Transmission capacity of CDMA ad hoc networks

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***Abstract*— Spread spectrum technologies are appropriate for ad hoc networking because they permit interference averaging and tolerate co-located simultaneous transmissions. We develop analytic results on the transmission capacity of a CDMA ad hoc network. Transmission capacity is defined as the maximum permissible density of simultaneous transmissions that allows a certain probability of successful reception. Three models of increasing generality are analyzed: a trivial model with two transmitters, a Poisson point process model where each node transmits with fixed power, and a Poisson point process model where nodes use variable transmission powers. We obtain upper and lower bounds on the transmission capacity for both frequency hopped (FH-CDMA) and direct sequence (DS-CDMA) implementations of CDMA for the latter two models. Our analysis shows that FH-CDMA obtains a higher transmission capacity than DS-CDMA on the order of $M^{1-\frac{2}{\alpha}}$, where M is the spreading factor and $\alpha > 2$ is the path loss exponent. The interpretation is that FH-CDMA is generally preferable to DS-CDMA for ad hoc networks, particulary when the path loss exponent is large.**

I. Introduction

Ad hoc networks offer the benefit of wireless communication without requiring planned infrastructure. Spread spectrum technologies, such as CDMA, are appropriate for ad hoc networking because they permit interference averaging and tolerate co-located simultaneous transmissions, e.g., [1], [2]. Interference averaging permits a receiver to successfully decode its intended transmission provided the aggregate interference power from other transmissions is sufficiently small relative to the received power of the intended transmission.

We study both frequency hopped (FH-CDMA) and direct sequence (DS-CDMA) implementations of CDMA. We let M denote the spreading factor for both. FH-CDMA divides the available bandwidth, W, into M sub-channels, each of bandwidth $\frac{W}{M}$. A receiver attempting to decode a signal from a transmitter on sub-channel m only sees interference from other simultaneous transmissions on sub-channel m. Whereas FH-CDMA uses the spreading factor, M, to thin out the set of interfering transmitters, DS-CDMA uses the spreading factor to reduce the minimum SINR required for successful reception. If the nominal SINR requirement for FH-CDMA is β, then DS-CDMA reduces the SINR requirement to $\frac{\beta}{M}$, assuming a typical PN code cross-correlation [3]. Thus, a receiver on a network using FH-CDMA only sees interference from transmitters on sub-channel m and the aggregate sub-channel interference must be such that the received SINR exceeds β, while a receiver on a network using DS-CDMA sees interference from all transmitters but the aggregate interference must be such that the received SINR exceeds $\frac{\beta}{M}$.

Important recent results [4] demonstrate that per node transport capacity, measured in bit-meters/second, decreases in the node density. We use a different definition of capacity, termed *transmission capacity*, defined as the maximum permissible density of simultaneous transmissions that satisfies a constraint on the probability of successful reception. Formally, let $\Pi = \{X_i\}$ denote a homogeneous Poisson point process on a plane, where the points X_i denote the locations of transmitting nodes. We define the transmission capacity, λ, as the maximum density of points in Π such that the probability a typical receiver is unable to decode its transmission is less than ϵ, for some $0 < \epsilon \ll 1$. Here, ϵ is the outage probability requirement. Essentially, the transmission capacity is an intuitive and practical measure of the usefulness of an ad hoc network, since it determines how many users can be supported at a given data rate and bandwidth.

In this work the transmission capacity is derived for three increasingly sophisticated models of a CDMA ad hoc network. The first is a trivial model consisting of a single receiver and two transmitters. The purpose for studying this model is to introduce some intuition behind why FH-CDMA yields a superior system capacity over DS-CDMA. The second model assumes the nodes comprising the ad hoc network form a homogeneous Poisson point process on the plane. The third model utilizes a *marked* Poisson point process, where the marks denote the transmission distance for each node. As will be explained, we assume nodes utilize *pairwise power control*, meaning that each transmitter chooses its transmission power such that the received signal power will be constant.

To our knowledge, this research is the first to analytically compare the system capacity of DS-CDMA and FH-CDMA in the ad hoc network scenario. Some previous results compare aspects of FH-CDMA and DS-CDMA for ad hoc networks [5] and cellular networks [6], [7], [8]. Specifically, previous results in [7], [8], which focused on the bit-error rate (BER) performance under various channel models, have shown FH-CDMA achieves better performance than DS-CDMA when centralized power control is not available. Instead of focusing on BER like prior research, we directly approach the comparison of DS-CDMA and FH-CDMA from the perspective of system capacity and show that FH-CDMA offers capacity improvement above DS-CDMA on the order of $M^{1-\frac{2}{\alpha}}$ for interference–constrained ad hoc wireless networks.

II. Base model: two transmitters and one receiver

Consider the case shown in Figure 1 where we have a receiver at the origin and two transmitters at random locations within a circle of radius $\bar{r}$ of the receiver. Transmitter 1 is trying to communicate with the receiver