as all applicants a choose $m_a = m$, and all employers choose $\alpha_e = \alpha$. Note that once we fix m and α, we have removed any strategic element from the evolution of the n-th system; these results are limit theorems about a certain sequence of stochastic processes. Second, we use the preceding results to show that any MFE is an approximate equilibrium in sufficiently large but finite markets.

We require the following notation. We let $\text{Binomial}(n, p)$ denote the binomial distribution with n trials and probability of success p, and let $\text{Binomial}(n, p)_k = \mathbb{P}(\text{Binomial}(n, p) = k)$. We let $\text{Poisson}(a)$ denote the Poisson distribution with mean a, and let $\text{Poisson}(a)_k = \mathbb{P}(\text{Poisson}(a) = k)$.

In the following theorem, we show that Mean Field Assumption 1 holds as $n \to \infty$. In the process, we also show half of Mean Field Assumption 3: that in the limit the number of applications received by an employer is Poisson distributed.

Theorem 3. *Fix r, β, m, and α. Suppose that the n-th system is initialized in its steady state distribution. Consider any employer e that arrives at $t_b \geq 0$. Let $R_b^{(n)}$ denote the number of applications received by employer e in the n-th system, and let $A_b^{(n)}$ be the number of these applicants that are still available when the employer screens.*

Then as $n \to \infty$, the pair $(R_b^{(n)}, A_b^{(n)})$ converges in total variation distance to (R, A), where $R \sim \text{Poisson}(rm)$, and conditional on R, we let $A \sim \text{Binomial}(R, q)$.

Analogously, we have the following theorem, where we show that Mean Field Assumption 2 holds as $n \to \infty$. In the process, we also show the other half of Mean Field Assumption 3: that in the limit the number of applications sent by an applicant is Poisson distributed.

Theorem 4. *Fix r, β, m, and α and any $m_0 < \infty$. Suppose that the n-th system is initialized in its steady state distribution. Consider any applicant a arriving at time $t_s \geq 0$, denote the value chosen by applicant a by m_a (all other applicants are assumed to choose m, and all employers are assumed to follow ϕ^α). Let $T_s^{(n)}$ denote the number of applications sent by applicant a in the n-th system, and let $Q_s^{(n)}$ be the number of these applications that generate offers. Let $T \sim \text{Poisson}(m_a)$, and conditional on T, we let $Q \sim \text{Binomial}(T, p)$. Then*

$$(8) \qquad \lim_{n \to \infty} \left\{ \max_{m_a \in [0, m_0]} d_{\text{TV}}\big((T_s^{(n)}, Q_s^{(n)}), (T, Q)\big) \right\} = 0\,,$$

where $d_{\text{TV}}(X, Y)$ denotes the total variation distance between the distributions of random variables X and Y that take values in the same countable set.

In establishing these results, the fundamental result that we prove is that the evolution of the "state" of the n-th system satisfies a stochastic contraction condition, and therefore remains "close" to an appropriately defined fixed point for all time. This basic result allows us to establish that the mean field assumptions hold asymptotically.

We conclude by using the preceding results to establish the following corollary, which establishes (in an appropriate sense) that any MFE is an approximate equilibrium in sufficiently large finite markets.

Corollary 1. *Fix r, β, c_s and c_a. Let the MFE be (m^*, α^*). For any $\varepsilon > 0$, and any nonnegative integer R_0, there exists n_0 such that for all $n \geq n_0$ the following hold:*

(1) For all a, if all employers and applicants other than a follow the prescribed mean field strategies, then applicant a can increase her expected payoff by no more than ε by deviating to any $m \neq m^*$.

(2) For all e, if e receives no more than R_0 applications, and if all employers and applicants other than e follow the prescribed mean field strategies, then in any state in the corresponding

dynamic optimization problem solved by employer e , employer e can increase her expected payoff by no more than ε by deviating to any strategy other than ϕ^{α^*}.

The preceding result shows that for sufficiently large n, the mean field equilibrium is (essentially) an ε-approximate equilibrium in the finite market of size n. The first statement in the result states that applicants cannot appreciably gain by changing the number of applications they send. The second statement in the result makes an analogous claim for employers. In particular, if employers follow ϕ^{α^*}, they will either exit immediately; or sequentially screen and make offers to compatible candidates until such an offer is accepted, or the applicant pool is exhausted. In doing so, they will obtain information about the compatibility and availability of the subset of applicants they have already screened. Our result states that at any stage in this dynamic optimization problem, the employer cannot appreciably increase their payoff by deviating.

An apparent limitation of the second statement is that it applies only to employers who receive no more than R applications. The fraction of employers who receive more than R applications scales as $\exp(-\Omega(R))$; thus by choosing R large enough, this fraction can be made as small as desired. Since we show that the n-th system satisfies an appropriate stochastic contraction condition, we expect that in fact, for sufficiently large R, both statements in Corollary 1 hold even under arbitrary behavior by employers who receive more than R applications. In the interest of brevity we choose to omit this slightly stronger result.

5. Welfare analysis

One of the chief advantages of MFE is that they are amenable to analysis. In particular, we can gain significant insight into employer and applicant welfare, and the potential value of intervention for each side of the market. In this section, we study employer and applicant welfare in MFE, and quantify the effects of limiting the number of applications sent by each applicant. We have two main goals. *First*, by studying the equilibrium market outcome, we characterize the welfare losses due to the two aspects of congestion described in the introduction: (1) many applications may be sent that are never even screened; and (2) many applicants may be screened that prove to be unavailable. The former leads to welfare losses for applicants; the latter leads to welfare losses for employers. *Second*, we study the effect of market intervention, and show that limiting the number of applications can lead to significant welfare gains for both sides of the market.

5.1. **Welfare Benchmarks: Efficiency and Constrained Efficiency.**

Fix parameter values r, β, c_a, and c_s. For given agent strategies m and α, we let $\Pi_a(m, \alpha)$ and $\Pi_e(m, \alpha)$ denote the mean field applicant and employer welfare, respectively (Formal expressions for these payoffs can be found in Appendix E). We define Π_a^*, Π_e^* to be the expected applicant and employer surplus in mean-field equilibrium, respectively. In other words,

$$(9) \qquad \Pi_a^* = \Pi_a(m^*, \alpha^*), \qquad \Pi_e^* = \Pi_e(m^*, \alpha^*),$$

Analogously, let Π_a^ℓ and Π_e^ℓ be expected applicant and employer payoffs in the equilibrium of the market with application limit ℓ. In other words,

$$(10) \qquad \Pi_a^\ell = \Pi_a(m_\ell^*, \alpha_\ell^*), \qquad \Pi_e^\ell = \Pi_e(m_\ell^*, \alpha_\ell^*).$$

When evaluating employer and applicant welfare in equilibrium, we need a point of comparison: what constitutes "good" welfare for each side of the market? In the search literature, there are two common comparison points: the "efficient" outcome and the "constrained efficient" outcome. In our setting, the former corresponds to the hypothetical situation in which the platform directly observes compatibility and costlessly matches individuals, so that applicants get $\min(1, 1/r)$ and employers $\min(1, r)$. While platform operators often have information that they use to infer compatibility and suggest matches, it is not reasonable to assume that they can costlessly and perfectly match agents. This motivates the concept of *constrained efficiency*, where the designer is subject to the informational constraints and costs of the decentralized market, but has the power to specify application and screening strategies for each side. Motivated by this approach, we define

$$\Pi_e^{\mathrm{opt}} = \sup_{m, \alpha} \ \Pi_e(m, \alpha); \quad \Pi_a^{\mathrm{opt}} = \sup_{m, \alpha} \ \Pi_a(m, \alpha).$$

It is clear that Π_e^{opt} (resp., Π_a^{opt}) is an upper-bound for employer (resp., applicant) welfare in equilibrium. These bounds are optimistic in at least two ways. First, Π_e^{opt} is the best that the social planner can obtain *when optimizing for employers* and Π_a^{opt} is the best that the social planner can obtain *when optimizing for applicants*; there is in general no reason to believe that the choices of m and α that induce employer welfare of Π_e^{opt} are the same strategies that yield a payoff of Π_a^{opt} for applicants. Second, the value Π_a^{opt} does not incorporate the screening cost c_s'. If this cost is high, then the employer strategy required to produce applicant welfare of Π_a^{opt} may not be individually rational (and thus not attainable in practice). Similarly, the value Π_e^{opt} does not

depend on the application cost c_a', and it may be that no individually rational applicant strategy can yield employer surplus of Π_e^{opt}.

We employ Π_e^{opt} and Π_a^{opt} as our constrained efficient benchmarks.[9] For most of this section, we will discuss employer and applicant welfare in terms of the *fraction* of Π_e^{opt} and Π_a^{opt} earned by employers and applicants, respectively. In Section 5.2 we focus on employer welfare, and turn our attention to applicants in Section 5.3. In both cases, we conclude that equilibrium payouts are (often substantially) below the constrained efficient benchmarks, and provide conditions under which the simple intervention of employing an application limit can raise employer welfare to Π_e^{opt} or applicant welfare to Π_a^{opt}. Although the employer-optimal and applicant-optimal limits do not coincide, in Section 5.4 we provide conditions under which an application limit can deliver Pareto improvements. Furthermore, we prove that for a wide range of parameters, the tradeoff between optimizing for employers and for applicants is never too severe. Finally, in Section 5.5 we consider the effect of an alternative intervention: *raising* the application cost c_a. We show that this intervention is equivalent to imposing an application limit from the perspective of employers, but always reduces applicant welfare.

Before moving on, we note that although our model has four parameters (r, c_a, c_s, β), it turns out that there is some redundancy, in that the quantities Π_a^*, Π_e^*, Π_a^{opt}, Π_e^{opt}, $\sup_\ell \Pi_a^\ell$, and $\sup_\ell \Pi_e^\ell$ depend only on $r, c_a' = c_a/\beta$, and $c_s' = c_s/\beta$. We characterize welfare in terms of this reduced parameter set.

5.2. **Welfare Analysis: Employers.**

In this section, we prove several theorems regarding equilibrium employer welfare, and the welfare that is attainable through the intervention of enforcing an application limit. We preview our results using Figure 1, which displays $\Pi_e^*/\Pi_e^{\mathrm{opt}}$ as r and c_s' vary. The plot depicts a rather sharp transition for employers: for most parameter values, they either do very well or quite poorly. Furthermore, in the regions where equilibrium employer welfare is low, suitably limiting applications can substantially improve employer welfare. We formalize these observations in the following proposition.

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[9]One possible measure for the performance of the market is the *aggregate* welfare given by $r\Pi_a + \Pi_e$. However, note that we have not imposed any constraints on the relative scale of employer and applicant payoffs; indeed, their units may not even be comparable. By focusing on Π_a^{opt} and Π_e^{opt} individually, we provide welfare analysis that holds for any relative weighting of employer and applicant welfare.
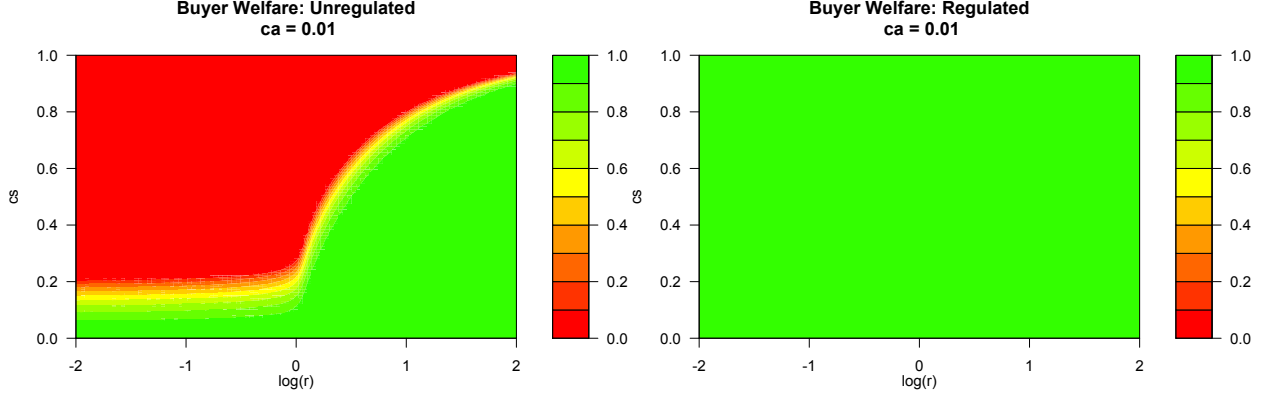
FIGURE 1. Normalized employer welfare in equilibrium ($\Pi_e^*/\Pi_e^{\text{opt}}$, left) and with an employer-optimal application limit ($\sup_\ell \Pi_e^\ell/\Pi_e^{\text{opt}}$, right). We fix the value $c_a' = 0.01$, and vary the values of $\log r$ (x-axis) and c_s' (y-axis). Note that there are effectively two regions: one in which employers do very well in equilibrium, and another in which they do very poorly. In the latter case, intervention can substantially raise employer surplus. See Proposition 3.

Proposition 3.

Employer welfare in equilibrium:

There exists a function $f(r, c_a') \in (0, 1)$, increasing in both arguments, such that $\Pi_e^* = 0$ if and only if $c_s' \geq f(r, c_a')$.

Employer welfare with an application limit:

If $\Pi_e^* = 0$, there exists ℓ such that $\Pi_e^\ell = \Pi_e^{\text{opt}}$.

The preceding result partitions the parameter space into two regions, via the function $f(r, c_a')$. The region where $c_s' > f(r, c_a')$ corresponds to the red area in the left image in Figure 1. Here employers obtain zero welfare in equilibrium, but intervention (in the form of an appropriate application limit) can raise employer welfare all the way to the constrained efficient benchmark Π_e^{opt}. We note that equilibrium employer welfare may be zero even if the screening cost c_s' is small. Indeed, if $r < 1$, then $f(r, c_a') \to 0$ as $c_a' \to 0$, implying that regardless of c_s', employer welfare may be zero if applicants send sufficiently many applications.

The intuition behind the proposition is as follows. When applicants send more applications, the effects on employers are two-fold. First, each employer expects to receive a higher number of qualified applicants, and is therefore more likely to match. However, each of these applicants is less likely to be available, as they have applied to many other employers; this is precisely the congestion effect (negative externality) on employers due to additional applicant applications. Thus, each

employer pays a higher expected screening cost per successful hire. The proposition captures the phenomenon that when this expected screening cost is too high, employers obtain zero welfare in equilibrium.

The preceding discussion motivates the definition of the function f. In particular, as a thought experiment, consider *fixing* the employer screening strategy to be ϕ^1: i.e., suppose employers always screen applicants before exiting. For fixed r and c_a', we can study the resulting market via a "partial" equilibrium where applicants optimally choose m. Note that this partial equilibrium *does not* depend on c_s, since the employer strategy is already fixed. Let $\tilde{q}$ denote the applicant availability in this partial equilibrium.[10]

The function $f(r, c_a')$ is exactly the availability $\tilde{q}$ derived in the preceding partial equilibrium analysis. Following Proposition 1, we see that if availability is high enough (i.e., if $\tilde{q} > c_s'$), then the strategy ϕ^1 is optimal for employers, and the partial equilibrium is in fact a general equilibrium; thus when $f(r, c_a') > c_s'$, employers receive positive welfare in equilibrium. On the other hand, if availability is too low (i.e., if $\tilde{q} < c_s'$), then employers would strictly prefer to exit immediately, and thus the partial equilibrium is not a general equilibrium. In this case, the market equilibrates just to the point that employers are indifferent between exiting and screening. In other words, when $f(r, c_a') < c_s'$, the general equilibrium will involve employers receiving zero welfare, and mixing between exiting and screening. This is exactly the dichotomy captured by Proposition 3.

We now provide intuition for the monotonicity of f. The larger the value of r, the more likely it is that applicants are available. Thus, large values of r imply that employers become indifferent about whether or not to screen only if c_s' is also large. Meanwhile, when c_a' is larger, applicants send fewer applications and are therefore more likely to be available; hence, f is increasing in c_a'.

Due to the search and screening costs c_a' and c_s', agents on both sides of the market remain unmatched in equilibrium.[11] One way to interpret the preceding definition of f is in terms of *which cost* (screening or application) constrains the equilibrium number of matches formed. If $c_s' > f(r, c_a')$, then the market is *screening-limited*: a marginal decrease in c_s' will increase the number of matches formed in equilibrium, but a marginal change in c_a' will have no effect on the number of matches. This is because an increase in the number of applications sent (due to

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[10]Formally, such a partial equilibrium can be defined as a solution to $m = \mathcal{M}(\mathcal{P}(m, 1))$. We show in Appendix C that a unique solution $\tilde{m}$ exists to this fixed point equation; we then let $\tilde{q} = \mathcal{Q}(\tilde{m}, 1)$.

[11]Indeed, one of the leading motivations behind the modeling of search frictions in labor markets is to explain the simultaneous presence of vacancies and unemployed workers.
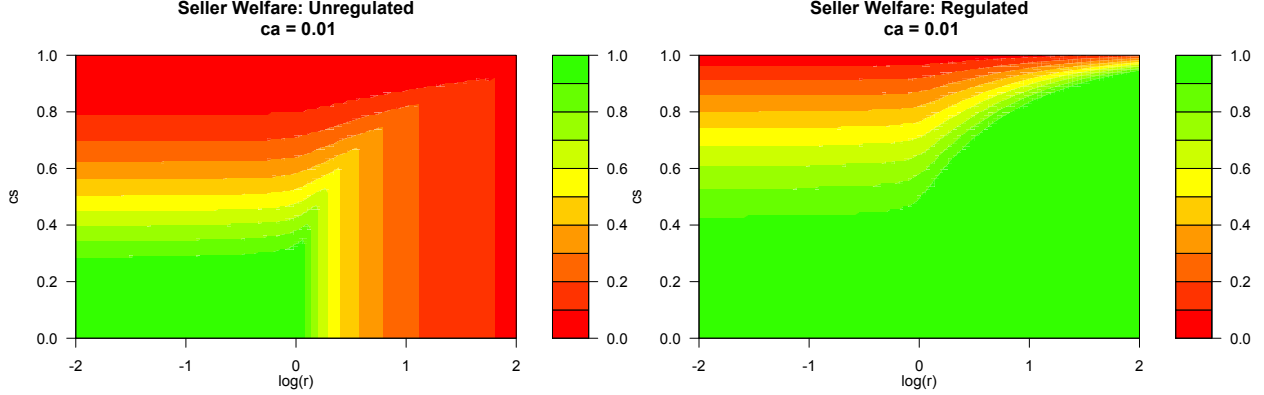
FIGURE 2. Normalized applicant welfare in equilibrium ($\Pi_a^*/\Pi_a^{\mathrm{opt}}$, left) and with an applicant-optimal application limit ($\sup_\ell \Pi_a^\ell/\Pi_a^{\mathrm{opt}}$, right). We fix the value $c_a' = 0.01$, and vary the values of $\log r$ (x-axis) and c_s' (y-axis). Note that normalized equilibrium applicant welfare is low whenever c_s' is high or r is large. In the latter case, intervention can substantially raise applicant welfare. We formalize these statements in Proposition 4.

lower application costs) is exactly offset by a reduction in screening by employers, such that the availability q (and the total number of matches formed) remains the same. On the other hand, if $c_s' < f(r, c_a')$, the market is *application-limited*: a marginal change in c_s' will have no effect on the number of matches formed, but a marginal decrease of c_a' will increase the equilibrium number of matches. This is because in this region employers receive positive welfare and never exit; reducing the screening cost thus cannot improve the likelihood an employer hires successfully, but increasing the number of applications can improve the likelihood of a match.

The fact that intervention ran raise employer welfare from zero to Π_e^{opt} follows from Proposition 2, which states that whenever a limit $\ell < m^*$ is enforced, applicants select $m = \ell$ in the equilibrium of the resulting game. If $\Pi_e^* = 0$, it follows that the employer-optimal choice of m is less than m^*, and thus Π_e^{opt} can be attained via appropriate intervention.

5.3. Welfare Analysis: Applicants.

In this section, we prove several theorems regarding equilibrium applicant welfare, and the welfare that is attainable through the intervention of enforcing an application limit. We preview our results using Figure 2, which displays $\Pi_a^*/\Pi_a^{\mathrm{opt}}$ as r and c_s' vary.

Figure 2 suggests that normalized applicant welfare in equilibrium is low whenever r is large. Of course, when $r > 1$, the fraction of applicants who eventually match is at most $1/r$. However, we normalize applicant welfare by Π_a^{opt}, which accounts for this fact. Applicants do badly *relative*

to Π_a^{opt} when r is large because in this case, they compete fiercely for the limited number of open positions, and many of their applications are not even read. This is the more widely studied congestion effect in matching markets: applicants find themselves in a "tragedy of the commons" due to the negative externality their applications impose on each other.

Figure 2 also suggests that equilibrium normalized applicant welfare is low whenever c_s' is large. There are two reasons for this effect. First, high screening costs ensure that relatively few employers screen and hire applicants (a fact that is not accounted for in the computation of Π_a^{opt}). Second, because many employers leave the marketplace without screening, the "effective" market imbalance (i.e. r/α^*) is large, so applicants again compete fiercely for limited spots. While enforcing an application limit cannot cause more applicants to match, it can eliminate the wasteful competition among applicants. For this reason, an appropriate application limit can always boost applicant welfare.

We formalize these observations in Proposition 4, which roughly states that applicant welfare is low whenever c_s' is large or r is notably above one. Proposition 4 also states that whenever employers all screen in equilibrium, intervention can boost applicant welfare to Π_a^{opt}.

Proposition 4.

Applicant welfare in equilibrium:

   (1) Let γ be the unique solution to $(1 - e^{-\gamma})/\gamma = c_s'$. Then $\Pi_a^* \leq 1 - (1 + \gamma)e^{-\gamma}$.
   (2) If $r > 1$, then $\Pi_a^* \leq \frac{1}{r}\left(1 - (r-1)\ln\left(\frac{r}{r-1}\right)\right)$ (Trivially, $\Pi_a^* \leq 1$ for $r \leq 1$.)

Applicant welfare with an application limit:

   (1) There always exists ℓ such that $\Pi_a^\ell > \Pi_a^*$.
   (2) If $c_s' \leq f(r, c_a')$, there exists ℓ such that $\Pi_a^\ell = \Pi_a^{\mathrm{opt}}$.

To illustrate the magnitude of this effect, we consider an example where $r = 1.4$ and application costs are near-zero. In this case, Proposition 4 implies that $\Pi_a^* \leq \frac{1}{2r}$, whereas an appropriate application limit ℓ can ensure that applicants match at minimal costs, so that $\Pi_a^\ell \approx \frac{1}{r}$. Thus, in this case, intervention can roughly double applicant welfare (without notably changing the number of matches that form). If $r = 1.9$, the same reasoning implies that intervention can triple applicant welfare.

5.4. **Pareto Improvements.**

The previous section separately addressed the employer and applicant welfare attainable through intervention. One natural concern is that while there may exist choices of ℓ that result in high employer welfare and choices of ℓ that result in high applicant welfare, these choices of ℓ might not coincide. In this section, we address this concern.

Proposition 5 states that whenever the market is "screening-limited", it is possible to choose a single application limit such that both employers and applicants are better off than they would be without any intervention. In such cases, we show that the tradeoff between optimizing for employers and for applicants cannot be too severe. In particular, Proposition 1 shows that if $c'_s \leq f(r, c'_a)$, there exists ℓ' such that $\Pi_e^{\ell'} \geq \frac{3}{4} \sup_\ell \Pi_e^\ell$ and $\Pi_a^{\ell'} \geq \frac{3}{4} \sup_\ell \Pi_a^\ell$. These bounds are tight in the limit where $c'_a \to 0$ and $c'_s \to 1$, but Figure 3 presents numerical results indicating that for many parameter values, this bound is overly pessimistic.

Proposition 5. *If $c'_s \geq f(r, c'_a)$, then there exists ℓ such that $\Pi_e^\ell > \Pi_e^*$ and $\Pi_a^\ell > \Pi_a^*$.*

One might wonder whether Pareto improvements are possible when the market is "application limited" (i.e. $c'_s < f(r, c'_a)$). The answer turns out to be, "not always." Indeed, if either r or c'_a is large enough, then availability never becomes a pressing concern to employers, and *any* binding application limit lowers employer welfare.[12] While one might conclude that enforcing an application limit is undesirable in these cases, Figure 3 shows numerically that an application limit can often substantially improve applicant welfare at little cost to employers.

In this figure, we plot the largest fraction δ such that there exists a single limit ℓ' where employers earn an expected surplus of $\delta \sup_\ell \Pi_e^\ell$ and applicants (simultaneously) earn an expected surplus of $\delta \sup_\ell \Pi_a^\ell$. Observe that δ is reasonably high across the entire figure. Conjecture1 provides a formal characterization of this observation: in particular, it states that $\delta \geq 3/4$.

Conjecture 1. *For all r, c'_a, c'_s, there exists ℓ' such that $\min\left(\frac{\Pi_a^{\ell'}}{\sup_\ell \Pi_a^\ell}, \frac{\Pi_e^{\ell'}}{\sup_\ell \Pi_e^\ell}\right) \geq \frac{3}{4}$.*

The intuition behind this conjecture is as follows. When the employer-optimal choice of m is less than the applicant-optimal value, then decreasing c'_a (thereby increasing the applicant-optimal m) or increasing c'_s (thereby decreasing the employer-optimal m) increases the tension between the two sides and decreases δ. Conversely, when the employer-optimal choice of m exceeds the applicant-optimal level, then increasing c'_a lowers δ. We have proofs that $\lim_{c'_s \to 1} \lim_{c'_a \to 0} \delta = 3/4$, and

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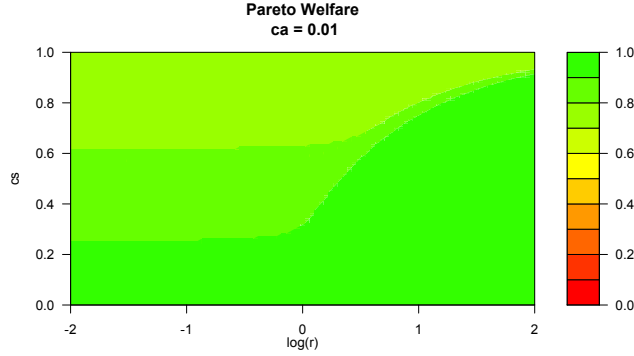[12]This holds, for example, if $c'_s < \max\{1 - 1/r, c'_a\}$.

FIGURE 3. For $c'_a = 0.01$ and varying values of c'_s and r, this figure shows the largest fraction δ such that there exists a single limit ℓ' for which employers earn an expected surplus of $\delta \sup_\ell \Pi^\ell_e$ and applicants (simultaneously) earn an expected surplus of $\delta \sup_\ell \Pi^\ell_a$. We conjecture that for all parameter values, $\delta \geq 3/4$.

$\lim_{c'_a \to 1} \delta = \frac{2+r}{1+r} - \frac{1}{4} \left(\frac{2+r}{1+r} \right)^2 \geq 3/4$. The result that $\delta \geq 3/4$ for all parameter values remains a conjecture because we lack a formal proof that these limiting cases are, in fact, worst cases (though numerical methods indicate that they are).

5.5. **An Alternate Intervention.** For reasons not modeled in this paper, placing a limit on the number of applications sent by each applicant may be impractical or undesirable. For example, this limit may be unenforcable if applicants can easily create multiple anonymous accounts. Additionally, a "one size fits all" limit may be too coarse if applicants vary in quality or in the number of jobs that they wish to hold. These thoughts motivate a second type of intervention that reduces the number of applications sent: raising the application cost.[13] This intervention would dispense with any incentive to create multiple accounts, and would allow for applicants to tailor their strategy to their own characteristics.

From the perspective of the employer, limiting applications and raising application costs have similar effects. To applicants, however, they look different. Although both interventions reduce the amount of competition for each opening, raising the application cost also directly harms applicants. A priori, it is not obvious which effect dominates. Proposition 6 shows that within our model,

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[13]This is effectively a "tax" on the externality that applicants impose on employers. This tax could take the form of an explicit monetary payment to the platform (presumably benefiting the platform, though we do not model the operator as a strategic agent), or could simply be an additional form or questionnaire that the applicant must complete for each employer contacted.

raising application costs can only hurt applicants.[14] Additionally, we formalize the idea that the two interventions are equivalent from the perspective of employers.

Proposition 6. *Fix values of r, c_s, β satisfying $c_s < \beta$, and let $\Pi_e^*(c)$ and $\Pi_a^*(c)$ be employer and applicant welfare in the MFE when $c_a' = c$. Then*

- *For each $\ell \in \mathbb{R}_+$, there exists a unique $\tilde{c}_a' \geq c_a'$ such that $\Pi_e^\ell = \Pi_e^*(\tilde{c}_a')$.*
- *For each $\tilde{c}_a' > c_a'$, there exists a unique $\ell \in \mathbb{R}_+$ such that $\Pi_e^\ell = \Pi_e^*(\tilde{c}_a')$.*
- *$\Pi_a^*(c)$ is a decreasing function.*

Though we focus primarily on the intervention of imposing an application limit, Proposition 6 establishes that in cases where the intent of the intervention is to help employers, raising application costs is a viable alternative approach. In practice, the relative merits of these two forms of intervention depend on a number of considerations not modeled in this paper, and choice between them should be determined by the operator's objectives and capabilities.

6. Discussion

In this section we interpret our analytical results, and also discuss a range of observations, extensions, and open directions related to our analysis. The following are our main insights.

(1) *No intervention: Employer welfare can drop to zero.* A major contribution of our paper is to study the effect of a lack of availability information. In congested matching markets, our results show this effect can be severe enough to drive employer welfare to zero. In particular, Proposition 3 reveals that if either r is low or c_s' is sufficiently high, then employers waste so much screening effort trying to find an available applicant, that equilibrium welfare is zero. Note that in particular, this effect is exacerbated when applicants are in short supply.

(2) *No intervention: The applicants' see a "tragedy of the commons".* Proposition 4 reveals that applicant welfare is bounded away from $\min\{1, 1/r\}$, *regardless* of application costs. The loss of welfare is particularly severe when either r or c_s' is large. This recovers the first order effect of congestion among applicants in a matching market: they do not internalize their competition externality on each other. This loss of welfare remains even as application

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[14]We assume that the costs paid by applicants cannot be redistributed. This is reasonable if the increased cost is not a monetary transfer, but instead an additional barrier to application (such as additional questions that each applicant must answer). Even when the additional cost is monetary, redistribution may be logistically challenging, create incentives for individuals to create multiple accounts, and may be undesirable from the point of view of the platform operator.

costs go to zero, because the increase in applications sent offsets the reduction in application costs.

(3) *The value of intervention: Constrained efficiency.* If we are able to limit the number of applicant applications, a dramatically different picture emerges. Whenever employer welfare is zero in the market without intervention, we can use an application limit to raise employer welfare to their constrained efficient benchmark (cf. Proposition 3). For applicants, Proposition 4 reveals that application limits always help, and in appropriate regimes can raise applicant welfare to their constrained efficient benchmark as well.

(4) *Pareto improvements.* In light of the preceding results, a natural question arises: how does helping one side of the market affect the other? Proposition 5 shows that whenever employer welfare is zero in the market without intervention, there exists a *single* choice of application limit that is Pareto improving: both sides of the market are strictly better off.

In fact, we show numerically that a large fraction (3/4) of the constrained efficient benchmark can be obtained for both sides of the market at once. In other words, tradeoffs between optimizing for employers and applicants are minimal.

We conclude the paper by discussing the implications of some of our modeling assumptions, as well as some open directions for future work.

Dynamic vs. static models. Ours is not the first work to conclude that restricting visibility may prove beneficial, but our dynamic model provides quantitative insights that differ substantially from those of the static models that preceded it.[15] One important distinction between this line of work and our own is that in most existing work on availability, firms are uncertain about the *preferences* of workers, and risk having their offers turned down by workers who prefer other firms.[16] In our setting, buyers and sellers are ex-ante homogeneous, and this effect is absent. The uncertainty facing buyers is one of *timing*; they know that their offer will be accepted, so long as the seller is not already employed. Our work demonstrates that interventions similar to those proposed to address preference-related frictions can also reduce timing-related frictions and enhance the welfare of agents on both sides of the market.

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[15]See, for example, Halaburda and Piskorski (2010). Further, Coles et al. (2010) discuss the recent introduction of a signaling mechanism with limited messaging into the market for students graduating with PhDs in economics, and empirically analyze its effects. Lee et al. (2011) study the benefits of a related intervention in the context of an online dating market. Meanwhile, the idea of credible signaling mechanisms has been explored theoretically by Coles et al. (2013) and Lee and Schwarz (2007).

[16]For example, in Coles et al. (2013), one driver of improved worker welfare is that when signals are used, workers receive "better" offers.

Even in the absence of preference heterogeneity, our conclusions are substantially different than those obtained in a static setting. In particular, in some parameter regimes, we find a stark dichotomy: our model concludes that lowering application costs can reduce employer welfare to zero, while a static model would conclude that lowering application costs only improves employer welfare. Here we briefly present a comparison to illustrate this point.

For concreteness, we consider a "one-shot" version of our dynamic model: applicants choose m, and apply to (in expectation) m employers; employers screen applicants, and make at most one offer; and applicants accept an offer at random among those received. A mean field analysis of this model reveals that if $c'_s < e^{-1/r}$, then employer welfare is increasing in m; see Appendix B. This suggests that minimizing search frictions (the costs c_a, c_s) is a reasonable proxy for maximizing welfare. Our results contrast with this conclusion in a fundamental manner: whenever $r \leq 1$, employer welfare is unimodal in the number of applications sent by applicants, and becomes zero if the number of applications is sufficiently large. Correspondingly, we conclude that reductions in application costs can severely reduce the marketplace's efficiency.

This dichotomy arises because static models, while admitting the possibility that employer offers will be rejected, severely understate the likelihood of this event. Note that in the static model, employers send at most one offer (regardless of the number of applications sent by applicants). Because agents have limited ability to coordinate, unless there are far more employers than applicants, a large number of applicants will go unmatched. From the perspective of any particular employer, if he makes an offer to a random applicant, there is a reasonable chance that she has no other offers and therefore will accept.

In practice, many labor markets (and more generally, matching markets) operate asynchronously, and it seems reasonable to expect that employers whose initial offers are turned down will make an offer to a second candidate. Our model, because it is dynamic, allows for this possibility; as discussed in Section 5, this effect imposes a negative externality on future employers who may choose to screen the same applicant. Because employers pay a cost for each applicant screened, this can dramatically lower employer welfare.[17]

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[17]It's worth noting that Kircher (2009) incorporates the idea of allowing employers to make multiple offers into a static model by assuming that applicants send applications, and then a stable match is formed (so that in particular, there is no employer-applicant pair (b, s) such that a applied to e and both remain unmatched). One key difference between his model and ours is that he assumes that the stable match forms costlessly (viewed in our setting, he sets $c_s = 0$).

Applicant behavior. Applicant behavior in our model has two features that can be relaxed, though at the cost of significant additional technical complexity. First, in our model applicants cannot directly choose the number of applications that they send; rather, they apply to each employer independently with probability m/n. Our primary motivation for this choice was technical convenience; allowing applicants to exactly select the number of employers to whom they apply introduces an additional source of correlation that further complicates the proofs of Theorems 3 and 4. While allowing applicants to exactly select how many applications to send complicates the approximate equilibrium result, the mean-field analysis for this alternative model remains quite tractable: the key distinction is that while in our model the number of applications an applicant sends is Poisson distributed with mean m, in the alternate model the number of applications an applicant sends is exactly m. As a result, the probability an applicant receives an offer becomes $1 - (1-p)^m$, instead of $1 - e^{-mp}$. This does not substantively change our results.

Second, note that in our model applicants cannot condition on the number of other applicants that apply to a given job. Again, this simplifies our technical arguments. If applicants do see the number of other applicants on a given job, then in the mean field analysis each employer would receive *exactly rm* applications (while in our model, the number of applications received by an employer is Poisson distributed with mean rm). Analogous to our discussion above, the probability that an employer matches in the alternate model is $1 - (1 - \beta q)^{rm}$, as opposed to $1 - e^{-rm\beta q}$ in our model. Again, this does not substantively change our results.

Heterogeneity. In our model, the probability of compatibility β is assumed to be constant across employers and applicants. In practice, we expect that employers do not look identical to applicants, and that applicants (or the platform) can direct applications to the most "promising" openings; in this case an application limit encourages applicants to apply to the openings for which they are the best fit. Under suitable assumptions, this could be captured in our model by having $\beta = \beta(m)$ being a decreasing function of m. In such a setting, smaller m would lead to a larger β, leading to an additional improvement in welfare. It is somewhat surprising that limiting visibility helps even when the platform has no agent-specific or match-specific information about compatibility, and applications are sent randomly by uninformed agents; this only strengthens our belief in the potential benefits of limiting applications.

Compatibility. In our model, we assume that each applicant is compatible with a given employer with probability β, i.i.d. across all employer-applicant pairs. In fact, we can relax this

definition, and all our results hold if we only assume that *conditional on applicant a having applied to an employer e*, applicant a is compatible with probability β, independent of other applicants to employer e. In this description of the model, with a larger β, the compatibility "graph" is biased so that a larger fraction of applicants to an employer are compatible with her.

With this view of compatibility in our model, β naturally captures the role of improved recommendation and/or search algorithms used by the platform. In other words, as β rises, we can view the resulting application dynamics as capturing the notion that an employer receives a greater number of applicants who are likely to be a good fit for her. In our paper we have not focused on the role of compatibility, but given the central role of search and recommendations in online matching platforms, it would be worthwhile to compare the potential welfare benefits of improved compatibility with the benefits of application limits.

Symmetric behavior. In our model, the applicants make the applications and employers do the screening. An important direction for future work is to instead consider a symmetric model where any agent on *either* side of the market can initially submit applications as well as post her availability, and later return to screen applicants if she does not receive any offers. While we believe our current model is appropriate for a range of markets where asymmetric screening and application behavior is inherent to the platform, a symmetric model may be more appropriate for settings where no such constraint exists *a priori*.

Transfers. Our model does not capture the possibility of monetary transfers, e.g., wages. Numerous papers (Moen, 1997; Acemoglu and Shimer, 1999; Kircher, 2009) have observed that endogenously determined wages can guide the market towards efficiency. These results do not extend to our setting directly. For example, Moen (1997) and Acemoglu and Shimer (1999) assume that workers apply to a single firm, precluding the possibility that they might be unavailable. Kircher (2009) allows workers to apply to multiple firms, but the efficiency of equilibrium crucially relies on the assumption that firms can (costlessly) contact all applicants to find one who is available. Nevertheless, we believe a model that also includes endogenously determined wages could prove to be an illuminating direction for future work.

Observability of availability. Our model explores the implications of the fact that availability is unobservable. This prompts a natural potential solution: the platform could just expose the fact that the match has formed, and that the applicant is "unavailable." One challenge with this approach is that in many online platforms, properly detecting availability is a difficult problem.

For example, on the online labor market oDesk, Horton (2014) shows that workers have an extremely wide distribution of hours worked per week on the platform, so that the amount of work a freelancer wants to take on can be difficult to forecast. Similar issues arise in other examples such as dating markets, where the platform has imperfect observability of availability. In particular, the simple intervention studied in our paper—limiting applications—is often much more feasible than a (potentially quite costly and imperfect) determination of the availability of an applicant.

Rather than trying to infer availability, the platform could solicit this information from its users. While some sites (for example, AirBnB) do just that, this approach comes with its own challenges. To encourage truthful reporting, there must be some form of punishment for those who turn down offers after declaring themselves available. However, in markets with heterogeneous preferences, punishing users who decline invitations may prove unpalatable.

Furthermore, even if availability is inferred by the platform (or self-reported by users), this information may arrive too late to influence the screening process. Although our model represents screening as happening instantaneously, this is largely a technical device; in practice of course, screening takes time, and there may be several employers screening the same applicant at the same time. In such a situation, the platform can't inform other employers that an applicant is unavailable, because a match has not yet been consummated. As a result, employers will expend effort screening applicants who may ultimately be off the market. In our model, we capture this phenomenon by ensuring that employers do not see their applicants' arrival times, and must screen (at some cost) before learning availability.

If instead we wanted to study a market where availability information is easily acquired, we might consider a model where employers learn the arrival time of an applicant who has applied to them, or even instantaneously learn when applicants have matched. Our focus in this paper is instead on modeling the large range of practical situations where employers do not have reliable information on availability at the time of screening. As noted through our discussion above, a lack of availability information is a prevalent issue affecting a range of marketplaces; our work highlights both the negative consequences of this issue in congested markets, as well as simple interventions to alleviate these effects.

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Appendix A. A utility-theoretic model of employer screening

In the model presented in the paper, we have assumed that employers must screen a candidate before they can make an offer to that applicant. This is a "mechanical" restriction, made because it is plausible and simplifies the technical presentation of our results. Notably, it does not explicitly model features of employer utility that lead to such behavior. In this section, we provide a utility-theoretic model of employers, together with reasonable assumptions on employer utility, that ensure that they would always screen before making an offer.

Let c_h be the cost to the employer of hiring an applicant (irrespective of compatibility). Let u be the benefit to an employer of hiring at least one compatible applicant. Then the net utility from hiring a single compatible applicant (and not hiring anyone else) is $v = u - c_h$, whereas the utility of hiring an applicant who is not compatible is $-c_h$.

Suppose employers can choose to make an offer without screening (at the risk of hiring someone incompatible). We argue that an employer would never want to do this if $\beta u < c_h$. Consider the dynamic decision problem faced by an employer upon exit. The state of this problem consists of the outcomes on all applicants a screened thus far: whether a was screened, and if so, whether a was compatible; and whether a accepted or rejected an offer (if an offer was made). The employer chooses whether to exit, or to skip, screen, or make an offer to the next applicant in her list.

Suppose the next applicant in the list is a'. We assume the continuation value of not exiting is nonnegative; otherwise the result is vacuous. We argue that the myopic net utility from making an offer to a' without screening is negative; and further, the continuation value cannot be increased by making an offer without screening.

- The myopic expected utility from hiring an applicant a' without screening her is $\beta u - c_h$, if no compatible applicant has been hired yet (otherwise the utility is even smaller). Recall that compatibility with the applicant is independent of everything else (including whether the applicant will accept an offer), and only the employer can learn compatibility. It follows that the myopic expected utility from making an offer to an applicant without screening is negative if $\beta u < c_h$, as long as there is non-zero likelihood that the offer will be accepted.

- Now consider the continuation value of making an offer to a without screening. We argue that even if the employer were to learn the compatibility of a at no cost after the offer has been made, the continuation value cannot increase. If a rejects the offer or is incompatible, the continuation value (weakly) decreases since the employer now has one fewer applicant

remaining; if a accepts the offer and is compatible, then the continuation value is zero, which again means that it does not increase.

In the dynamic decision problem faced by the employer, therefore, it cannot be optimal to make an offer without screening, since the resulting expected utility in the dynamic decision problem will be less than the (optimal) continuation value at the current state.

This justifies our mechanical restriction that an employer must screen before making an offer to an applicant, under the reasonable condition $\beta u < c_h$. Informally, this condition will hold when the cost of matching to an applicant is high. In particular, this cost is likely to be quite high when the employer is only interested in matching to at most one applicant.

Appendix B. Static model

Consider a "one-shot" version of our dynamic model: applicants choose m, and apply to each employer independently with probability m/n; employers screen applicants, and make at most one offer; and applicants accept an offer at random among those received.

A mean field analysis of this model can be carried out as before. We make mean field assumptions very similar to those in Section 3: Assumptions 2 and 3 are unchanged, and Assumption 1 is slightly modified to "Each applicant in an employer's applicant set will accept an offer from the employer with probability q, independently across applicants in the applicant set." Again assume that each employer screens with probability α. The consistency condition for p is slightly modified to

$$p = \alpha\beta g(rm\beta)\,,$$

where $g(x) = (1 - e^{-x})/x$ and the argument of $g(\cdot)$ now does not contain the term q since employers do not screen for availability (the absence of q from this condition causes the static model to behave very differently from our dynamic model, as we see below). The consistency condition for q remains

$$q = g(mp)\,,$$

leading to

$$q = g(\alpha(1 - e^{-rm\beta})/r)\,.$$

Employer welfare, when $\alpha = 1$ is

$$(1 - e^{-rm\beta})(q - c'_s) = (1 - e^{-rm\beta})(g((1 - e^{-rm\beta})/r) - c'_s) = r(1 - e^{-(1-e^{-rm\beta})/r}) - c'_s(1 - e^{-rm\beta})$$

$$= r(1 - e^{-y/r}) - c'_s y,$$

where $y = 1 - e^{-rm\beta} < 1$. Note that y is monotone increasing in m. The derivative of the employer welfare with respect to y is $e^{-y/r} - c'_s > e^{-1/r} - c'_s$. Thus, under the relatively mild condition $c'_s < e^{-1/r}$, the employer welfare is monotone increasing with respect to y, and hence with respect to m, for $\alpha = 1$. (For values of m such that the employer welfare is negative with $\alpha = 1$, the equilibrium value of α will be less than 1 such that the resulting employer welfare is 0.) This is in contrast with our dynamic model, where whenever $r \leq 1$, employer welfare is unimodal in m, and becomes zero if m is sufficiently large. Thus, the static model does not capture that reductions in application costs can severely reduce the marketplace's efficiency. See Section 6 for a further discussion of this issue.

Appendix C. Proofs: Section 3

C.1. **Section 3.2: Mean Field Steady States.** In this appendix we study mean field steady states (MFSS), i.e., solutions (p, q) to the equations (4) and (5). We prove Theorem 1, which states that for any r, β, m, α, there exists a unique MFSS. We also show in Lemma 3 that the MFSS values p and q are monotonic in m and α. This will be useful in proving results about mean field equilibria in Section C.2.

For notational convenience, we define the function g as follows:

$$(11) \qquad g(0) = 1, \quad g(x) = \frac{1 - e^{-x}}{x} \text{ for } x > 0.$$

Lemma 1. *The function $g : [0, \infty) \to (0, 1]$ defined by (11) is continuous and strictly decreasing. Further:*

(1) $g'(x) \geq -g(x)/x$ *and* $g''(x)/g'(x) > -1$ *for* $x > 0$;
(2) $\lim_{x \to 0} g'(x) = -1/2$, *and* $\lim_{x \to 0} g''(x) = 1/3$;
(3) $e^{-g^{-1}(c)} \leq c^2$ *for* $0 < c < 1$.

Proof of Lemma 1. Note first that g is continuous at zero. Differentiating, for $x > 0$ we have

$$(12) \qquad g'(x) = \frac{(1 + x)e^{-x} - 1}{x^2} = \frac{e^{-x} - g(x)}{x}.$$

Monotonicity of g follows from applying the inequality $1 + x < e^x$ for $x > 0$, which implies (on rearranging terms) that $e^{-x} < g(x)$. Having established that the expressions in (12) are negative, it follows that $g'(x) > -g(x)/x$ for $x > 0$.

We now prove that $g''(x)/g'(x) > -1$, which is equivalent to $g'(x) + g''(x) < 0$ (the inequality has reversed because $g' < 0$). Basic algebra reveals that

$$(13) \qquad -x(g'(x) + g''(x)) = (g(x) + 2g'(x))$$

$$(14) \qquad = \frac{1}{x^2}\left((2+x)e^{-x} + x - 2\right)$$

Thus, $g'(x) + g''(x) < 0$ if and only if $(2+x)e^{-x} + x - 2 > 0$. Differentiating this expression, we see that

$$\frac{d}{dx}\left((2+x)e^{-x} + x - 2\right) = 1 - e^{-x}(1+x) \geq 0,$$

so the expression is minimized at $x = 0$, when it takes the value zero.

If we apply L'Hospital's rule to (12), we obtain

$$\lim_{x \to 0} g'(x) = \lim_{x \to 0} \frac{e^{-x} - g(x)}{x} = \lim_{x \to 0} -e^{-x} - g'(x).$$

Rearranging, we see that $2\lim_{x \to 0} g'(x) = -1$, implying that $\lim_{x \to 0} g'(x) = -1/2$. Rearranging (13) reveals that

$$(15) \qquad \lim_{x \to 0} g''(x) = \lim_{x \to 0} -g'(x) - \frac{g(x) + 2g'(x)}{x} = \frac{1}{2} - \lim_{x \to 0}(g'(x) + 2g''(x)),$$

where we have used the fact that $\lim_{x \to 0} g'(x) = -1/2$ and applied L'Hospital's rule. Rearranging (15), we see that $3\lim_{x \to 0} g''(x) = 1$.

Finally, by rearranging we observe that $e^{-g^{-1}(c)} \leq c^2$ if and only if $1 - c^2 + 2c \log c \geq 0$. It is straightforward to check that the expression $1 - c^2 + 2c \log c$ is decreasing in c, and is equal to zero at $c = 1$. $\qquad \square$

Proof of Theorem 1. Recall from (4)-(5) that given r, m, β, α, we define a MFSS as any solution (p, q) to

$$(16) \qquad p = \alpha \beta g(rm\beta q), \qquad q = g(mp).$$

Note that the preceding system trivially has a single solution when $m = 0$: namely, $p = \alpha\beta$ and $q = 1$. Therefore we focus on the case where $m > 0$. We proceed by showing that the preceding system has a unique solution.

For this purpose it is useful to rewrite (16) as providing two functions that yield p in terms of q. In particular, for $m > 0$, we note that (p, q) is an MFSS if and only if $p = p_1(q) = p_2(q)$, where $p_1, p_2 : [g(m), 1] \to [0, 1]$ are defined by

$$(17) \qquad p_1(x; m, \alpha) = \alpha\beta g(rm\beta x), \qquad p_2(x; m) = g^{-1}(x)/m.$$

Here the notation $f(x; y)$ indicates that the function f is parameterized by y. The chosen lower bound of $g(m)$ for the domain arises from the fact that any mean-field steady-state (p, q) corresponding to (m, α) must satisfy $q = g(mp) \geq g(m)$ (since g is strictly decreasing).

We show in Lemma 2 that for $m > 0$ and any r, β, α, there is a unique point where p_1 and p_2 intersect; this point is the unique MFSS. $\qquad\square$

The following lemma, used in the preceding proof, also establishes some useful properties of MFSS.

Lemma 2. *Fix r, β, and α, and $m > 0$. Then there exists a unique pair $(p, q) \in [0, \alpha\beta] \times [g(m), 1)$ such that $p = p_1(q; m, \alpha) = p_2(q; m)$.*

Furthermore: (1) the functions p_1 and p_2 defined in (17) are monotonically decreasing in q; and (2) for $q' < q$, we have $p < p_1(q'; m, \alpha) < p_2(q'; m)$; and for $q' > q$, we have $p > p_1(q'; m, \alpha) > p_2(q'; m)$.

Proof. For the duration of the proof we suppress the dependence of p_1 and p_2 on α and m.

Note that on the domain $[g(m), 1)$, p_1 begins "below" p_2 and finishes "above" it, i.e. (because $g(\cdot) \leq 1$ by Lemma 1), we have $p_1(g(m)) \leq \alpha\beta < 1 = p_2(g(m))$ and $p_1(1) > 0 = p_2(1)$. Since p_1 and p_2 are continuous, this implies that a mean-field steady-state exists. This is illustrated in Figure 4.

Once we know that the intersection of p_1 and p_2 is unique, the final claim of the lemma follows immediately. Thus, all that remains to show is that p_1 and p_2 have a unique intersection point.

Note that because g is decreasing by Lemma 1, so are both p_1 and p_2. To show uniqueness, we will show that whenever p_1 and p_2 intersect, the curve p_2 is decreasing more "steeply," i.e. $\frac{\partial p_1}{\partial x} > \frac{\partial p_2}{\partial x}$. Equivalently, we wish to show that whenever $p_1(q) = p_2(q) = p$, we have $\frac{\partial p_1}{\partial x}(q)/\frac{\partial p_2}{\partial x}(q) < 1$ (the
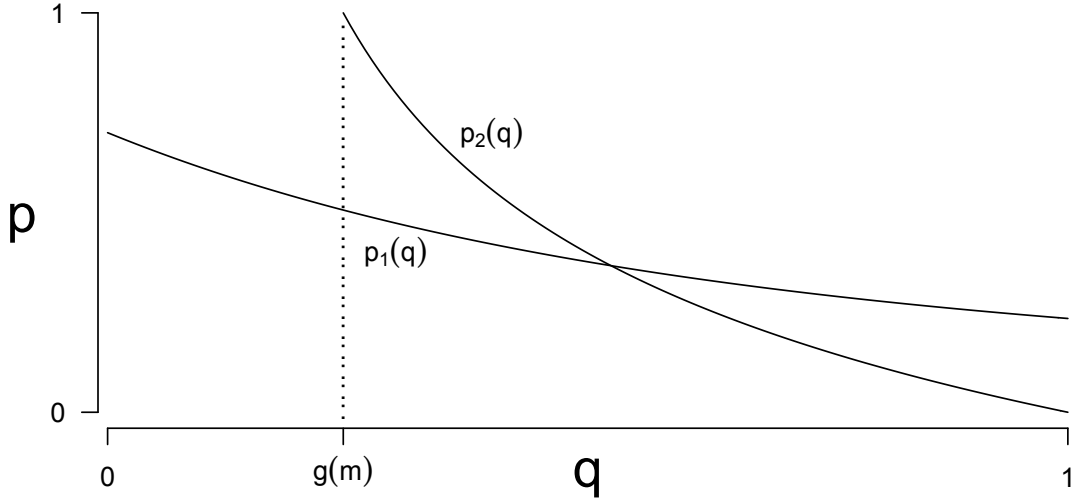
FIGURE 4. A depiction of $p_1(q)$ and $p_2(q)$; we prove in Lemma 2 that they have a unique intersection point.

inequality reverses because both partials are negative). But

$$(18) \qquad \frac{\partial p_1}{\partial x}(q) \Big/ \frac{\partial p_2}{\partial x}(q) = \big(\alpha \beta r m \beta g'(rm\beta q)\big)\big(mg'(mp)\big)$$

$$(19) \qquad < \left(\frac{\alpha \beta g(rm\beta q)}{q}\right)\left(\frac{g(mp)}{p}\right).$$

$$(20) \qquad = \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = 1.$$

The first line follows from implicit differentiation of p_2; the second from the inequality $|g'(x)| < g(x)/x$ proven in Lemma 1; and the third from the fact that (p,q) is a mean-field steady-state, i.e. $p_1(q) = p_2(q) = p$. $\qquad\square$

Having established the existence of a unique mean-field steady-state, it will be useful when proving later results to understand how $\mathcal{P}(m,\alpha)$ and $\mathcal{Q}(m,\alpha)$ vary with m and α. We require the following identity, which can be easily derived by rearranging (4)-(5):

$$(21) \qquad \alpha(1 - e^{-rm\beta q}) = r(1 - e^{-mp})$$

There is a straightforward interpretation of this identity. Each employer matches if and only if they screen (which they do with probability α), and they have a qualified available applicant (which occurs with probability $1 - e^{-rm\beta q}$, because each applicant is available with probability q and qualified with probability β). On the other hand, each applicant matches if and only if one of

her applications receives an offer (which occurs with probability $1 - e^{-mp}$). Because the number of employers and applicants who match must be equal, we conclude that if p and q are the MFSS consistent with strategies m and α, then (21) must hold.

The following lemma provides some monotonicity properties of $\mathcal{P}(m, \alpha)$ and $\mathcal{Q}(m, \alpha)$.

Lemma 3. *For $(m, \alpha) \in (0, \infty) \times (0, 1]$, the function $\mathcal{P}(m, \alpha)$ is strictly decreasing in m and strictly increasing in α, and $\mathcal{Q}(m, \alpha)$ is striclty decreasing in m and in α. For $\alpha > 0$,*

$$\lim_{m \to \infty} \mathcal{P}(m, \alpha) = \alpha \beta f_2(\alpha/r), \quad \mathcal{P}(0, \alpha) = \alpha \beta.$$

$$\lim_{m \to \infty} \mathcal{Q}(m, \alpha) = f_2(r/\alpha), \quad \mathcal{Q}(0, \alpha) = 1,$$

where f_2 is defined by

(22)
$$f_2(x) = \begin{cases} 0 & : x \leq 1 \\ g(\log \frac{x}{x-1}) & : x > 1. \end{cases}$$

Proof. Define the functions p_1 and p_2 as in (17). Recall from Lemma 2 that $\mathcal{P}(m, \alpha)$ and $\mathcal{Q}(m, \alpha)$ are the unique solution (p, q) to $p_1(q; m, \alpha) = p_2(q; m) = p$. For $x < \mathcal{Q}(m, \alpha)$, we have $p_1(x) < p_2(x)$ and for $x > \mathcal{Q}(m, \alpha)$, we have $p_1(x) > p_2(x)$.

We first prove the statements about monotonicity in α. Suppose that $\alpha < \alpha'$. Then $p_2(\mathcal{Q}(m, \alpha); m) = p_1(\mathcal{Q}(m, \alpha); m, \alpha) < p_1(\mathcal{Q}(m, \alpha); m, \alpha')$. It follows immediately from Lemma 2 that $\mathcal{Q}(m, \alpha) > \mathcal{Q}(m, \alpha')$. Furthermore, since $\mathcal{P}(m, \alpha) = p_2(\mathcal{Q}(m, \alpha); m)$ and $\mathcal{P}(m, \alpha') = p_2(\mathcal{Q}(m, \alpha'); m)$ and p_2 is monotonically decreasing, this implies that $\mathcal{P}(m, \alpha) > \mathcal{P}(m, \alpha')$.

We now prove monotonicity with respect to m. It follows from (21) that for fixed r and $\alpha > 0$,

(23)
$$m'\mathcal{Q}(m', \alpha) < m\mathcal{Q}(m, \alpha) \Leftrightarrow m'\mathcal{P}(m', \alpha) < m\mathcal{P}(m, \alpha).$$

Let $m' > m$, and suppose that $m'\mathcal{P}(m', \alpha) < m\mathcal{P}(m, \alpha)$. It follows that

$$\mathcal{Q}(m', \alpha) = g(m'\mathcal{P}(m', \alpha)) > g(m\mathcal{P}(m, \alpha)) = \mathcal{Q}(m, \alpha),$$

which contradicts (23). Therefore, our supposition was incorrect; it must be that both $m'\mathcal{P}(m', \alpha) > m\mathcal{P}(m, \alpha)$ and $m'\mathcal{Q}(m', \alpha) > m\mathcal{Q}(m, \alpha)$. By definition (see (4) and (5)), we have $\mathcal{Q}(m, \alpha) = g(m\mathcal{P}(m, \alpha))$ and $\mathcal{P}(m, \alpha) = \alpha \beta g(r\beta m\mathcal{Q}(m, \alpha))$. Applying the fact that g is decreasing (see Lemma 1), we conclude that $\mathcal{Q}(m', \alpha) < \mathcal{Q}(m, \alpha)$ and $\mathcal{P}(m', \alpha) < \mathcal{P}(m, \alpha)$, as claimed.

Having established monotonicity of $\mathcal{P}$ and $\mathcal{Q}$, we move on to evaluating their limits as $m \to \infty$. When $r < \alpha$, (21) implies that $1 - e^{r\beta m \mathcal{Q}(m,\alpha)} \le r/\alpha$, so $m\mathcal{Q}(m,\alpha)$ is bounded above. This implies both that $\mathcal{Q}(m,\alpha) \to 0$ as $m \to \infty$ and that $\mathcal{P}(m,\alpha) = \alpha\beta g(r\beta m \mathcal{Q}(m,\alpha))$ is bounded away from zero. It follows from (21) that

$$\lim_{m\to\infty} 1 - e^{-r\beta m \mathcal{Q}(m,\alpha)} = \lim_{m\to\infty} \frac{r}{\alpha}(1 - e^{-m\mathcal{P}(m,\alpha)}) = \frac{r}{\alpha},$$

so $r\beta m \mathcal{Q}(m,\alpha) \to -\log(1 - \frac{r}{\alpha}) = \log\left(\frac{\alpha/r}{\alpha/r-1}\right)$ and $\mathcal{P}(m,\alpha) = \alpha\beta g(r\beta m \mathcal{Q}(m,\alpha)) \to \alpha\beta f_2(\alpha/r)$.

Analogously, when $r > \alpha$, (21) implies that $1 - e^{-m\mathcal{P}(m,\alpha)} \le \alpha/r$. This means that $m\mathcal{P}(m,\alpha)$ is bounded above, so $\mathcal{P}(m,\alpha) \to 0$ and $\mathcal{Q}(m,\alpha) = g(m\mathcal{P}(m,\alpha))$ is bounded away from zero. Applying (21) we get that

$$\lim_{m\to\infty} 1 - e^{-m\mathcal{P}(m,\alpha)} = \lim_{m\to\infty} \frac{\alpha}{r}(1 - e^{-rm\beta \mathcal{Q}(m,\alpha)}) = \frac{\alpha}{r},$$

so $m\mathcal{P}(m,\alpha) \to \log\left(\frac{r/\alpha}{r/\alpha-1}\right)$ and $\mathcal{Q}(m,\alpha) = g(m\mathcal{P}(m,\alpha)) \to f_2(r/\alpha)$.

Finally, when $r = \alpha$, (21) implies that $\alpha\beta\mathcal{Q}(m,\alpha) = \mathcal{P}(m,\alpha) = \alpha\beta g(r\beta m \mathcal{Q}(m,\alpha))$. Since g is strictly decreasing, $g(r\beta mq) \le g(r\beta m)$, and $g(r\beta m) \to 0$ as $m \to \infty$. We conclude that $g(r\beta mq) \to 0$ uniformly in q as $m \to \infty$; since $\mathcal{Q}(m,\alpha)$ is the solution to $q = g(r\beta mq)$, it follows that $\mathcal{Q}(m,\alpha) \to 0$ as well as $m \to \infty$. Since $\mathcal{P}(m,\alpha) = \alpha\beta\mathcal{Q}(m,\alpha)$, this implies $\mathcal{P}(m,\alpha) \to 0 = f_2(1)$, completing the proof. $\qquad\square$

C.2. **Section 3.3: Mean Field Equilibrium.** We now prove Theorem 2, which states that mean field equilibria exist and are unique.

Proof of Theorem 2. We prove the theorem in two steps. First, we fix the employer strategy to be ϕ^α, and allow applicants to respond optimally. We show in Proposition 4 that for any $\alpha \in [0,1]$, there exists a unique strategy m such that

(24) $$m = \mathcal{M}(\mathcal{P}(m,\alpha)).$$

Let m_α denote the unique choice of m satisfying (24).

Second, we endogenize the employer's choice of α; Lemma 7 shows that there is exactly one value of α such that $\alpha \in \mathcal{A}(\mathcal{Q}(m_\alpha,\alpha))$, i.e. such that (m_α,α) is a mean field equilibrium. $\qquad\square$

We start with the following lemma, regarding "partial" equilibrium among the applicants given a fixed value of α.
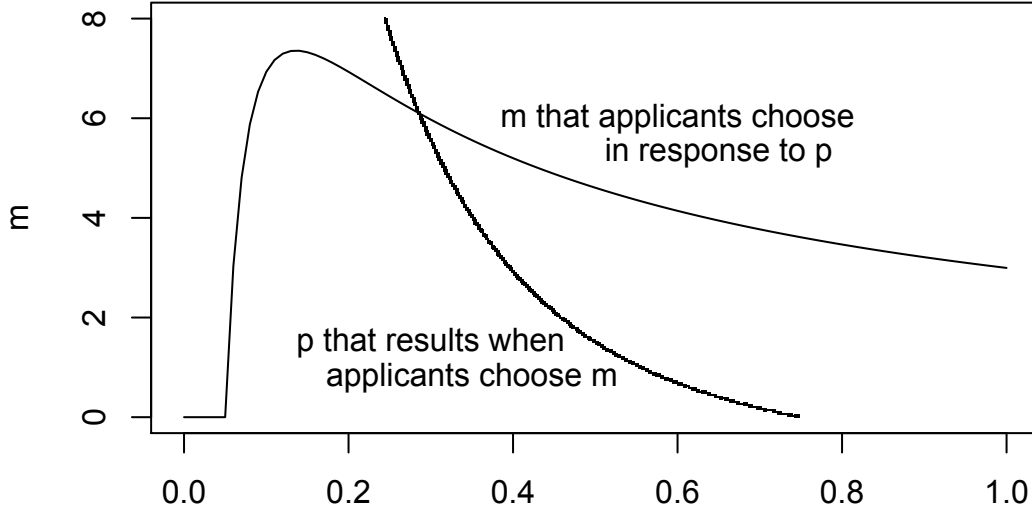
FIGURE 5.   A visualization of Proposition 4. The value $\mathcal{P}(m, \alpha)$ that results when applicants choose m is the solution to (26). Proposition 4 states that this line intersects with the applicant best response function $\mathcal{M}(p)$ at a unique point.

Lemma 4. *Given $\alpha \in [0, 1]$, there exists a unique value m_α satisfying:*

$$(25) \qquad\qquad m_\alpha = \mathcal{M}(\mathcal{P}(m_\alpha, \alpha)).$$

Furthermore: (1) $m_\alpha = 0$ if and only if $\alpha \leq c'_a$; and (2) $\mathcal{P}(m_\alpha, \alpha)$ is strictly increasing in α.

Proof. Refer to Figure 5. Any solution to (24) is equivalent to finding a solution to the following system of equations:

$$p = \mathcal{P}(m, \alpha); \quad m = \mathcal{M}(p).$$

As discussed in Section 3.2, the value $\mathcal{P}(m, \alpha)$ is the (unique) solution to

$$(26) \qquad \mathcal{P}(m, \alpha) = \alpha\beta g(rm\beta g(m\mathcal{P}(m, \alpha))) = \alpha\beta g(r\beta(1 - e^{-m\mathcal{P}(m,\alpha)})/\mathcal{P}(m, \alpha)).$$

We now incorporate the fact that m should be optimally chosen by applicants. Because $g(x) \leq 1$ (see Lemma 1), we have $\mathcal{P}(m, \alpha) \leq \alpha\beta$. Thus, if $\alpha \leq c'_a = c_a/\beta$, then $\mathcal{P}(m, \alpha) \leq c_a$ for all m, and thus $\mathcal{M}(\mathcal{P}(m, \alpha)) = 0$ for all m, so $m = 0$ is the unique solution to (24).

Thus, we can substitute (24) into (2) to obtain:

$$(27) \qquad\qquad p = \alpha\beta g(r\beta(1 - \beta c'_a/p)/p).$$

It suffices to show that (27) has a unique solution $p \in (0, \alpha\beta)$, as the desired (unique) m_α is then $\mathcal{M}(p)$.

To see this, multiply each side by $r/(\alpha p)$ and substitute $x = \beta c_a'/p$ to get

$$(28) \qquad \frac{r}{\alpha} = \frac{r}{c_a'} x g(r x(1-x)/c_a') = \frac{1 - e^{-(r/c_a')x(1-x)}}{(1-x)}.$$

By Lemma 5 (see below), the right side of (28) is strictly increasing in x, takes the value zero at $x = 0$, and approaches r/c_a' as $x \to 1$. Because $\alpha > c_a'$, this implies that there is a unique solution x to (28) and therefore a unique solution p to (27). Furthermore, evaluating the right side of (28) at $x = c_a'/\alpha$ indicates that the solution x to (28) is greater than c_a'/α and thus the solution $p = \beta c_a'/x$ to (27) is less than $\alpha\beta$.

All that remains to show is that $\mathcal{P}(m_\alpha, \alpha)$ is strictly increasing in α. For $\alpha \le c_a'$, we have $m_\alpha = 0$, so $\mathcal{P}(m_\alpha, \alpha) = \mathcal{P}(0, \alpha) = \alpha\beta$, which is strictly increasing. When $\alpha > c_a'$, we have $\mathcal{P}(m_\alpha, \alpha) = \beta c_a'/x(\alpha)$, where $x(\alpha)$ is the solution to (28). Lemma 5 implies that the right side is strictly increasing in x, so $x(\alpha)$ is strictly decreasing in α and thus $\mathcal{P}(m_\alpha, \alpha)$ is strictly increasing. □

Our proof of Lemma 4 used the following fact.

Lemma 5. *For any $a > 0$ the function $y(x) = \frac{1 - e^{-ax(1-x)}}{1-x}$ for $x \in [0, 1)$ is strictly increasing in x, taking the value 0 when $x = 0$ and approaching a as $x \to 1$.*

Proof. Evaluating $y(0)$ and $\lim_{x\to 1} y(x)$ is straightforward, so we move on to proving that $y(x)$ is strictly increasing. Differentiating with respect to x and rearranging terms, we get

$$(29) \qquad \frac{dy}{dx} = \frac{a e^{-ax(1-x)}}{(1-x)^2} \left(2x^2 + \frac{1}{a}(e^{ax(1-x)} - 1) - 3x + 1 \right).$$

We wish to show that this expression is positive for $x \in (0, 1)$. The first term is clearly positive, so let's consider the second term. Since $e^{ax(1-x)} \ge 1 + ax(1-x)$, we have for $x \in (0, 1)$:

$$(30) \qquad \left(2x^2 + \frac{1}{a}(e^{ax(1-x)} - 1) - 3x + 1 \right) \ge 2x^2 + x(1-x) - 3x + 1 = (1-x)^2 > 0.$$

□

Before proceeding, we require some properties of the solution m_α to (25).

Lemma 6. *$\mathcal{Q}(m_\alpha, \alpha)$ is nonincreasing in α. If $c_a' < 1$, $\mathcal{Q}(m_\alpha, \alpha) = 1$ for $\alpha \in [0, c_a']$ and is strictly decreasing for $\alpha \in (c_a', 1]$. If $c_a' \ge 1$, $\mathcal{Q}(m_\alpha, \alpha) = 1$ for all $\alpha \in [0, 1]$.*

Proof. From Lemma 4, we have that $m_\alpha = 0$ if and only if $\alpha \le c'_a$. When $m_\alpha = 0$, we have $\mathcal{Q}(m_\alpha, \alpha) = g(0) = 1$. If $c'_a < 1$, for $\alpha > c'_a$, Lemma 4 implies that $m_\alpha > 0$ and thus by (2) we have:

$$(31) \qquad m_\alpha \mathcal{P}(m_\alpha, \alpha) = \log\left(\frac{\mathcal{P}(m_\alpha, \alpha)}{\beta c'_a}\right).$$

Lemma 4 implies that the right hand side (and therefore the left hand side) is strictly increasing in α. But

$$(32) \qquad \mathcal{Q}(m_\alpha, \alpha) = g(m_\alpha \mathcal{P}(m_\alpha, \alpha)).$$

Because g is a strictly decreasing function (see Lemma 1), this completes the proof. $\qquad \square$

The following lemma completes the proof of Theorem 2, by endogenizing the employers' choice of α.

Lemma 7. *Define m_α to be the unique solution to (24). If $c'_s < 1$, there is a unique value of $\alpha \in [0, 1]$ such that*

$$(33) \qquad \alpha \in \mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)),$$

i.e. such that (m_α, α) is a mean field equilibrium.

Proof. Our proof leverages the following intuition. Consider fixing the employers' strategy to be ϕ^1, i.e., employers always enter and screen. If in that case the resulting applicant availability is high enough (under the optimal seller response), then this will be an MFE. On the other hand, if the resulting applicant availability is too low, some employers will choose to exit the market. The key phenomenon that we exploit is that in this case, applicant availability is monotonically increasing as employers leave the market, increasing until exactly the point where employers are indifferent between entering and exiting. This indifference point is precisely where applicant availability q is equal to the (scaled) screening cost c'_s.

Formally, recall from Section 3.1.2 that

$$(34) \qquad \mathcal{A}(q) = \begin{cases} \{0\}, & \text{if } q < c'_s; \\ [0, 1], & \text{if } q = c'_s; \\ \{1\}, & \text{if } q > c'_s. \end{cases}$$

If $\mathcal{Q}(m_1, 1) \geq c'_s$, then $\alpha = 1$ solves (33). For any $\alpha < 1$, since $c'_s < 1$, it follows by Lemma 6 that regardless of the value of c_a, we have $\mathcal{Q}(m_\alpha, \alpha) > c'_s$. Thus, for any $\alpha < 1$, $\alpha \notin \mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)) = \{1\}$, so $\alpha = 1$ is the unique solution to (33).

Now suppose that $\mathcal{Q}(m_1, 1) < c'_s$. By Lemma 6, for any $\alpha \leq c'_a$, we have $\mathcal{Q}(m_\alpha, \alpha) = 1 > c'_s$. By continuity and monotonicity of $\mathcal{Q}$ (see Lemma 6), there exists exactly one value $\alpha' \in (c'_a, 1)$ such that $\mathcal{Q}(m_{\alpha'}, \alpha') = c'_s$. Clearly, $\alpha' \in \mathcal{A}(\mathcal{Q}(m_{\alpha'}, \alpha')) = \mathcal{A}(c'_s) = [0, 1]$, so α' solves (33). Furthermore, for any $\alpha < \alpha'$, we have $\mathcal{Q}(m_\alpha, \alpha) > c'_s$ and thus $\mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)) = \{1\}$. For any $\alpha > \alpha'$, we have $\mathcal{Q}(m_\alpha, \alpha) < c'_s$ and thus $\mathcal{A}(\mathcal{Q}(m_\alpha, \alpha)) = \{0\}$. Therefore, α' is the unique solution to (33).

$\square$

C.3. **Section 3.4: The Regulated Market.**

In this section we prove Proposition 2, which states that there is a unique equilibrium in the regulated market. It is the same as the equilibrium of the unregulated market if the limit ℓ does not bind, and otherwise involves applicants selecting $m_a = \ell$.

Our analysis requires a partial equilibrium characterization of the employers' behavior given a choice of m by applicants. In particular, for a fixed m, we show there exists a unique value α_m satisfying:

$$\alpha_m \in \mathcal{A}(\mathcal{Q}(m, \alpha_m)), \tag{35}$$

where we recall from Section 3.1.2 that:

$$\mathcal{A}(q) = \begin{cases} \{0\}, & \text{if } q < c'_s; \\ [0, 1], & \text{if } q = c'_s; \\ \{1\}, & \text{if } q > c'_s. \end{cases} \tag{36}$$

We have the following lemma.

Lemma 8. *For any fixed m, there exists a unique solution α_m to (35).*

Proof. If $\mathcal{Q}(m, 1) \geq c'_s$, then $\alpha_m = 1$ solves (35). Furthermore, for any $\alpha < 1$, $\mathcal{Q}(m, \alpha) > \mathcal{Q}(m, 1)$ by Lemma 3, and thus $\alpha \notin \mathcal{A}(\mathcal{Q}(m, \alpha)) = \{1\}$.

If $\mathcal{Q}(m, 1) < c'_s$, then $1 \notin \mathcal{A}(\mathcal{Q}(m, 1)) = \{0\}$, and $0 \notin \mathcal{A}(\mathcal{Q}(m, 0)) = \mathcal{A}(1) = \{1\}$. It follows that any solution α_m to (35) must satisfy $0 < \alpha_m < 1$, and thus must satisfy $\mathcal{Q}(m, \alpha_m) = c'_s$. Lemma 3 states that $\mathcal{Q}(m, \alpha)$ strictly increases continuously to 1 as α decreases, implying that there is a unique α_m such that $\mathcal{Q}(m, \alpha_m) = c'_s$.

$\square$

In addition, note that the pair (m, α_m) is an MFE if and only if $m \in \mathcal{M}(\mathcal{P}(m, \alpha_m))$. For $m > 0$, from (2), this holds if and only if $h(m) = -\log c_a'$, where

$$(37) \qquad\qquad h(m) = m\mathcal{P}(m, \alpha_m) - \log\left(\mathcal{P}(m, \alpha_m)/\beta\right).$$

The following lemma gives some basic properties of the function h.

Lemma 9. *The function $h(m)$ given in (37) is strictly increasing. It takes the value zero at $m = 0$, and increases without bound as $m \to \infty$.*

Proof. Note that a mean-field equilibrium is exactly a pair (m, α_m) such that $m = \mathcal{M}(\mathcal{P}(m, \alpha_m))$. From the definition of $\mathcal{M}$ it follows that the pair (m, α_m) with $m > 0$ is an equilibrium if and only if $m = \frac{1}{\mathcal{P}(m, \alpha_m)} \log\left(\frac{\mathcal{P}(m, \alpha_m)}{c_a}\right)$, or equivalently

$$(38) \qquad\qquad m\mathcal{P}(m, \alpha_m) - \log(\mathcal{P}(m, \alpha_m)/\beta) = -\log c_a'.$$

We already know from Theorem 2 that for any $c_a' < 1$ and any β, there is a unique equilibrium with $m > 0$, i.e. a unique solution to $h(m) = -\log c_a'$. This implies that h is invertible (if $h(m) = h(m')$ for $m \neq m'$, then for some c_a', there would be multiple mean-field equilibria). Because h is continuous, this also implies that h is monotonic. Furthermore, the existence of a solution to $h(m) = -\log c_a'$ for any $c_a' < 1$ (by Theorem 2) implies that $h(0) = 0$ and that h is unbounded. $\qquad\qquad\square$

We now prove the desired result.

Proof of Proposition 2. Let (m^*, α^*) be the mean-field equilibrium in the original game, and let $(m_\ell^*, \alpha_\ell^*)$ be any equilibrium of the game with application limit ℓ, meaning that $m_\ell^* = \mathcal{M}_\ell(\mathcal{P}(m_\ell^*, \alpha_\ell^*))$, and $\alpha_\ell^* \in \mathcal{A}(\mathcal{Q}(m_\ell^*, \alpha_\ell^*))$.

Immediately from (6) (which states that $\mathcal{M}_\ell(p) = \min(\ell, \mathcal{M}(p))$), we get that if there is an equilibrium of the regulated market with $m_\ell^* < \ell$, it must be that $(m_\ell^*, \alpha_\ell^*)$ is an equilibrium in the unregulated market and thus $(m_\ell^*, \alpha_\ell^*) = (m^*, \alpha^*)$. Therefore, the only candidate equilibria in the game with application limit ℓ are (m^*, α^*) and (ℓ, α_ℓ).

It is clear that (m^*, α^*) is an equilibrium of the game with application limit ℓ if and only if $m^* \leq \ell$. To complete the proof of Proposition 2, we claim that the pair (ℓ, α_ℓ) is an equilibrium of the regulated market if and only if $m^* \geq \ell$.

To prove this claim, first suppose that $m^* < \ell$. Then $-\log c_a' = h(m^*) < h(\ell)$ by Lemma 9. Conversely, if $m^* \geq \ell$, then $-\log c_a' = h(m^*) \geq h(\ell)$. In other words, for $c_a' \leq 1$,

$$(39) \qquad m^* \geq \ell \Leftrightarrow h(\ell) \leq -\log c_a'.$$

Straightforward manipulation of the equation (37) defining h and an application of (2) reveals that for $\ell > 0$ and $c_a' \leq 1$,

$$(40) \qquad \ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell)) \Leftrightarrow h(\ell) \leq -\log c_a'.$$

Combining (39) and (40), we get that $m^* \geq \ell$ if and only if $\ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$. Furthermore, $\ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$ implies $\mathcal{M}_\ell(\mathcal{P}(\ell, \alpha_\ell)) = \ell$, and conversely $\ell > \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$ implies $\mathcal{M}_\ell(\mathcal{P}(\ell, \alpha_\ell)) = \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell))$ (using $\mathcal{M}_\ell(p) = \min(\ell, \mathcal{M}(p))$ from (6)). Putting it all together, we get that

$$(41) \qquad m^* \geq \ell \Leftrightarrow \ell \leq \mathcal{M}(\mathcal{P}(\ell, \alpha_\ell)) \Leftrightarrow \ell = \mathcal{M}_\ell(\mathcal{P}(\ell, \alpha_\ell)).$$

In other words, $m^* \geq \ell$ if and only if (ℓ, α_ℓ) is an equilibrium of the game with application limit ℓ. □

Appendix D. Proofs: Section 4

In this section we develop the technology required to prove the approximation theorems (Theorems 3 and 4).

We begin by formalizing the stochastic process of interest, when m and α are fixed. Note that in our original model, applicants decide where to apply when they arrive to the system; however, for purposes of stochastic analysis, we obtain an equivalent system if we *realize applicant applications only when employers depart.* In particular, we consider the following stochastic system parameterized by n. Individual applicants arrive at intervals of length $1/rn$, as before. Let $S(t)$ denote the number of applicants in the system at time t. In addition, we define $\Sigma(t)$ as the *normalized* number of applicants in the system:

$$(42) \qquad \Sigma(t) = S(t)/(rn);$$

note that $\Sigma(t) \leq 1$ for any $N(t)$ that can arise. At intervals of length $1/n$ (corresponding to employer departures), there are opportunities for at most a single applicant in the system to depart

early. At each such employer departure time t, the probability of a departure of an applicant is:

$$(43) \qquad \alpha \left(1 - \left(1 - \frac{\beta m}{n} \right)^{S(t)} \right) = \alpha(1 - \rho^{\Sigma(t)}),$$

where we define $\rho := (1 - \beta m/n)^{rn}$. Note that $\rho \to \exp(-\eta)$ as $n \to \infty$, where we define $\eta := rm\beta$.

The preceding equation (43) is derived as follows. As before, with probability α an employer screens using strategy ϕ, and exits immediately otherwise (in which case no applicant departs early). Every applicant that arrived in the last one time unit applied to the departing employer with probability m/n. Any such applicant that has already departed cannot match to the given employer. On the other hand, among the remaining employers, if even one of them is compatible with the employer, then employer following ϕ is sure to find a match. Thus at least one departure occurs as long as there is at least one available, compatible applicant that applied to the departing employer. Note that under ϕ, each of the applicants in the system at time t is equally likely to depart, so for each applicant the probability of departure is $\alpha(1 - \rho^{\Sigma(t)})/(rn\Sigma(t))$.

Note that to capture the state at time t, we must track the residual lifetimes of all applicants in the system. To simplify this tracking, a key instrument in our analysis is a "binned" version of the stochastic process $S(t)$, defined as follows. Fix an integer k, and let $S_j(t)$ be the number of applicants that have been in the system for a time between j/k and $(j + 1)/k$ units, for $j = 0, 1, \ldots, k - 1$. Let $X_j(t) = S_j(t)/(rn)$; note that $\Sigma(t) = \sum_{j=0}^{k-1} X_j(t)$. Our fundamental result proves a concentration result for the vector-valued stochastic process $X(t)$.

What does $X(t)$ concentrate around? To develop intuition, let's think of the matching process from the perspective of the applicants that arrive in a fixed interval of length $1/k$. In the large market limit, each of these applicants should match in successive intervals of length $1/k$ with a constant probability; or equivalently, their survival probability is a constant γ in each such interval. Looking back in time, then, in steady state we should expect that the vector $X(t)$ satisfies $X_j(t) = \gamma X_{j-1}(t)$ for $j = 1, \ldots, k - 1$, with $X_0(t) = 1/k$. With this inspiration (and in an abuse of notation), we define $\Sigma(\gamma)$ as:

$$(44) \qquad \Sigma(\gamma) = \frac{1}{k} \sum_{j=0}^{k-1} \gamma^j = \frac{1 - \gamma^k}{k(1 - \gamma)}.$$

(Note that $\Sigma(1) = 1$.)

On the other hand, as in our mean field analysis, we can develop a "consistency check" that γ must satisfy using (43). Assume that $k \geq \eta/r$ and define $\gamma(\Sigma)$ as follows:

$$(45) \qquad \gamma(\Sigma) = 1 - \alpha \left(\frac{1 - e^{-\eta\Sigma}}{rk\Sigma} \right),$$

where we take $\gamma(0) = 1 - \alpha\eta/(rk)$ (this is the limit of the preceding quantity as $\Sigma \to 0$). This equation is an approximate version of (43): $1 - \gamma(\Sigma)$ represents an estimate of the probability that an individual applicant in the system is matched in the next $1/k$ time units, if the current (normalized) number of available applicants is Σ, and k and n are "large" (specifically $k = \omega(1)$ and $n = \omega(k)$).

The following two results are critical to our analysis: they establish the uniqueness of a solution to the preceding two equations, and relate this solution to the unique MFSS (p, q) guaranteed by Theorem 1. The proofs are in Section D.1.

Lemma 10. *Suppose that $k \geq \eta/r$. There is a unique pair of real numbers (γ^*, Σ^*) that simultaneously solve (44) and (45).*

Lemma 11. *Suppose that $k \geq \eta/r$. Let (γ_k^*, Σ_k^*) denote the unique solution to (44) and (45) guaranteed by Lemma 10. Then as $k \to \infty$, $k(1 - \gamma_k^*) \to mp$, and $\Sigma_k^* \to q$, where (p, q) is the unique MFSS guaranteed by Theorem 1.*

The interpretation is as follows. Note that under the mean field assumptions, each applicant sends a Poisson distributed number of applications, with mean m; and each application independently succeeds with probability p. Since the applicant's applications are independent to each employer, it follows that in the mean field model the applicant's lifetime in the system is an exponential random variable of mean $1/mp$ truncated to be less than or equal to 1 (since applicants only live for at most a unit lifetime). In other words, the rate of applicant departure is mp. On the other hand, for fixed k the rate of departure is approximated by $k(1 - \gamma_k^*)$, so we should expect the latter quantity to approach mp. Similarly, observe that Σ^* is meant to be an estimate of the steady state (normalized) number of applicants in the system; we should expect that this approaches the applicant availability q in the mean field model.

Let X^* be the vector given by $X_j^* = (\gamma^*)^j/k$ for $j = 0, \ldots, k-1$, so that $\Sigma^* = \sum_{j=0}^{k-1} X_j^*$ from Eq. (44). Our main result is the following theorem, that shows that $X(t)$ concentrates around X^*. Note that this is a very strong result, because it precludes drift of the $X(t)$ process away from X^*

as time grows. We achieve this result by using a stochastic concentration argument on the process $X(t)$.[18]

Theorem 5. *Fix $m_0 \in [1, \infty)$ and $\alpha \in [0, 1]$. There exists $C = C(r, m_0, \beta, \alpha) < \infty$ such that for any $m \in [1/m_0, m_0]$, for any $n > C$, $k = \lfloor n^{1/3} \rfloor$ the following is true: For any $t > C \log n$, and any starting state at time 0, we have $\mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6}$. If the starting state at time 0 is drawn from the steady state distribution, we have $\mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6}$ for all $t \geq 0$.*

We use this theorem to establish that the mean field assumptions hold asymptotically (Theorems 3 and 4).

The remainder of this section is organized as follows. In Section D.1, we prove Lemmas 10 and 11. In Section D.2, we prove our main Theorems 3 and 4. In Section D.3 we prove Corollary 1. In Section D.4, we prove Theorem 5 (together with the supporting lemmas required).

D.1. Proofs of Lemmas 10 and 11.

Proof of Lemma 10. For $\alpha = 0$ we immediately find that $\gamma^* = 1$ and $\Sigma^* = 1$ is the unique solution.

Assume $\alpha > 0$. Rearranging (45), we get

$$(46) \qquad r = \frac{\alpha(1 - e^{-\eta\Sigma})}{k(1 - \gamma)\Sigma} = \frac{\alpha(1 - e^{-\eta\Sigma})}{1 - \gamma^k},$$

where the final expression follows from substituting (44). Differentiate the expression on the right with respect to γ, thinking of Σ as the function of γ specified by Eq. (44): the result is

$$(47) \qquad \frac{\alpha}{(1 - \gamma^k)^2}\left(k\gamma^{k-1}(1 - e^{-\eta\Sigma}) + \eta e^{-\eta\Sigma}(1 - \gamma^k)\frac{d\Sigma}{d\gamma}\right).$$

Since $\frac{d\Sigma}{d\gamma} > 0$ for Σ given by Eq. (44), the expression above is positive, so the right side of (46) is increasing in γ. Further, since

$$(48) \qquad \lim_{\gamma \to 0}\frac{\alpha(1 - e^{-\eta\Sigma})}{1 - \gamma^k} = \alpha(1 - e^{-\eta/k}) < \alpha\eta/k \leq r < \infty = \lim_{\gamma \to 1}\frac{\alpha(1 - e^{-\eta\Sigma})}{1 - \gamma^k},$$

there is a unique solution γ^* to (46), leading to a unique value Σ^* from Eq. (44). $\qquad \square$

Proof of Lemma 11. Note that $0 \leq \Sigma_k^* \leq 1$. Further:

$$(49) \qquad k(1 - \gamma_k^*) = \alpha\left(\frac{1 - e^{-\eta\Sigma_k^*}}{r\Sigma_k^*}\right).$$

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[18]Note that technically, the result only holds at rational time points t, given the definition of our process $X(t)$. However, we suppress this in the presentation for clarity.

This remains bounded for all $\Sigma_k^* \in [0,1]$. Taking subsequences if necessary, therefore, we assume without loss of generality that $k(1 - \gamma_k^*) \to m\hat{p}$ and $\Sigma_k^* \to \hat{q}$ as $k \to \infty$.

To establish the result, we show that $(\hat{p}, \hat{q})$ must satisfy (4) and (5). Rewriting and taking limits in (44), we have:

$$\Sigma_k^* = \frac{1 - \left(1 - \frac{k(1-\gamma_k^*)}{k}\right)^k}{k(1 - \gamma_k^*)} \qquad \to \frac{1 - e^{-m\hat{p}}}{m\hat{p}}.$$

And similarly, taking limits in (49) and substituting for η we have:

$$m\hat{p} = \alpha \left(\frac{1 - e^{-rm\beta\hat{q}}}{r\hat{q}}\right).$$

Thus $(\hat{p}, \hat{q})$ satisfy (4) and (5), as required. □

D.2. **Proofs of Theorems 3 and 4.** We require the following lemma.

Lemma 12 (Le Cam's inequality). *Let $X_1, \ldots, X_n$ be Bernoulli random variables with $\mathbb{P}(X_i = 1) = p_i$. Let $S_n = \sum_j X_j$, and $\lambda_n = \sum_i p_i$. Then:*

$$(50) \qquad \sum_{j \geq 0} \left| \mathbb{P}(S_n = j) - \frac{e^{-\lambda_n} \lambda_n^j}{j!} \right| \leq 2 \sum_{i=1}^n p_i^2.$$

The preceding inequality is a concentration result for the Poisson approximation to the binomial distribution; in particular it implies the following result.

Corollary 2. *Fix $\zeta_0 < \infty$. For any $\zeta < \infty$, as $n \to \infty$, we have that $\mathrm{Binomial}(n, \zeta/n)$ converges in total variation distance to $\mathrm{Poisson}(\zeta)$. Further, this convergence is uniform over $\zeta \in [0, \zeta_0]$.*

Proof. Let $p_i = \zeta/n$ for all i in Lemma 12; then (50) bounds the total variation distance between $\mathrm{Binomial}(n, \zeta/n)$ and $\mathrm{Poisson}(\zeta)$ by $\zeta^2/n \leq \zeta_0^2/n \to 0$ as $n \to \infty$. □

Proof of Theorem 3. We proceed as follows:

$$\sum_{\ell,a} \left| \mathbb{P}(R_b^{(n)} = \ell, A_b^{(n)} = a) - \mathbb{P}(R = \ell, A = a) \right|$$

$$= \sum_{\ell,a} \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell)\mathbb{P}(R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell)\mathbb{P}(R = \ell) \right|$$

$$\leq \sum_{\ell,a} \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) \left| \mathbb{P}(R_b^{(n)} = \ell) - \mathbb{P}(R = \ell) \right|$$

$$(51) \qquad + \sum_{\ell,a} \mathbb{P}(R = \ell) \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell) \right|.$$

Since the system begins in steady state, any employer arriving to the system is visible to rn applicants that each apply to the given employer with probability m/n; i.e., the number of applications $R_b^{(n)}$ received by such an employer e follows a Binomial$(rn, m/n)$ distribution. Thus $R_b^{(n)}$ converges in total variation distance to R by Corollary 2, so the first summation in (51) approaches zero as $n \to \infty$ (since for every ℓ, $\sum_a \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) = 1$).

We thus focus on the second summation in (51). Note that for all a such that $0 \leq a \leq \ell$, $\mathbb{P}(A = a | R = \ell) = \binom{\ell}{a} q^a (1 - q)^{\ell - a}$. To simplify notation, let $t = t_b + 1$ denote the exit time of employer e.

Given $\varepsilon > 0$, let $C_n(\varepsilon)$ be the event that $\|X(t) - X^*\|_1 \leq \varepsilon$ in the n-th system. Recall that we realize the randomness as follows: at the time of exit, we independently determine whether each applicant that arrived in the last time unit applied to this employer. Thus in particular the number of applications that employer e receives is independent of the state $X(t)$ at her exit time t, and so we conclude that $R_b^{(n)}$ is independent of $C_n(\varepsilon)$. Letting $k = \lfloor n^{1/3} \rfloor$, from Theorem 5, for n sufficiently large it follows that:

$$\mathbb{P}(C_n(\varepsilon) | R_b^{(n)} = \ell) \geq 1 - \frac{C}{\varepsilon n^{1/6}}$$

for an appropriate constant C, by Markov's inequality. Note that on $C_n(\varepsilon)$ it follows that $|\Sigma(t) - \Sigma^*| \leq \varepsilon$ as well, since $\Sigma(t) = \|X(t)\|_1$ and $\Sigma^* = \|X^*\|_1$.

Now again, since we realize applications at the time of exit of the employer, note that conditional on the value of $S(t)$ as well as $R_b^{(n)} = s$, the employer receives exactly a applications from available

applicants with the following probability:

$$(52) \qquad \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, S(t)) = \frac{\binom{S(t)}{a}\binom{rn-S(t)}{\ell-a}}{\binom{rn}{\ell}}.$$

For the moment, assume that $0 < q < 1$, and choose n large enough and ε small enough so that $0 < \Sigma^* - \varepsilon < \Sigma^* + \varepsilon < 1$ (cf. Lemma 11). On the event $C_n(\varepsilon)$, we know that $\Sigma^* - \varepsilon < \Sigma(t) < \Sigma^* + \varepsilon$; since $S(t) = rn\Sigma(t)$, we conclude that on $C_n(\varepsilon)$ both $S(t)$ and $rn - S(t)$ are $\Theta(n)$. Thus for fixed ℓ and a, we can approximate the preceding probability with the equivalent calculation as if the available and unavailable applicants were sampled with replacement. It follows that:

$$\left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, C_n(\varepsilon)) - \binom{\ell}{a}(\Sigma^*)^a (1 - \Sigma^*)^{\ell-a} \right| \leq f(\varepsilon),$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, taking n large enough, we can assume that $|\Sigma^* - q| \leq \varepsilon$, from which it follows that:

$$\left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, C_n(\varepsilon)) - \binom{\ell}{a}q^a (1 - q)^{\ell-a} \right| \leq \hat{f}(\varepsilon),$$

where $\hat{f}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Putting the steps together, we find that:

$$\begin{aligned}
|\mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) &- \mathbb{P}(A = a | R = \ell)| \\
&\leq \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell, C_n(\varepsilon)) - \binom{\ell}{a}q^a(1-q)^{\ell-a} \right| + |\mathbb{P}(C_n(\varepsilon)^c| \\
&\leq \hat{f}(\varepsilon) + \frac{C}{\varepsilon n^{1/6}}.
\end{aligned}$$

Take $n \to \infty$, then $\varepsilon \to 0$ to conclude that:

$$\lim_{n \to \infty} \left| \mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell) \right| = 0.$$

In the case where $q = 0$ or $q = 1$, a direct analysis of (52) can be used to establish the preceding result; we omit the details.

Finally, to conclude the proof, note that for each ℓ:

$$\sum_{a=0}^{\ell} |\mathbb{P}(A_b^{(n)} = a | R_b^{(n)} = \ell) - \mathbb{P}(A = a | R = \ell)|$$

converges to zero as $n \to \infty$. Further, the preceding quantity is bounded above by $\ell + 1$, which is integrable against the Poisson(rm) distribution; so by the dominated convergence theorem we conclude that the second term in (51) converges to zero as $n \to \infty$, as required.   □

Proof of Theorem 4. The result is trivial for $m_a = 0$ so we assume $m_a > 0$ henceforth.

We start with a similar approach to the proof of Theorem 3, as follows:

$$\sum_{\ell,a} \left| \mathbb{P}(T_a^{(n)} = \ell, Q_a^{(n)} = a) - \mathbb{P}(T = \ell, Q = a) \right|$$

$$\leq \sum_{\ell,a} \mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) \left| \mathbb{P}(T_a^{(n)} = \ell) - \mathbb{P}(T = \ell) \right|$$

(53)
$$+ \sum_{\ell,a} \mathbb{P}(T = \ell) \left| \mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) - \mathbb{P}(Q = a | T = \ell) \right|.$$

Since the system begins in steady state, any arriving applicant will find n employers in the system upon arrival, and applies to each of these employers independently with probability m_a/n; i.e., the number of applications $T_a^{(n)}$ sent by such an applicant a follows a Binomial($n, m_a/n$) distribution. Thus $T_a^{(n)}$ converges in total variation distance to T by Corollary 2, so the first summation in (53) approaches zero as $n \to \infty$, uniformly for all $m_a \in (0, m_0]$.

As before, therefore, we focus our attention on the second summation in (53). Note that for all a such that $0 \leq a \leq \ell$, we have $\mathbb{P}(Q = a | T = \ell) = \binom{\ell}{a} p^a (1-p)^{\ell-a}$.

First, we fix an upper bound L on the number of applications that we consider by applicant a. In particular, note that we have:

(54)
$$\sum_{\ell > L, a} \mathbb{P}(T = \ell) \left| \mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) - \mathbb{P}(Q = a | T = \ell) \right| \leq \sum_{\ell \geq L} (\ell + 1) \mathbb{P}(T = \ell),$$

and the last expression does not depend on n and goes to zero as $L \to \infty$, uniformly for all $m_a \leq m_0$. Thus it suffices to show that for fixed L, the sum:

(55)
$$\sum_{\ell \leq L, a} \mathbb{P}(T = \ell) \left| \mathbb{P}(Q_a^{(n)} = a | T_a^{(n)} = \ell) - \mathbb{P}(Q = a | T = \ell) \right|$$

goes to zero as $n \to \infty$, uniformly for all $m_a \leq m_0$.

Let $\mathcal{I}(\ell)$ denote the set of all 2^ℓ possible outcome vectors $\mathbf{I}$ that are possible when the applicant sends ℓ applications. Note that (55) is less than or equal to:

$$(56) \qquad \sum_{\ell \leq L, \mathbf{I} \in \mathcal{I}(\ell)} \mathbb{P}(T = \ell) \left| \mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i} \right|,$$

where we take advantage of the fact that Q is distributed as Binomial(ℓ, p) when $T = \ell$. Since $\ell \leq L$, it suffices to show that for any ℓ and each $\mathbf{I} \in \mathcal{I}(\ell)$, the quantity

$$(57) \qquad \left| \mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i} \right|$$

goes to zero as $n \to \infty$. Now note that the preceding quantity is less than or equal to:

$$(58) \qquad \int \left| \mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell, \mathbf{t}) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i} \right| \, d\mathbb{P}_n(\mathbf{t}|T_a^{(n)} = \ell).$$

Here the integral is over all possible vectors of departure times for employers to whom the applicant submitted an application. Note that $\mathbf{t}$ has an atomic distribution that varies with n, so the integral reduces to a sum over feasible $\mathbf{t}$. Note also that this quantity does not depend on m_a at all (since we are conditioning on applicant a making ℓ applications). In fact, it turns out that we do not need to consider m_a for the rest of the proof.

We argue as follows. Fix $k = \lfloor n^{1/3} \rfloor$. When $T_a^{(n)} = \ell$, we use $b_1, \ldots, b_\ell$ to denote the employers that a applied to, and without loss of generality we let $t_1 \leq t_2 \leq \cdots \leq t_\ell$ denote the departure times of these employers, and let $\mathbf{t}$ denote the vector of departure times. We define "rounded" departure times by rounding the true departure times up to the nearest multiple of $1/k$; denote these as $t_i' = \lceil kt_i \rceil / k$ for $i = 1, 2, \ldots, \ell$, and let $\mathbf{t}'$ denote the vector of rounded departure times. We use the rounded departure times so that we can apply Corollary 3; that result applies to the "binned" process of available applicants, binned on time increments of length $1/k$.

In our analysis, we make use of a coupled process of available applicants, but with a particular employer b_i and applicant a removed during $(t_{i-1}', t_i']$ (with $t_0' = 0$). Let $S^{-i}(t)$ denote the state of this process at time t for $t \leq t_i'$. We couple this process to the original process so that each employer (besides b_i) receives the same set of applications, and screens them in the same order (or does not screen at all in both systems). So $S^{-i}(t)$ is identical to $S(t)$ for $t \leq t_{i-1}'$, and further for $t \in (t_{i-1}', t_i)$ except for the removal of a. The states are further nearly identical until t_i' if none of the applicants who applied to b_i applied to another employer who departed in (t_i, t_i') (the event

$E_n(i+1)$ defined below): in this case the systems can differ only on applicant a and any applicant who received an offer from b_i.

We also use I_i to denote the outcome on the i'th application; i.e., $I_i = 1$ if applicant a receives an offer from employer b_i, and $I_i = 0$ otherwise. Let $\mathbf{I}$ denote the vector of application outcomes. Fix $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots \varepsilon_\ell > 0$. Let $C_n(0)$ be the event that the $\mathbf{t}'$ are all distinct. Let $D_n(J)$ be the event that no more than $2mrn/k$ distinct applicants apply to employers who depart during (t_J, t'_J); this will hold with high probability for large n. Define $C_n(1)$ be the event that $C_n(0)$ occurs; and in the $S^{-1}(\cdot)$ process, $D_n(1)$ occurs and $\|X(t'_1) - X^*\|_1 \le \varepsilon_1$. Let $C_n(J)$ for $J > 1$ be iteratively defined as the event that $C_n(J-1)$ occurs; and in the $S^{-J}(\cdot)$ process, $D_n(J)$ occurs, $\|X(t'_J) - X^*\|_1 \le \varepsilon_J$, and no applicant applies to both b_{J-1} and some other employer who departs in the interval $(t_{J-1}, t'_{J-1}]$. We remark here that we consider the modified process $S^{-i}(\cdot)$, and employ event $D_n(J)$ in the definition of $C_n(J)$ for convenience; these can be viewed as minor technical details.

In what follows, to economize on notation for a vector $\mathbf{x}$, we let $\mathbf{x}_{i:j} = (x_i, x_{i+1}, \ldots, x_j)$.

We proceed as follows. First, we condition on $\mathbf{I}_{1:i-1}$ and $\Sigma(t_i)$, and consider the probability employer b_i makes applicant a an offer. Recall that the employer only learns compatibility but not availability by screening; a receives an offer from b_i if and only if she is screened by employer b_i and is also compatible.

Given that the employer follows ϕ^α, $\Sigma(t_i)$ and $\mathbf{I}_{1:i-1}$ form a sufficient statistic to determine whether employer i screens a, as follows. Let d denote the number of competing applicants that employer i receives. Applicant a receives an offer from employer b_i if she is compatible with b_i, and no compatible, available applicant is screened by employer b_i before a. The scaled number of available applicants besides a at time t_i is $\Sigma(t_i)$ (possibly with a $O(1/n)$ adjustment if applicant a is also available; we skip this minor detail). It follows that the probability a receives an offer is:

$$\beta \sum_{d=0}^{rn\Sigma(t_i)} \binom{rn\Sigma(t_i)}{d} \left(\frac{m}{n}\right)^d \left(1 - \frac{m}{n}\right)^{rn\Sigma(t_i)-d} \left\{ \frac{\alpha}{d+1} \sum_{j=0}^{d} (1-\beta)^j \right\}.$$

The expression in brackets simplifies to:

$$\frac{\alpha}{d+1} \cdot \frac{1 - (1-\beta)^{d+1}}{\beta},$$

so the probability that a receives an offer becomes:

$$(59) \qquad \sum_{d=0}^{rn\Sigma(t_i)} \text{Binomial}\left(rn\Sigma(t_i), \frac{m}{n}\right)_d \left\{\frac{\alpha(1-(1-\beta)^{d+1})}{d+1}\right\}.$$

At this point we recall the derivation of (5); from that calculation it follows that:

$$(60) \qquad \sum_{d=0}^{\infty} \text{Poisson}(rm\Sigma)_d \cdot \frac{\alpha(1-(1-\beta)^{d+1})}{d+1} = \frac{\alpha(1-e^{-rm\beta\Sigma})}{rm\Sigma} = \alpha\beta g(rm\beta\Sigma),$$

where $g(x) = (1-e^{-x})/x$. Further, observe that if $\Sigma = q$ in the preceding expression, then (5) implies the right hand side of the preceding expression is equal to p.

We proceed by showing that on $C_n(i)$, (59) is well approximated by (60) with $\Sigma = q$. This requires three steps: first, showing that the binomial distribution in (59) is well approximated by a Poisson distribution; second, exploiting the fact that $\Sigma(t_i)$ is close to Σ^*; and third, taking n large enough so that Σ^* is close to q.

We proceed as follows. Let $h(d) = \alpha(1-(1-\beta)^{d+1})/(d+1)$. Note that $0 \le h(d) \le 1$ for all d. Further, note that we have:

$$\left|\left(\sum_{d=0}^{rn\Sigma(t_i)} h(d)\text{Binomial}\left(rn\Sigma(t_i), \frac{m}{n}\right)_d\right) - p\right|$$

$$\le \sum_{d=0}^{\infty} h(d)\left|\text{Binomial}\left(rn\Sigma(t_i), \frac{m}{n}\right)_d - \text{Poisson}(rm\Sigma(t_i))_d\right|$$

$$+ \left|\sum_{d=0}^{\infty} h(d)\left(\text{Poisson}(rm\Sigma(t_i))_d - \text{Poisson}(rm\Sigma^*)_d\right)\right|$$

$$+ \left|\sum_{d=0}^{\infty} h(d)\left(\text{Poisson}(rm\Sigma^*)_d - \text{Poisson}(rmq)_d\right)\right|.$$

By Lemma 12, the first summation on the right is bounded above by a K/n for some constant K. We now use (60) to simplify the second and third summations. The second summation reduces to $|\alpha\beta(g(rm\beta\Sigma(t_i)) - g(rm\beta\Sigma^*))|$, which for large enough n on $C_n(i)$, using that $|\Sigma(t_i) - \Sigma(t_i')| = O(t_i' - t_i) = O(1/k)$, is bounded above by $f(\varepsilon_i + 1/k)$ for some f such that $f(x) \to 0$ as $x \to 0$. Finally, the third summation above reduces to $|\alpha\beta(g(rm\beta\Sigma^*) - g(rm\beta q))|$, which can be made less than or equal to ε_i for n large enough. Summarizing then, by taking n large enough, we can ensure

that on $C_n(i)$ we have:

$$\left|\left(\sum_{d=0}^{rn\Sigma(t_i)} h(d)\text{Binomial}\left(rn\Sigma(t_i), \frac{m}{n}\right)_d\right) - p\right| \leq \hat{f}(\varepsilon_i + 1/k),$$

for some $\hat{f}$ such that $\hat{f}(\varepsilon_i) \to 0$ as $\varepsilon_i \to 0$.

Analogously, in the case where employer i does not make an offer to applicant a, it immediately follows that $\mathbb{P}(I_i = 0 | T_a^{(n)} = \ell, \mathbf{t}, \mathbf{I}_{1:i-1}, C_n(i))$ is well approximated by $1 - p$.

To simplify notation, let $A_{\ell,\mathbf{t}}^{(n)}$ be the event that $T_a^{(n)} = \ell$ and the vector of employer departure times this applicant applied to is $\mathbf{t}$. For now, consider only those $A_{\ell,\mathbf{t}}^{(n)}$ such that $C_n(0)$ holds (this occurs with high probability, cf. Eq. (68) below). To summarize then, we conclude that by taking n large enough, we have:

$$(61) \qquad \left|\mathbb{P}(I_i | A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)) - p^{I_i}(1-p)^{1-I_i}\right| \leq \hat{f}(\varepsilon_i + 1/k).$$

Next, we consider the conditional probability $\mathbb{P}(C_n(i) | A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1))$. As a preliminary step, note that $\mathbb{P}(D_n(i) | A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1)) \leq 1 - \exp(-\Theta(n/k))$ by the Chernoff bound, since each of $(1 + 1/k)rn$ applicants has, independently, applied to some employer who departs in (t_i, t_i') with probability no more than $(m/n)(n/k) = m/k$, so the likelihood that over $2mrn/k$ distinct applicants have applied to this set of employers is $\exp(-\Theta(n/k))$. Also, let $E_n(i)$ be the event that none of the applicants who applied to an employer who departs in (t_{i-1}, t_{i-1}') also applied to employer b_i. By the union bound, we have $\mathbb{P}(E_n(i) | A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), D_n(i)) \leq (m/n)(2mrn/k) = 2m^2/k$ under $D_n(i-1)$. It is straightforward to check that this bound remains $O(1/k)$ even if we further condition on I_{i-1}, because the conditional probability $\mathbb{P}(I_{i-1} | A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1), D_n(i))$ is bounded away from zero and one for sufficiently large n.

We now rely on Corollary 3 to control $\mathbb{P}(C_n(i) | A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i-1))$. Now observe that on $C_n(i-1)$, we know that $\|X(t_{i-1}') - X^*\|_1 \leq \varepsilon_{i-1}$ for the $S^{-(i-1)}(\cdot)$ process. To iterate our argument to the departure of the i'th employer, we need to control the state $X(t_i')$ in the process $S^{-i}(\cdot)$, i.e., for the system where employer b_i and applicant a are removed during (t_{i-1}', t_i') (but employer b_{i-1} is included). For this, we only need to make an adjustment to $X(t_{i-1}')$ if employer b_{i-1} matched to an applicant a' other than a in the original system. If this indeed happened, then, on $E_n(i)$, we only need to adjust one coordinate of $X(t_{i-1}')$ downward by $1/n$ corresponding to a' being unavailable in $S^{-i}(t_{i-1}')$, and this small adjustment will not affect our analysis (we omit this detail below). For

n large, it follows that an analogous result to Corollary 3 holds for the evolution of $X(t)$ between t'_i and t'_{i-1}, from which we can conclude that for large enough n there holds:

$$\mathbb{E}[\|X(t'_i) - X^*\|_1 | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)]$$

$$\leq \varepsilon_{i-1} + \frac{C'}{\kappa'} \frac{1}{n^{1/6}} + O(1/n).$$

The first term follows from the contraction term in Corollary 3; the second term follows because $k = \lfloor n^{1/3} \rfloor$. By Markov's inequality we obtain:

$$(62) \qquad \mathbb{P}(\|X(t'_i) - X^*\|_1 < \varepsilon_i | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)) \geq 1 - \frac{\varepsilon_{i-1}}{\varepsilon_i} - \frac{2C'}{\kappa'\varepsilon_i} \frac{1}{n^{1/6}}.$$

Combining, we obtain

$$\mathbb{P}(C_n(i) | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1))$$

$$\geq \mathbb{P}(C_n(i) | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i))(1 - \mathbb{P}(E_n(i)^c))$$

$$\geq \mathbb{P}(C_n(i) | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1), E_n(i)) - \mathbb{P}(E_n(i)^c)$$

$$(63) \qquad\qquad \geq 1 - \frac{\varepsilon_{i-1}}{\varepsilon_i} - \frac{3C'}{\kappa'\varepsilon_i} \frac{1}{n^{1/6}}$$

for large enough n. This bound applies for any $i > 1$. For $i = 1$—the first employer that applicant a applies to—we can directly apply Theorem 5 together with Markov's inequality to conclude that for sufficiently large n, we have:

$$(64) \qquad\qquad \mathbb{P}(C_n(1) | A^{(n)}_{\ell,\mathbf{t}}, C_n(0)) \geq 1 - \frac{2}{\varepsilon_1 n^{1/6}}.$$

Observe that by the definition of conditional probability, together with the fact that $C_n(i) \subset C_n(i-1)$, we have:

$$\mathbb{P}(\mathbf{I}_{i:\ell} | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1))$$

$$= \mathbb{P}(\mathbf{I}_{i:\ell} | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i)) \cdot \mathbb{P}(C_n(i) | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1))$$

$$(65) \qquad + \mathbb{P}(\mathbf{I}_{i:\ell} | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i)^c, C_n(i-1)) \cdot \mathbb{P}(C_n(i)^c | A^{(n)}_{\ell,\mathbf{t}}, \mathbf{I}_{1:i-1}, C_n(i-1)).$$

If we consider the first term in the expansion above, we have again by the definition of conditional probability that:

$$\mathbb{P}(\mathbf{I}_{i:\ell}|A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i))$$

(66)
$$= \mathbb{P}(\mathbf{I}_{i+1:\ell}|A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i}, C_n(i)) \cdot \mathbb{P}(I_i|A_{\ell,\mathbf{t}}^{(n)}, \mathbf{I}_{1:i-1}, C_n(i)).$$

The last piece we need is the following simple bound on the difference of two products, where $0 \leq a_i, b_i \leq 1$:

(67)
$$|a_1 b_1 - a_2 b_2| \leq |a_1 - a_2| + |b_1 - b_2|.$$

We now return to our overall goal: namely, we wish to show that the absolute value

$$\left|\mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i}\right|$$

becomes small for $A_{\ell,\mathbf{t}}^{(n)}$ such that $C_n(0)$ holds. To show this, we iterate and combine (61), (63), (64), (65), (66), and (67). For example, after the first step of this iteration, we obtain that:

$$
\begin{aligned}
\left|\mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i}\right| &= \left|\mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}, C_n(1))\mathbb{P}(C_n(1)|A_{\ell,\mathbf{t}}^{(n)}) \right. \\
&\quad + \mathbb{P}(\mathbf{I}|A_{\ell,\mathbf{t}}^{(n)}, C_n(1))\mathbb{P}(C_n(1)^c|A_{\ell,\mathbf{t}}^{(n)}) \\
&\quad \left. - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i}(\mathbb{P}(C_n(1)|A_{\ell,\mathbf{t}}^{(n)}) + \mathbb{P}(C_n(1)^c|A_{\ell,\mathbf{t}}^{(n)}))\right| \\
&\leq \left|\mathbb{P}(\mathbf{I}_{2:\ell}|A_{\ell,\mathbf{t}}^{(n)}, I_1, C_n(1))\mathbb{P}(I_1|A_{\ell,\mathbf{t}}^{(n)}, C_n(1)) \right. \\
&\quad \left. - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i}\right| + \mathbb{P}(C_n(1)^c|A_{\ell,\mathbf{t}}^{(n)}) \\
&\leq \left|\mathbb{P}(\mathbf{I}_{2:\ell}|A_{\ell,\mathbf{t}}^{(n)}, I_1, C_n(1)) - p^{\sum_{i>1} I_i}(1-p)^{\ell - \sum_{i>1} I_i}\right| \\
&\quad + \hat{f}(\varepsilon_1 + 1/n^{1/3}) + O(n^{-1/6}).
\end{aligned}
$$

The first step follows by conditioning, i.e., (65); the second step follows by (66), together with the fact that the absolute value of the difference of two probabilities is bounded above by one; and the third step follows by applying (67) to the absolute value, and then using (61) and (64). Continuing

in this manner, we obtain:

$$|\mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell, \mathbf{t} \text{ s.t. } C_n(0)) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i}|$$

$$\leq \sum_i \hat{f}(\varepsilon_i + 1/n^{1/3}) + \sum_{i=2}^{\ell} \frac{\varepsilon_{i-1}}{\varepsilon_i} + O(n^{-1/6}).$$

We know that

(68) $$\mathbb{P}(C_n(0)^c | T_a^{(n)} = \ell) \leq L^2/k = O(n^{-1/3})$$

using an elementary union bound over events that two rounded departure times are identical. To complete the proof, returning to (58), and recalling the integral is in fact a sum over feasible $\mathbf{t}$, we deduce that:

$$\int \left| \mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell, \mathbf{t}) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i} \right| d\mathbb{P}_n(\mathbf{t}|T_a^{(n)} = ell)$$

$$\leq \sum_i \hat{f}(\varepsilon_i + 1/n^{1/3}) + \sum_{i=2}^{\ell} \frac{\varepsilon_{i-1}}{\varepsilon_i} + O(n^{-1/6}) + \mathbb{P}(C_n(0)^c | T_a^{(n)} = \ell)$$

$$= \sum_i \hat{f}(\varepsilon_i + 1/n^{1/3}) + \sum_{i=2}^{\ell} \frac{\varepsilon_{i-1}}{\varepsilon_i} + O(n^{-1/6}).$$

Note that $\varepsilon_1, \ldots, \varepsilon_\ell$ were arbitrary; thus if we first take $n \to \infty$, and then take $\varepsilon_i \to 0$ in such a way that every ratio $\varepsilon_{i-1}/\varepsilon_i \to 0$ as well, then we conclude that the right hand side of the preceding expression approaches zero as $n \to \infty$. Returning to (57), we conclude that for each ℓ and $\mathbf{I} \in \mathcal{I}(\ell)$ we have:

$$\left| \mathbb{P}(\mathbf{I}|T_a^{(n)} = \ell) - p^{\sum_i I_i}(1-p)^{\ell - \sum_i I_i} \right| \to 0$$

as $n \to \infty$, as required. $\qquad\square$

D.3. **Proof of Corollary 1.** We give a proof of Corollary 1, using Theorems 3 and 4.

Proof of Corollary 1. We first prove the result for applicants. Consider an applicant a who can choose the value of m_a, while other agents play their prescribed mean field strategies. Clearly, choosing $m_a = 0$ leads to a utility of 0, whereas choosing $m_a > 1/c_a$ leads to a negative expected utility since the expected application cost exceeds 1. (In particular, note that $m^* < 1/c_a$.) Hence, it suffices to show that:

Claim 1. *For large enough n, playing $m_a = m^*$ is additively ε-optimal for applicant a among $m_a \in [0, 1/c_a]$.*

(Since $m_a = 0$ has greater utility than any $m_a > 1/c_a$, this will imply that $m_a = m^*$ is additively ε-optimal among $m_a \in [0, \infty)$.)

We prove the claim using Theorem 4 with $m_0 = 1/c_a$. Now, the expected utility of applicant a who chooses m_a is

$$\Pr(\text{Applicant } a \text{ gets at least one offer under } m_a) - c_a m_a \, .$$

Since m^* is a best response under mean field assumptions, it follows that

$$(69) \qquad\qquad 1 \cdot \Pr(Q(m_a) > 0) - c_a m_a \;\leq\; \Pr(Q(m^*) > 0) - c_a m^*$$

for any m_a. Theorem 4 implies that there exists n_0 such that for any $n > n_0$, we have

$$\max_{m_a \in [0, m_0]} d_{\mathrm{TV}}\big((T_a^{(n)}(m_a), Q_a^{(n)}(m_a)), (T(m_a), Q(m_a))\big) \;\leq\; \varepsilon/2 \, .$$

where we have made the dependence on m_a explicit for convenience. It follows that

$$|\Pr(Q(m_a) > 0) - \Pr(Q_a^{(n)}(m_a) > 0)| \;\leq\; \varepsilon/2 \qquad \forall m_a \in [0, m_0] \, .$$

Combining with Eq. (69), we obtain that

$$\Pr(Q_a^{(n)}(m_a) > 0) - c_a m_a \;\leq\; \Pr(Q_a^{(n)}(m^*) > 0) - c_a m^* + \varepsilon \qquad \forall m_a \in [0, m_0] \, ,$$

implying the claim.

We now prove the result for employers who receive no more than R_0 applications. Think of employer e as first selecting a uniformly random order among her applicants, and then screening them (or not) in that order. With this order, denote the applicants by $s_1, s_2, \ldots, s_\ell$ (here ℓ denotes the number of applications) and let $\mathcal{S}$ denote the set of all applicants. Let $\mathbf{I}_b^{(n)}$ denote the applicant availability vector at the time that e departs, with this order. (So $\mathbf{I}_b^{(n)} \in \{0, 1\}^\ell$, where a 1 represents that the corresponding applicant is available.) Let $\mathbf{I}$ denote an analogously defined vector for a hypothetical employer who receives $R \sim \mathrm{Poisson}(rm)$ applications of which $A \sim \mathrm{Binomial}(R, q)$ are from applicants who are still available. It follows from Theorem 3, and the fact that applicants are considered in a uniformly random order, that:

Claim 2. $(R_b^{(n)}, A_b^{(n)}, \mathbf{I}_b^{(n)})$ *converges in total variation distance to* $(R, A, \mathbf{I})$.

Let $\mathcal{S}_i = \{s_j : j \leq i\}$. Let $\mathcal{S}'_i$ denote the subset of $\mathcal{S}_i$ that e makes offers to. (So, for $i < \ell$, employer e does not screen s_{i+1} if one of $\mathcal{S}_{i'}$ accepts an offer.) The next claim follows from Claim 2 above.

Claim 3. *Fix* $\varepsilon' > 0$ *and* $R_0 < \infty$. *There exists* $n_1 < \infty$ *such that the following holds. For any non-negative integer* $\ell \leq R_0$, *any* $i < \ell$ *and any subset* $S'_i \subseteq \mathcal{S}_i$ *with fixed indices in* $\{1, 2, \ldots, i\}$ *we have*

$$(70) \qquad \left| \Pr\left(s_{i+1} \text{ is available} \,\middle|\, |\mathcal{S}| = \ell, \, \mathcal{S}'_i = S'_i, \, \mathcal{S}'_i \text{ are not available}\right) - q \right| \leq \varepsilon'.$$

The prescribed mean field strategy for employer e is ϕ^{α^*}. For $\alpha^* < 1$, we argue that not screening at all is additively ε optimal (for large enough n): First, note that $c'_s \geq \beta q$, since not screening is a best response under the employer mean field assumption. Then we know, from Claim 3, that each time the employer chooses to screen an applicant, the net expected benefit is no more than $\beta \varepsilon' \leq \varepsilon'$. Using an elementary martingale stopping argument, it follows that the overall net utility of any other strategy (relative to the utility of 0 obtained by not screening) is no more than $\mathbb{E}[\bar{R}\varepsilon'] \leq R_0 \varepsilon'$, where $\bar{R}$ is the number of applicants screened under ϕ^1, using $\bar{R} \leq \ell \leq R_0$. Choosing $\varepsilon' \leq \varepsilon/R_0$ we obtain the desired result.

Similarly for $\alpha^* > 0$, we argue that ϕ^1 (keep screening until you hire or run out of applicants) is additively ε-optimal (for large enough n): First, note that $c'_s \leq \beta q$, since screening is a best response under the employer mean field assumption. Then we know, from Claim 3, that each time the employer chooses to screen the next applicant, the net expected benefit is at least $-\beta \varepsilon' \geq -\varepsilon'$, irrespective of the realization of $\mathcal{S}'_i$ (all these applicants rejected their offers). Any other strategy can be better than ϕ^1 by at most ε' in expectation for each applicant screened under ϕ^1 but not under the alternative strategy Using an elementary martingale stopping argument, it follows that ϕ^1 is additively $\mathbb{E}[\underline{R}]\varepsilon' \leq R_0 \varepsilon'$ optimal, where $\underline{R}$ is the number of applicants screened under ϕ^1, but not under the alternative strategy. Choosing $\varepsilon' \leq \varepsilon/R_0$ we obtain the desired result.

Finally, since ϕ^0 is ε-optimal for $\alpha^* < 1$ and ϕ^1 is ε-optimal for $\alpha^* > 0$, we conclude that ϕ^{α^*} is ε-optimal, irrespective of the value of α^*, for large enough n. $\qquad\qquad \square$

D.4. **Proof of Theorem 5.** Throughout our proof of Theorem 5, we think of the state of the system at t as being the subset of applicants who are still available at t (with their arrival times) from among the applicants who arrived during the last 1 time unit. The applications to a specific employer are 'revealed' just before the employer exits. If at time 0, the set of applications by applicants in the system has been revealed, we simply need to move forward 1 time unit before we can use our above description of the state. Thus, our analysis holds for times $t \geq 1$.

In order to prove Theorem 5, we use the following lemma.

Lemma 13. *Fix $m_0 \in [1, \infty)$ and $\alpha \in [0, 1]$. There exists $\kappa' = \kappa'(r, m_0, \beta, \alpha) > 0$ and $C' = C'(r, m_0, \beta) < \infty$ such that for any $n > C'$, for any k, any $m \in [1/m_0, m_0]$, and any starting state, using $k = \lfloor n^{1/3} \rfloor$ we have*

$$(71) \qquad \mathbb{E}\|X(t + 1/k) - X^*\|_1 \leq (1 - \kappa'/k)\|X(t) - X^*\|_1 + C'\left(\frac{1}{\sqrt{n}} + \frac{1}{k^{3/2}}\right).$$

The following corollary is immediate.

Corollary 3. *Fix $m_0 \in [1, \infty)$ and $\alpha \in [0, 1]$. There exists $\kappa' = \kappa'(r, m_0, \beta, \alpha) > 0$ and $C' = C'(r, m_0, \beta) < \infty$ such that for any $n > C'$, for any k, any $m \in [1/m_0, m_0]$, and any starting state,*

$$(72) \qquad \mathbb{E}\|X(t + i/k) - X^*\|_1 \leq (1 - \kappa'/k)^i \|X(t) - X^*\|_1 + C'\left(\frac{1}{\sqrt{n}} + \frac{1}{k^{3/2}}\right)k/\kappa'.$$

With these results in hand, we are ready to prove Theorem 5.

Proof of Theorem 5. Note that $\|X(t_0) - X^*\|_1 \leq \|X(t_0)\|_1 + \|X^*\|_1 = \Sigma(X(t_0)) + \Sigma(X^*) \leq 2$ for any t_0. Use Corollary 3 to track the evolution from $t_0 = t - i/k$ to t where $i = \lfloor kC \log n \rfloor$. For $C = (1 + 2C')/\kappa'$ and using $(1 - \kappa'/k)^i\|X(t_0) - X^*\|_1 \leq \exp(-C\kappa' \log n) \cdot 2 \leq 1/n$, we have

$$(73) \qquad \mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6},$$

yielding the first part of the theorem.

Now consider the second part of the theorem. For any starting state, the system reaches steady state as $t \to \infty$. Let X_{ss} be the steady state distribution of X. Then we have $\lim_{t \to \infty} X(t) = X_{\text{ss}}$. It follows from the dominated convergence theorem that $\mathbb{E}[\|X_{\text{ss}} - X^*\|_1] = \lim_{t \to \infty} \mathbb{E}[\|X(t) - X^*\|_1] \leq Cn^{-1/6}$ using (73). The second part of the theorem follows immediately since if the starting state is distributed as per the steady state distribution, the state at time t is also distributed as per the steady state distribution, so the same bound holds for all $t \geq 0$. □

D.4.1. *Proof of Lemma 13.* All that remains is to prove Lemma 13. Define $\hat{X}(t + 1/k)$ by $\hat{X}_0(t + 1/k) = 1/k$, and for $0 \le j \le k - 1$, $\hat{X}_{j+1}(t + 1/k) = \gamma X_j(t)$, where γ is given by (45), i.e.,

$$(74) \qquad \gamma = 1 - \frac{\alpha(1 - e^{-\eta\|X(t)\|_1})}{rk\|X(t)\|_1}.$$

We use the following two lemmas to prove Lemma 13.

Lemma 14. *Fix $m_0 \in [1, \infty)$. There exists $\kappa = \kappa(r, m_0, \beta) > 0$ such that for any k, any $\alpha \in [0, 1]$, any $m \in [1/m_0, m_0]$, and any $X(t)$ we have*

$$(75) \qquad \|\hat{X}(t + 1/k) - X^*\|_1 \le (1 - \alpha\kappa/k)\|X(t) - X^*\|.$$

Lemma 15. *Fix $m_0 < \infty$. There exists $C' = C'(m_0, r, \beta) < \infty$ such that for $n > C'$, any $m \le m_0$, any k, any $\alpha \in [0, 1]$, and any starting state, we have*

$$(76) \qquad \mathbb{E}\|X(t + 1/k) - \hat{X}(t + 1/k)\|_1 \le C'(n^{-1/2} + k^{-3/2}).$$

Proof of Lemma 13. Using triangle inequality, we have

$$(77) \qquad \mathbb{E}\|X(t + 1/k) - X^*\|_1 \le \|\hat{X}(t + 1/k) - X^*\|_1 + \mathbb{E}\|X(t + 1/k) - \hat{X}(t + 1/k)\|_1.$$

Lemma 13 now follows immediately from Lemmas 14 and 15 with $\kappa' = \alpha\kappa$. □

To prove Lemmas 14 and 15, we use the following facts about Eq. (74).

Lemma 16. *Let $h(\Sigma) = (1 - e^{-\eta\Sigma})/\Sigma$, so that $\gamma = 1 - \alpha h(\Sigma)/(rk)$. Then*

(1) *We have $-\eta^2 \le \frac{dh}{dx} \le 0$.*
(2) *We have $0 \le \frac{d\gamma}{d\Sigma} \le \frac{\alpha\eta^2}{rk}$.*
(3) *We have $\frac{\alpha(1 - e^{-\eta})}{rk} \le 1 - \gamma \le \frac{\alpha\eta}{rk}$*
(4) *We have $\frac{d}{d\Sigma}(1 - \gamma)\Sigma \ge \frac{\alpha\eta e^{-\eta}}{rk}$.*

Proof. Note that $\frac{dh}{dx} = \frac{(1 + \eta x)e^{-\eta x} - 1}{x^2}$. The first inequality in item 1 comes from substituting $e^{-\eta x} \ge 1 - \eta x$, the second from substituting $1 + \eta x \le e^{\eta x}$. Item 2 follows from the fact that $\gamma = 1 - \frac{\alpha}{rk}h(\Sigma)$. The third item comes from the fact that $(1 - \gamma)$ is monotone in Σ (from item 2), and $\Sigma \in [0, 1]$. The final item comes from the fact that $\frac{d}{d\Sigma}(1 - \gamma)\Sigma = \frac{d}{d\Sigma}\frac{\alpha(1 - e^{-\eta\Sigma})}{rk} = \frac{\alpha\eta e^{-\eta\Sigma}}{rk}$, which is decreasing in Σ (and $\Sigma \le 1$). □

Lemma 17. *Recall $\rho = (1 - m\beta/n)^{rn}$. Let $\tilde{h}(\Sigma) = (1 - \rho^\Sigma)/\Sigma$. Then we have $(\log \rho)^2 \le \frac{d\tilde{h}}{dx} \le 0$.*

Proof. Analogous to the proof of Lemma 16 item 1.   □

D.4.2. *Proof of Lemma 14.* In this subsection we prove Lemma 14; we prove Lemma 15 in the next subsection.

Proof of Lemma 14. For simplicity, we use X to denote $X(t)$, $\hat{X}$ for $X(t + 1/k)$, Σ for $\Sigma(X)$, and γ for $\gamma(\Sigma)$.

We first prove Lemma 14 for the case $\Sigma \geq \Sigma^*$. By Lemma 16.2, this implies $\gamma \geq \gamma^*$. Note that

$$(78) \qquad |a - b| = (b - a) + 2[a - b]_+ ,$$

which we use to get that

$$(79) \qquad \|X - X^*\|_1 = \sum_{j=0}^{k-1}(X_j - X_j^*) + 2[X_j^* - X_j]_+ = \Sigma - \Sigma^* + 2\sum_{j=0}^{k-1}[X_j^* - X_j]_+ .$$

Recall that $\hat{X}_0 = X_0^* = 1/k$. It follows that

$$\|\hat{X} - X^*\|_1 = \sum_{j=1}^{k-1}\left|X_j^* - \hat{X}_j\right| = \sum_{j=0}^{k-2}\left|\gamma^* X_j^* - \gamma X_j\right|$$

$$= \sum_{j=0}^{k-2}\gamma X_j - \gamma^* X_j^* + 2[\gamma^* X_j^* - \gamma X_j]_+$$

$$= \gamma\Sigma - \gamma X_{k-1} - \gamma^*\Sigma^* + \gamma^* X_{k-1}^* + 2\sum_{j=0}^{k-2}[\gamma^* X_j^* - \gamma X_j]_+$$

$$(80) \qquad = \gamma\Sigma - \gamma^*\Sigma^* - \left|\gamma^* X_{k-1}^* - \gamma X_{k-1}\right| + 2\sum_{j=0}^{k-1}[\gamma^* X_j^* - \gamma X_j]_+ ,$$

where both the first and last lines use (78). Now, by Lemma 16, item 4,

$$\gamma\Sigma - \gamma^*\Sigma^* = \Sigma - \Sigma_* + [(1 - \gamma^*)\Sigma^* - (1 - \gamma)\Sigma]$$

$$(81) \qquad \leq (\Sigma - \Sigma^*)\left(1 - \frac{\alpha\eta e^{-\eta}}{rk}\right).$$

Also, using $\gamma \geq \gamma^*$ and the upper bound on γ^* from Lemma 16, item 3, we have

$$[X^*_{j+1} - \hat{X}_{j+1}]_+ = [\gamma^* X^*_j - \gamma X_j]_+$$

$$\leq [\gamma^* X^*_j - \gamma^* X_j]_+ = \gamma^*[X^*_j - X_j]_+$$

(82)
$$\leq \left(1 - \frac{\alpha(1 - e^{-\eta})}{rk}\right)[X^*_j - X_j]_+ .$$

If we define $\kappa = \min_{m \in [1/m_0, m_0]} \frac{1}{r} \min(\eta e^{-\eta}, 1 - e^{-\eta}) > 0$, then substituting (81) and (82) into (80) yields

$$\|\hat{X} - X^*\|_1 \leq (1 - \alpha\kappa/k)(\Sigma - \Sigma^*) - |\gamma^* X^*_{k-1} - \gamma X_{k-1}| + 2(1 - \alpha\kappa/k)\sum_{j=0}^{k-1}[X^*_j - X_j]_+$$

(83)
$$\leq (1 - \alpha\kappa/k)\|X - X^*\|_1,$$

where the second line follows from dropping the absolute value term and applying (79).

The proof for the complementary case $\Sigma \leq \Sigma^*$ is analogous. $\qquad\square$

D.4.3. *Proof of Lemma 15.* Note that

(84)
$$\mathbb{E}\left|X_j - \hat{X}_j\right| \leq \left|\mathbb{E}[X_j] - \hat{X}_j\right| + \mathbb{E}\left|X_j - \mathbb{E}[X_j]\right| \leq \left|\mathbb{E}[X_j] - \hat{X}_j\right| + \sqrt{\mathrm{Var}(X_j)},$$

where we have applied the triangle inequality and Jensen's inequality in turn. Sum over j to get

(85) $\mathbb{E}\|X(t + 1/k) - \hat{X}(t + 1/k)\|_1 \leq \|\mathbb{E}[X(t + 1/k)] - \hat{X}(t + 1/k)\|_1 + \sum_{j=0}^{k-1}\sqrt{\mathrm{Var}(X_j(t + 1/k))}.$

Fix an applicant a who is in the system at time t and has arrived within the last $1 - 1/k$ time units. Let I_a be the indicator that a is matched in the next $1/k$ time units, and let $\tilde{\gamma} = \mathbb{E}[1 - I_a]$ be the probability that this applicant is still in the system at time $t + 1/k$. To prove Lemma 15 we use the following lemma.

Lemma 18. *Fix $m_0 < \infty$. There exists $C = C(m_0, r, \beta) < \infty$ such that for $n > C$, any $m \leq m_0$, any k, any $\alpha \in [0, 1]$, and any starting state, the following hold:*

(86)
$$|\tilde{\gamma} - \gamma| \leq C/k^2$$

For applicants a and a' who arrived between $t - 1 + 1/k$ and $t + 1/k$ we have

$$(87) \qquad\qquad \text{Var}(I_a) \le C/k$$

$$(88) \qquad\qquad |\text{Cov}(I_a, I_{a'})| \le 1/k^3 \text{ for } a \ne a'$$

We defer the proof of Lemma 18 below.

We use Lemma 18 to prove the following lemma.

Lemma 19. *Fix $m_0 < \infty$. There exists $C = C(m_0, r, \beta) < \infty$ such that for $n > C$, any $m \le m_0$, any k, any $\alpha \in [0,1]$, and any starting state, the following hold*

$$(89) \qquad \|\mathbb{E}[X(t + 1/k)] - \hat{X}(t + 1/k)\|_1 \le (C + \eta/r)/k^2 \le (C + \eta/r)k^{-3/2},$$

and

$$(90) \qquad \sum_{j=0}^{k-1} \sqrt{\text{Var}(X_j(t + 1/k))} \le \sqrt{C/r}\, n^{-1/2} + k^{-3/2}.$$

Here C is the same as in Lemma 18.

Proof of Lemma 15. Note that Lemma 19 implies that

$$(91) \quad \|\mathbb{E}[X(t + 1/k)] - \hat{X}(t + 1/k)\|_1 + \sum_{j=0}^{k-1} \sqrt{\text{Var}(X_j(t + 1/k))} \le \sqrt{C/r}\, n^{-1/2} + (\eta/r + C + 1)k^{-3/2},$$

using Lemma 18. If we take $C' = \max(\sqrt{C/r}, \eta/r + C + 1)$, then (85) and (91) imply Lemma 15. $\qquad\square$

Proof of Lemma 19. To establish (89), note that for $j = 0, 1, \ldots, k - 1$

$$(92) \qquad \left| \mathbb{E}[X_j(t + 1/k)] - \hat{X}_j(t + 1/k) \right| = |\tilde\gamma - \gamma|\, X_{j-1}(t) \le |\tilde\gamma - \gamma|/k \le C/k^3,$$

with the final inequality coming from Lemma 18 Eq. (86). Meanwhile, using the fact that newly arrived applicants are matched with probability no greater than $\alpha\eta/(rk)$ in time $(1/k)$, we have that

$$(93) \qquad \left| \mathbb{E}[X_0(t + 1/k)] - \hat{X}_0(t + 1/k) \right| \le (1/k)\max(\alpha\eta/(rk), 1 - \gamma) = \alpha\eta/(rk^2),$$

where we used Lemma 16 item 2. Summing (92) over j and combining it with (93) yields (89).

We now turn our attention to the variance term, i.e. (90). For $j \geq 1$,

$$
\text{Var}(X_j(t + 1/k)) = \frac{1}{(rn)^2} \text{Var}\left(\sum_{s \in N_{j-1}(t)} I_a \right)
$$

$$
= \frac{1}{(rn)^2} \left(|N_{j-1}(t)| \text{Var}(I_a) + 2\binom{|N_{j-1}(t)|}{2} \text{Cov}(I_a, I_{a'}) \right)
$$

$$
\leq X_{j-1}(t)\text{Var}(I_a)/rn + X_{j-1}(t)^2 \text{Cov}(I_a, I_{a'}).
$$

$$
(94) \qquad\qquad\qquad \leq \frac{C}{rnk^2} + \frac{1}{k^5},
$$

where the final line comes from the fact that $X_{j-1}(t) \leq 1/k$ and Lemma 18 Eqs. (87) and (88). By the concavity of the square root function, this implies

$$
(95) \qquad\qquad \sqrt{\text{Var}(X_j(t + 1/k))} \leq \sqrt{C}/(k\sqrt{rn}) + 1/k^{5/2}
$$

Summing (95) over j yields (90). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 18. Note that for all $t' \in (t, t + 1/k)$, we have

$$
(96) \qquad\qquad \Sigma(t) - 1/(rk) \leq \Sigma(t') \leq \Sigma(t) + 1/k,
$$

since the number of new arrivals in the system is rn/k, the number of "match" departures is at most n/k, and $\Sigma(t') = N(t')/(rn)$.

Fix a, and suppose that there are $rn\Sigma$ available applicants when an employer possibly screens and exits. The probability that a exits at this opportunity is $\tilde{h}(\Sigma)/(rn) = \frac{\alpha(1-\rho^\Sigma)}{rn\Sigma}$ and this is a monotone decreasing function of Σ by Lemma 17 item 1. For convenience we define $\tilde{h}(x) = \alpha \cdot (-\log \rho)/rn$ for $x \leq 0$. Since there are n/k opportunities to depart, Eq. (96) implies

$$
(97) \qquad \left(1 - \frac{\alpha}{rn}\tilde{h}(\Sigma(t) + 1/k)\right)^{n/k} \leq \mathbb{E}[1 - I_a] = \tilde{\gamma} \leq \left(1 - \frac{\alpha}{rn}\tilde{h}(\Sigma(t) - 1/(rk))\right)^{n/k}
$$

Recall that $\lim_{n\to\infty} \rho = \exp(-\eta)$. By Lemma 17 item 1 we have that

$$
(98) \qquad\qquad \tilde{h}(\Sigma(t) - 1/rk) \leq \tilde{h}(\Sigma(t)) + \frac{2\eta^2}{rk}, \quad \text{and}
$$

$$
(99) \qquad\qquad \tilde{h}(\Sigma(t) + 1/k) \geq \tilde{h}(\Sigma(t)) - \frac{2\eta^2}{k}.
$$

for large enough n. It follows by substitution into (97) that

$$(100) \qquad \left(1 - \frac{1}{rn}\tilde{h}(\Sigma(t)) - \frac{2\eta^2}{rnk}\right)^{n/k} \leq \tilde{\gamma} \leq \left(1 - \frac{1}{rn}\tilde{h}(\Sigma(t)) + \frac{2\eta^2}{r^2nk}\right)^{n/k}.$$

Using the inequality

$$(101) \qquad 1 - m\varepsilon \leq (1-\varepsilon)^m \leq 1 - m\varepsilon + \frac{1}{2}m^2\varepsilon^2,$$

we get

$$(102) \qquad 1 - \frac{\alpha}{rk}\tilde{h}(\Sigma(t)) - \frac{2\alpha\eta^2}{rk^2} \leq \tilde{\gamma} \leq 1 - \frac{\alpha}{rk}\tilde{h}(\Sigma(t)) + \frac{2\alpha\eta^2}{r^2k^2} + \frac{1}{2}\left(\frac{\alpha}{rk}\right)^2\tilde{h}(\Sigma(t))^2.$$

for large enough k. Now $\rho = \exp(-\eta) + O(1/n)$. Also, $|\partial\tilde{h}/\partial\rho| = \rho^{\Sigma-1} \leq 1/\rho \leq 2\exp(\eta)$ for all $\Sigma \in [0,1]$, for large enough n. It follows that

$$(103) \qquad \tilde{h}(\Sigma) = h(\Sigma) + O(1/n).$$

Since we have $\gamma = 1 - \frac{1-e^{-\eta\Sigma}}{rk\Sigma} = 1 - \frac{1}{rk}h(\Sigma)$, combining Eqs. (102) and (103), we obtain $|\tilde{\gamma} - \gamma|$ is $\mathcal{O}(\alpha/k^2)$, establishing (86) in Lemma 18.

Note that

$$(104) \qquad \mathrm{Var}(I_a) = \tilde{\gamma}(1 - \tilde{\gamma}) \leq 1 - \tilde{\gamma}.$$

Using Eq. (102) and $\tilde{h}(\Sigma(t)) \leq \tilde{h}(0) = -\log(\rho) = \eta + O(1/n)$ from Lemma 17, we have

$$1 - \tilde{\gamma} \leq \frac{\alpha\eta}{rk} + O(\alpha/k^2),$$

which proves (87) from Lemma 18 for applicants that arrived between $t - 1 + 1/k$ and t, and were available at t. For any applicant a that arrived after t, the probability of still being available at $t + 1/k$ is even larger than $\tilde{\gamma}$, i.e., this probability is in $[\tilde{\gamma}, 1]$, leading to $\mathrm{Var}(I_a) \leq \tilde{\gamma}(1 - \tilde{\gamma})$ since $\tilde{\gamma} \geq 1/2$ leading to (87) for a.

Finally, we bound $\mathrm{Cov}(I_a, I_{a'})$. We define a bipartite 'interaction' graph $G = (V_S, V_B, E)$ whose vertex sets are $V_S = \mathcal{S}(t) \cup \mathcal{S}(t + 1/k)$, i.e., the applicants who arrive between $t - 1$ and $t + 1/k$, and $V_B = \mathcal{B}(t + 1/k)\backslash\mathcal{B}(t)$, i.e., the set of employers who arrive between t and $t + 1/k$. For $s \in V_S$ and $b \in V_B$, we decide on the presence of edge (s, b) as follows: Let τ denote the time of arrival of a. Then if $b \in \mathcal{B}(\tau)$, i.e., e is in the system when a arrives, we set $\mathbb{I}((s, b) \in E) = \mathbb{I}(s \in M_b)$, i.e.,

we include edge (s, b) if applicant a applies to employer e. If $b \notin \mathcal{B}(\tau)$, i.e., e is not in the system when a arrives, we draw $\mathbb{I}((s, b) \in E) \sim$ Bernoulli(m/n), independent of everything else.

Note that the interaction graph G as defined above is a bipartite Erdos-Renyi graph with $rn(1 + 1/k)$ vertices on one side, n/k vertices on the other, and edge probability m/n independently between vertices on the two sides. The following fact is immediate (see, e.g., Janson et al. (2011)) :

Fact 1. *Fix k such that $r(1 + 1/k)(1/k)m < 1$. There exists $C < \infty$ such that the following occurs. For any $\varepsilon > 0$ there exists $n_0 < \infty$ such that for all $n > n_0$, with probability at least $1 - \varepsilon$, no connected component in G has more than $C \log n$ vertices.*

Here the threshold for existence of a giant component is $r(1 + 1/k)(1/k)m = 1$. Thus it is sufficient to ensure that $r(1 + 1/k)(1/k)m \leq 2rm/k < 1$. In fact, fixing k, Fact 1 holds uniformly for all $m < m_0 = k/(2r)$, since the size of the largest connected component is monotone in the edge probability.

Clearly, I_a depends only on the connected component $\mathcal{C}_a$ containing a and similarly for $I_{a'}$. Choose arbitrary $\varepsilon > 0$. Let $\mathcal{E}_1$ be the event that no connected component has size more than $C \log n$. Fact 1 tells us that $\mathbb{P}(\mathcal{E}_1^c) \leq \varepsilon$. Let $\mathcal{E}_2$ be the event that $a' \notin \mathcal{C}_a$. Clearly, $\mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1) \leq C \log n / (rn) \leq \varepsilon$ for large enough n. We deduce that

$$(105) \qquad \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c) = \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c \cap \mathcal{E}_1) \leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1) \leq 2\varepsilon$$

Reveal $\mathcal{C}_a$. If $|\mathcal{C}_a| \geq C \log n$ or $a' \in \mathcal{C}_a$, declare 'failure'. Here $|\mathcal{C}_a|$ denotes the number of vertices in $\mathcal{C}_a$. Suppose failure does not occur.

Let $\mathcal{C}_s = \mathcal{C}$ be the revealed connected component. Since failure has not occurred we know that $\mathcal{C}$ contains no more than $C \log n$ nodes and does not contain a'. Now consider the conditional distribution of $\mathcal{C}_{a'}$ given $\mathcal{C}_s = \mathcal{C}$.

Claim 4. *Consider any candidate connected component $\mathcal{C}'$, containing a' and not overlapping with $\mathcal{C}$. We have*

$$\mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') = \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' | \mathcal{C}_s = \mathcal{C})(1 - m/n)^{|\mathcal{C}'|_a |\mathcal{C}|_b + |\mathcal{C}'|_b |\mathcal{C}|_a},$$

where $|\mathcal{C}|_a$ denotes the number of applicants in component $\mathcal{C}$, and $|\mathcal{C}|_b$ denotes the number of employers in component $\mathcal{C}$.

Proof. The distribution of the rest of G conditioned on $\mathcal{C}_s = \mathcal{C}$ has edge (s, b) present iid with probability m/n if both a and e are not in $\mathcal{C}$, and not present if one of a or e is present in $\mathcal{C}$. The result follows from a standard revelation argument on $\mathcal{C}_{a'}$. $\square$

We have

$$\text{(106)} \qquad \mathbb{E}(I_{a'}) = \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}')\mathbb{E}(I_{a'}|\mathcal{C}_{a'} = \mathcal{C}'),$$

and

$$\text{(107)} \qquad \mathbb{E}[I_{a'}|I_a = 1] = \sum_{\mathcal{C} \ni s, \mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_a = \mathcal{C}|I_a = 1)\mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}'|\mathcal{C}_a = \mathcal{C})\mathbb{E}[I_{a'}|\mathcal{C}_{a'} = \mathcal{C}']$$

using the fact that I_a—$\mathcal{C}_a$—$\mathcal{C}_{a'}$—$I_{a'}$ form a Markov chain. Below we argue that the sum in Eq. (107) is very close to the sum in Eq. (106).

Let $\pi \equiv \mathbb{E}[I_a] = \mathbb{E}[I_{a'}]$. If $\pi = 0$, which might occur for instance if a arrives just before $t + 1/k$, we immediately have $\text{Cov}(I_a, I_{a'}) = 0$. As such, we assume $\pi > 0$ in what follows. We deduce from Eq. (105) that

$$\text{(108)} \qquad \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c|I_s = 1) \leq 2\varepsilon/\pi$$

It follows from Eqs. (107) and (108) that

$$\mathbb{E}[I_{a'}|I_a = 1] = \sum_{\substack{\mathcal{C} \ni s, \mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}| \leq C \log n, |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_a = \mathcal{C}|I_a = 1)\mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}'|\mathcal{C}_a = \mathcal{C})\mathbb{E}[I_{a'}|\mathcal{C}_{a'} = \mathcal{C}']$$

$$\text{(109)} \qquad\qquad\qquad + \; \delta_1,$$

where $0 \leq \delta_1 \leq \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c|I_a) \leq 2\varepsilon/\pi$. Now using Claim 4, we know that for such $(\mathcal{C}, \mathcal{C}')$ we have

$$\mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}'|\mathcal{C}_a = \mathcal{C}) = \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}')(1 + \delta_{\mathcal{C}, \mathcal{C}'}),$$

where $0 \leq \delta_{\mathcal{C}, \mathcal{C}'} \leq \varepsilon$ for large enough n. It follows that

$$\sum_{\substack{\mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}'|\mathcal{C}_a = \mathcal{C})\mathbb{E}[I_{a'}|\mathcal{C}_{a'} = \mathcal{C}']$$

$$\text{(110)} \qquad = (1 + \delta_2) \sum_{\substack{\mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}')\mathbb{E}[I_{a'}|\mathcal{C}_{a'} = \mathcal{C}']$$

for some $0 \leq \delta_2 \leq \varepsilon$. Now,

$$(111) \qquad \mathbb{P}(|\mathcal{C}_{a'}| \geq C \log n) \leq \varepsilon$$

and

$$\mathbb{P}(v \in \mathcal{C}_{a'} \mid |\mathcal{C}_{a'}| \leq C \log n) \leq C \log n/(kn)$$

for any agent v, whether employer or applicant, hence for any $\mathcal{C}$ s.t. $a' \notin \mathcal{C}$ and $|\mathcal{C}| \leq C \log n$ we have

$$(112) \qquad \mathbb{P}(\mathcal{C} \cap \mathcal{C}_{a'} \neq \Phi \mid |\mathcal{C}_{a'}| \leq C \log n) \leq (C \log n)^2/(kn) \leq \varepsilon$$

for large enough n. Combining Eqs. (111) and (112), we obtain

$$(113) \qquad \mathbb{P}\big((|\mathcal{C}_{a'}| \geq C \log n) \cup (\mathcal{C} \cap \mathcal{C}_{a'} \neq \Phi)\big) \leq 2\varepsilon.$$

Plugging in to Eq. (110) we obtain

$$\sum_{\substack{\mathcal{C}' \ni a' \text{ s.t. } \mathcal{C} \cap \mathcal{C}' = \Phi, \\ |\mathcal{C}'| \leq C \log n}} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}' \mid \mathcal{C}_a = \mathcal{C}) \mathbb{E}[I_{a'} \mid \mathcal{C}_{a'} = \mathcal{C}']$$

$$= (1 + \delta_2)\left(-\delta_3 + \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} \mid \mathcal{C}_{a'} = \mathcal{C}']\right)$$

$$(114) \qquad = \delta_4 + \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} \mid \mathcal{C}_{a'} = \mathcal{C}']$$

for some $0 \leq \delta_3 \leq 2\varepsilon$, leading to $3\varepsilon \leq -2\varepsilon(1+\varepsilon) \leq \delta_4 \leq \varepsilon$. Plugging Eq. (114) back into Eq. (107), we obtain

$$\mathbb{E}[I_{a'} \mid I_a = 1] = \sum_{\substack{\mathcal{C} \ni s \text{ s.t.} \\ |\mathcal{C}| \leq C \log n}} \mathbb{P}(\mathcal{C}_a = \mathcal{C} \mid I_a = 1)\left(\delta_4 + \sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} \mid \mathcal{C}_{a'} = \mathcal{C}']\right)$$

$$(115) \qquad = \delta_5 + \left(\sum_{\mathcal{C}' \ni a'} \mathbb{P}(\mathcal{C}_{a'} = \mathcal{C}') \mathbb{E}[I_{a'} \mid \mathcal{C}_{a'} = \mathcal{C}']\right)\left(\sum_{\substack{\mathcal{C} \ni s \text{ s.t.} \\ |\mathcal{C}| \leq C \log n}} \mathbb{P}(\mathcal{C}_a = \mathcal{C} \mid I_a = 1)\right)$$

where $|\delta_5| \leq |\delta_4| \leq 3\varepsilon$. The first term in the product is simply $\pi' \equiv \mathbb{P}(I_{a'} = 1) = \mathbb{E}[I_{a'}]$, as noted in Eq. (106). The second term in the product is $\mathbb{P}(|\mathcal{C}_a| \leq C \log n \mid I_a = 1) \geq 1 - \mathbb{P}(\mathcal{E}_1 \mid I_a = 1) \geq 1 - \varepsilon/\pi$,

where again $\pi = \mathbb{P}(I_a = 1) \geq 1/(C_1 k)$. We deduce that

$$(116) \qquad \mathbb{E}[I_{a'}|I_a = 1] = \delta_6 + \pi'$$

where $|\delta_6| \leq \varepsilon(3 + 1/\pi)$ for large enough n. We have

$$\text{Cov}(I_a, I_{a'}) = \mathbb{E}[I_a I_{a'}] - \mathbb{E}[I_a]\mathbb{E}[I_{a'}]$$
$$= \mathbb{E}[I_a]\mathbb{E}[I_{a'}|I_a = 1] - \pi\pi'$$
$$= \pi(\pi' + \delta_6) - \pi\pi' = \pi\delta_6$$

Hence

$$|\text{Cov}(I_a, I_{a'})| \leq \varepsilon(3\pi + 1) \leq 4\varepsilon$$

Choosing $\varepsilon = 1/(4k^3)$ yields the desired result. $\qquad \square$

APPENDIX E. PROOFS: SECTION 5

Consider an applicant in the mean field environment, where each application yields an offer with probability p. If the applicant chooses to send an average of m applications, they match with probability $1 - e^{-mp}$ and incur expected application costs of $c_a m$. Thus, we define

$$(117) \qquad \Pi_a(m, \alpha) = 1 - e^{-m\mathcal{P}(m,\alpha)} - c_a m.$$

Similarly, we can consider an employer in the mean field environment, where each applicant is available with probability q. If this employer chooses to screen, they match if and only if they have a qualified available applicant (which occurs with probability $1 - e^{-rm\beta q}$). We claim that the expected number of applicants screened is equal to the probability that the employer matches, divided by $q\beta$. [19] Thus, we define

$$(118) \qquad \Pi_e(m, \alpha) = \alpha(1 - e^{-rm\beta\mathcal{Q}(m,\alpha)})(1 - c'_s/\mathcal{Q}(m,\alpha)).$$

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[19]To see this, let X be geometric with parameter βq (this represents the number of applicants that would need to be screened before matching), and let Y be Poisson with parameter rm (this represents the number of applications received). Then employers who choose to screen expect to screen $\mathbb{E}[\min\{X, Y\}] = \mathbb{E}[X] - \mathbb{E}[(X - Y)1_{X>Y}] = \mathbb{E}[X](1 - \mathbb{P}(X > Y))$ applicants, where we use the memoryless property of the geometric distribution in the second equality. But $1 - \mathbb{P}(X > Y)$ is the probability that a match is found (conditioned on deciding to screen).

Note that in the expression for Π_a, both m and $m\mathcal{P}(m, \alpha)$ appear. In the expression for Π_e, both $\mathcal{Q}(m, \alpha)$ and $m\mathcal{Q}(m, \alpha)$ appear. Because $\mathcal{P}(m, \alpha)$ and $\mathcal{Q}(m, \alpha)$ have no closed form, directly analyzing the expressions in (117) and (118) is difficult. For this reason, in Proposition 7, we re-express Π_a and Π_e as functions of only the model parameters (r, c_a', c_s'), the value α selected by employers, and the quantity $m\mathcal{P}(m, \alpha)$.

Proposition 7. *For any m and α, we have*

(119) $\qquad \Pi_a(m, \alpha) = 1 - e^{-m\mathcal{P}(m,\alpha)} - c_a' m\mathcal{P}(m, \alpha)/(\alpha g(-\log(1 - \frac{r}{\alpha}(1 - e^{-m\mathcal{P}(m,\alpha)}))))$

(120) $\qquad \Pi_e(m, \alpha) = r(1 - e^{-m\mathcal{P}(m,\alpha)} - c_s' m\mathcal{P}(m, \alpha)).$

Furthermore,

(121) $$\Pi_a^* = 1 - (1 + m^* p^*)e^{-m^* p^*}.$$

(122) $$\Pi_e^* = r(1 - e^{-m^* p^*} - c_s' m^* p^*).$$

Proof. Recall that the mean field equations (4) and (5) imply that $\mathcal{P}(m, \alpha)$ and $\mathcal{Q}(m, \alpha)$ solve the following equations

(123) $\qquad rm\mathcal{P}(m, \alpha)\mathcal{Q}(m, \alpha) = \alpha(1 - e^{-rm\beta\mathcal{Q}(m,\alpha)}) = r(1 - e^{-m\mathcal{P}(m,\alpha)}).$

Applying this to (118) yields (120), from which (122) follows immediately.

To get to (119), we note that $c_a m = (\alpha\beta)c_a' m/\alpha$. The mean-field equation (5) for $\mathcal{P}(m, \alpha)$ implies that

$$\alpha\beta = \mathcal{P}(m, \alpha)/g(rm\beta\mathcal{Q}(m, \alpha)) = \mathcal{P}(m, \alpha)/g(-\log(1 - \frac{r}{\alpha}(1 - e^{-m\mathcal{P}(m,\alpha)}))),$$

where the final equality follows from solving (123) for $rm\beta\mathcal{Q}(m, \alpha)$. Combining these facts and substituting into (117) yields (119). We obtain (121) by applying (2) (which gives the applicant's best response as a function of p) to (119). $\qquad \square$

From Proposition 7, we have the following corollary.

Proposition 8. *For any $\alpha < 1$ and any $m > 0$, there exists $m' < m$ such that $\Pi_a(m', 1) > \Pi_a(m, \alpha)$ and $\Pi_e(m', 1) = \Pi_e(m, \alpha)$.*

Proof. Note that by Lemma 3, $\mathcal{Q}(m,1) < \mathcal{Q}(m,\alpha) \leq \mathcal{Q}(0,\alpha) = 1$, and furthermore $\mathcal{Q}(\cdot,1)$ is continuously decreasing. It follows that for some $m' < m$, $\mathcal{Q}(m',1) = \mathcal{Q}(m,\alpha)$. Because the mean-field consistency equation (4) states that $\mathcal{Q} = g(m\mathcal{P})$, it follows that $m'\mathcal{P}(m',1) = m\mathcal{P}(m,\alpha)$. This fact, combined with (120) from Proposition 7, implies that $\Pi_e(m',1) = \Pi_e(m,\alpha)$. Furthermore, applying (117) from Proposition 7, we see that

$$\Pi_a(m',1) = 1 - e^{-m'\mathcal{P}(m',1)} - c_a m' = 1 - e^{-m\mathcal{P}(m,\alpha)} - c'_a m' > 1 - e^{-m\mathcal{P}(m,\alpha)} - c_a m = \Pi_a(m,\alpha).$$

$\square$

Proposition 8 states that the Pareto frontier of (Π_e, Π_a) consists only of points where $\alpha = 1$. When $\alpha = 1$, Proposition 7 gives Π_a and Π_e as functions of only r, c'_a, c'_s and the quantity $m\mathcal{P}(m,1)$. Motivated by this, for fixed r, c'_a, c'_s we define

(124) $$SW(x) = 1 - e^{-x} - c'_a x / g(-\log(1 - r(1 - e^{-x})))$$

(125) $$BW(x) = r(1 - e^{-x} - c'_s x)$$

Note that $\Pi_a(m,1) = SW(m\mathcal{P}(m,1))$, and $\Pi_e(m,1) = BW(m\mathcal{P}(m,1))$. Thus, we have reduced the problem of optimizing Π_a and Π_e to that of optimizing SW and BW over the set of values that the quantity $m\mathcal{P}(m,1)$ may attain.[20]

Lemma 20. *For any r, c'_a, c'_s, the functions $\Pi_a(\cdot,1)$, $\Pi_e(\cdot,1)$, $SW(\cdot)$, and $BW(\cdot)$ are unimodal. Π_a and SW have a unique local maximum, and Π_e and BW either have a unique local maximum or are strictly increasing.*

Proof. Proposition 7 establishes that $\Pi_e(m,1) = BW(m\mathcal{P}(m,1))$, and $\Pi_a(m,1) = SW(m\mathcal{P}(m,1))$. Lemma 3 states that $\mathcal{Q}(m,1) = g(m\mathcal{P}(m,1))$ is decreasing in m, implying that $m\mathcal{P}(m,1)$ is increasing in m (since $g(\cdot)$ is decreasing). Thus, the unimodality of Π_a and Π_e follows from the unimodality of SW and BW.

It is straightforward to show that BW is concave. Thus, all that remains is to prove that for all $r > 0$ and $c'_a \in (0,1)$, SW is unimodal. Because SW has a continuous first derivative, $SW'(0) = 1 - c'_a > 0$, and SW is negative for sufficiently large x, it suffices to show that there is a unique solution to $SW'(x) = 0$.

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[20]Lemma 3 implies that if $r \leq 1$, $m\mathcal{P}(m,1)$ is onto $[0,\infty)$; if $r > 1$, $m\mathcal{P}(m,1)$ is onto $[0, \log \frac{r}{r-1})$.

For the purposes of this proof, for fixed r we define $h(x) = -\log(1 - r(1 - e^{-x}))$, $b(x) = g(h(x))$, and $u(x) = x/b(x)$, so that $SW(x) = 1 - e^{-x} - c'_a u(x)$. Then

$$(126) \qquad SW'(x) = e^{-x} - c'_a u'(x) = 0 \Leftrightarrow u'(x)e^x = 1/c'_a.$$

Note that

$$(127) \qquad u'(x) = \frac{1}{b(x)} - x\frac{b'(x)}{b(x)^2},$$

so $u'(0)e^0 = 1$. It follows from (126) that $SW'(x) = 0$ has a unique solution for all $c'_a \in (0,1)$ if and only if $u'(x)e^x$ is (strictly) increasing and unbounded. We see that

$$\frac{d}{dx}u'(x)e^x = e^x(u'(x) + u''(x)).$$

To show that $u'(x)e^x$ is increasing and unbounded, we will show that $u'(x) + u''(x) > 1$.

By differentiating (127), we see that

$$u'(x) + u''(x) = \frac{1}{b(x)}\left(1 - \frac{xb'}{b} - \frac{xb''}{b} + 2x\left(\frac{b'}{b}\right)^2 - 2\frac{b'}{b}\right).$$

Note that $b'(x) = g'(h(x))h'(x) < 0$, so every term in the above sum except $-x\frac{b''}{b}$ is clearly positive. Since $b(x) \le 1$, to show that $u'(x) + u''(x) > 1$, it suffices to show that $-\frac{xb'}{b} - \frac{xb''}{b} > 0$, or equivalently, $b'(x) + b''(x) < 0$, or equivalently

$$(128) \qquad \frac{b''(x)}{b'(x)} > -1.$$

We note (omitting the algebra) that

$$(129) \qquad b'(x) = g'(h(x))h'(x)$$

$$(130) \qquad b''(x) = g''(h(x))h'(x)^2 + g'(h(x))h''(x).$$

$$(131) \qquad h'(x) = re^{h(x)-x}$$

$$(132) \qquad h''(x) = \frac{r-1}{r}e^x h'(x)^2 = (r-1)e^{h(x)}h'(x)$$

We apply (129), (130), followed by (132), to conclude that

$$(133) \qquad \frac{b''(x)}{b'(x)} = h'(x)\frac{g''(h(x))}{g'(h(x))} + \frac{h''(x)}{h'(x)}$$

$$(134) \qquad\qquad\quad = h'(x)\frac{g''(h(x))}{g'(h(x))} + (r-1)e^{h(x)}.$$

Note that $h'(x) > 0$, and by Lemma 1, $g''(h)/g'(h) > -1$. Hence,

$$\frac{b''(x)}{b'(x)} > -h'(x) + (r-1)e^{h(x)}.$$

Now apply (131) and rearrange to get that

$$-h'(x) + (r-1)e^{h(x)} = -e^{h(x)}(1 - r(1 - e^{-x})) = -1,$$

completing the proof of (128).

$$\square$$

Motivated by Lemma 20, we define

$$(135) \qquad\qquad m_a \in \arg\max_m \Pi_a(m, 1), \qquad m_e \in \arg\max_m \Pi_e(m, 1).$$

Note that Lemma 20 implies that there is a unique value of m that maximizes $\Pi_a(m, 1)$. However, in some cases, $\arg\max_m \Pi_e(m, 1)$ may be empty.[21] In this case, we define $m_e = \infty$ and $\Pi_e(\infty, 1) = \lim_{m\to\infty} \Pi_e(m, 1)$.

We are now prepared to prove the Propositions from Section 5.

Proof of Proposition 3. For fixed r, c'_a, define $f(r, c'_a) = \mathcal{Q}(m_1, 1)$ (recall that m_1 is the value of m chosen by applicants when they respond optimally to employers playing $\alpha = 1$). If $\mathcal{Q}(m_1, 1) \geq c'_s$, then $1 \in \mathcal{A}(\mathcal{Q}(m_1, 1))$ and thus $(m^*, \alpha^*) = (m_1, 1)$. By Proposition 7,

$$(136) \qquad\qquad \Pi_e(m_1, 1) = rm_1\mathcal{P}(m_1, 1)(\mathcal{Q}(m_1, 1) - c'_s),$$

where we have applied the mean-field consistency equation (4) : $\mathcal{Q}(m_1, 1) = g(m_1\mathcal{P}(m_1, 1))$. It follows that $\Pi_e^* > 0$ if and only if $\mathcal{Q}(m_1, 1) > c'_s$. If $\mathcal{Q}(m_1, 1) < c'_s$, then $(m_1, 1)$ is not an equilibrium, and thus $\alpha^* < 1$. This implies that employers are indifferent between screening and exiting, so we must have $\Pi_e^* = 0$.

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[21]This is precisely when the maximizer of BW, given by $x = -\log c'_s$, is outside of the domain of BW, as established by Lemma 3.

We now turn to proving the monotonicity of $f(r, c_a') = \mathcal{Q}(m_1, 1)$. So long as $c_a' < 1$, then $m_1 > 0$, so the applicant best response function (2) implies that $m_1 \mathcal{P}(m_1, 1) = -\log(c_a/\mathcal{P}(m_1, 1))$. Combining this with the consistency equation (4), we see that

$$(137) \qquad f(r, c_a') = \mathcal{Q}(m_1, 1) = g(-\log(c_a'\beta/\mathcal{P}(m_1, 1))).$$

Making use of (28) from Proposition 4, $c_a'\beta/\mathcal{P}(m_1, 1)$ is the solution y to

$$(138) \qquad c_a' = yg(ry(1-y)/c_a'),$$

or equivalently (after rearrangement),

$$(139) \qquad r = \frac{1 - e^{-\frac{r}{c_a'}y(1-y)}}{1 - y}.$$

Because g is decreasing (Lemma 1), (137) implies that in order to prove that $f(r, c_a')$ is increasing in r, it suffices to show that the solution y to (138) is increasing in r. This follows because (by Lemma 5) the right side of (138) is increasing in y (for any r), and decreasing in r (for fixed y). Similarly, to prove that $f(r, c_a')$ is increasing in c_a', it suffices to show that the solution y to (139) is increasing in c_a'. This follows because the right side of (139) is increasing in y (Lemma 5) and decreasing in c_a' (for fixed y).

All that remains is to show that whenever $\Pi_e^* = 0$, a suitably chosen limit ℓ can attain the employer-optimal outcome. By Proposition 2, we know that whenever $m_e < m^*$, this can be accomplished by setting $\ell = m_e$. Furthermore, by (136), $\Pi_e^* = 0$ implies $\mathcal{Q}(m^*, 1) \leq c_s'$ (whereas $\mathcal{Q}(m_e, 1) > c_s'$ by the definition of m_e). Using the fact that $\mathcal{Q}(m, 1)$ is decreasing in m (Lemma 3), we deduce that $m_e < m^*$.

$\square$

Proof of Proposition 4. We begin by noting that the expression for Π_a^* given by (121) is increasing in m^*p^*. Thus, to derive upper-bounds on Π_a^*, it suffices to provide upper-bounds on m^*p^*.

Our first bound comes from the fact that in any equilibrium, $g(m^*p^*) = q^* \geq c_s'$ (otherwise, employers would select $\alpha = 0$, implying $q = 1$). It follows that $m^*p^* \leq g^{-1}(c_s') = \gamma$, and therefore (121) implies that $\Pi_a^* \leq 1 - (1 + \gamma)e^{-\gamma}$.

Our second bound comes from Lemma 3, which states that when $r > 1$, $m\mathcal{P}(m, \alpha) \leq \log \frac{r}{r-1}$ for all m and α.[22]

The statements about applicant welfare in the regulated market are proven in Lemma 21. $\square$

Proof of Proposition 5. This proposition is restated and proven in Lemma 21. $\square$

Lemma 21.

(1) *If $c'_s > f(r, c'_a)$, then there exists ℓ such that $\Pi^\ell_a > \Pi^*_a$ and $\Pi^\ell_e > \Pi^*_e$.*

(2) *If $c'_s \leq f(r, c'_a)$, then there exists ℓ such that $\Pi^\ell_a = \Pi^{\mathrm{opt}}_a > \Pi^*_a$.*

Proof. As established in Proposition 3, if $c'_s \leq f(r, c'_a)$, then $\alpha^* = 1$ and $q^* = g(m^*\mathcal{P}(m^*, 1)) \geq c'_s$. Note that for any application limit ℓ, we have $m^*_\ell \leq m^*$, and thus $g(\ell\mathcal{P}(\ell, 1)) \geq g(m^*\mathcal{P}(m^*, 1)) \geq c'_s$, so $\alpha^*_\ell = 1$ for all ℓ. Furthermore,

$$\frac{d}{dm}\Pi_a(m, 1)\big|_{m=m^*} = \frac{d}{dm}(1 - e^{-m\mathcal{P}(m,1)} - c'_a m\beta)\big|_{m=m^*}$$
$$< \frac{d}{dm}(1 - e^{-mp^*} - c'_a m\beta)\big|_{m=m^*} = 0.$$

The inequality follows because $\frac{d}{dm}m\mathcal{P}(m, 1) = \mathcal{P}(m, 1) + m\frac{d}{dm}\mathcal{P}(m, 1) < \mathcal{P}(m, 1)$ (by Lemma 3), and the final equality follows because m^* is optimally chosen by applicants, who take p^* as given. Because $\Pi_a(\cdot, 1)$ is unimodal (Lemma 20), it follows that $m_a < m^*$. By Theorem 2, if we set limit $\ell = m_a$, then we obtain Π^{opt}_a for applicants.

By Proposition 3, if $c'_s > f(r, c'_a)$, then $\alpha^* < 1$ and $\Pi^*_e = 0$. It follows (from (120) in Proposition 7) that $g(m^*p^*) = q^* = c'_s$, and thus $g^{-1}(c'_s) = m^*\mathcal{P}(m^*, \alpha^*) < m^*\mathcal{P}(m^*, 1)$ (the inequality follows from Lemma 3). Since $\ell\mathcal{P}(\ell, 1)$ is increasing in ℓ (Lemma 3), there exists $\ell < m^*$ such that $\ell\mathcal{P}(\ell, 1) = g^{-1}(c'_s)$. Then $1 \in \mathcal{A}(\mathcal{P}(\ell, 1))$, and thus $\alpha^*_\ell = 1$. It follows from the definition of Π_a in (117) that

$$\Pi^\ell_a = \Pi_a(\ell, 1) = 1 - e^{-\ell\mathcal{P}(\ell,1)} - c_a\ell = 1 - e^{-m^*p^*} - c_a\ell > 1 - e^{-m^*p^*} - c_a m^* = \Pi^*_a.$$

By continuity, for all sufficiently small $\varepsilon > 0$, $\Pi^{\ell-\varepsilon}_a > \Pi^*_a$. For any such ε, $q^*_{\ell-\varepsilon} = g((\ell - \varepsilon)\mathcal{P}(\ell - \varepsilon, 1)) > c'_s$, so by (120), $\Pi^{\ell-\varepsilon}_e > 0 = \Pi^*_e$. $\square$

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[22]This has a simple interpretation. Of course, when $r > 1$, applicants match with probability at most $1/r$. But the proportion of applicants who match is $1 - e^{-m\mathcal{P}(m,\alpha)}$. From this, we conclude that $r(1 - e^{-m\mathcal{P}(m,\alpha)}) \leq 1$, which can be rearranged to give $m\mathcal{P}(m, \alpha) \leq \log \frac{r}{r-1}$.

Lemma 22. *Fix r, c'_s, and consider m^* and p^* as functions of c'_a. The quantity m^* is strictly decreasing in c'_a, with $m^*(1) = 0$ and $\lim_{c'_a \to 0} m^*(c'_a) = \infty$. The quantity m^*p^* is weakly decreasing in c'_a.*

Proof. We know that m^* is the unique solution to $h(m^*) = -\log c'_a$. Recall that by Lemma 9, the function $h : [0, \infty) \to [0, \infty)$ defined in (37) (which does not depend on c'_a) satisfies $h(0) = 0$ and is strictly increasing. From this, the statements about m^* follow.

We know from Propositions 3 and 7 that if $c'_s \geq f(r, c'_a)$, then $g(m^*p^*) = c'_s$, so m^*p^* is constant on the set $\{c'_a : c'_s \geq f(r, c'_a)\}$. For c'_a such that $c'_s < f(r, c'_a)$, we know that $\alpha^* = 1$, and (making use of (28) from Proposition 4) that $c'_a \beta / p^*$ is the solution y to

$$(140) \qquad r = \frac{1 - e^{-\frac{r}{c'_a} y(1-y)}}{1 - y}.$$

Furthermore, the applicant best-response function $\mathcal{M}$ given by (2) implies that $e^{-m^*p^*} = c'_a \beta / p^*$. Thus, to show that m^*p^* is decreasing in c'_a, it is enough to show that $c'_a \beta / p^*$ is increasing in c'_a. This holds because the right side of (140) is decreasing in c'_a (for fixed $y > 0$), and increasing in y for fixed c'_a (Lemma 5). $\qquad \square$

Proof of Proposition 6. The proof is straightforward. For a fixed applicant strategy, the value of c'_a is irrelevant for employers. Thus, it suffices to show that for each $\ell \leq m^*(c'_a)$, there exists $\tilde{c}'_a \in [c'_a, 1]$ such that $m^*(\tilde{c}'_a) = m^*_\ell$; and for each $\tilde{c}'_a > c'_a$, there exists a unique $\ell \leq m^*(c'_a)$ such that $m^*(\tilde{c}'_a) = m^*_\ell$.

This follows because Proposition 2 implies that for $\ell \leq m^*$, $m^*_\ell = \ell$, and Lemma 22 implies that for any desired application level $\ell \in [0, m^*]$, there exists a unique $\tilde{c}'_a \geq c'_a$, such that $m^*(\tilde{c}'_a) = \ell$.

As for applicant welfare, by Proposition 7, $\Pi^*_a = 1 - (1 + m^*p^*)e^{-m^*p^*}$, which is increasing in m^*p^*. Lemma 22 states that m^*p^* is decreasing in c'_a, implying that Π^*_a is, as well. $\qquad \square$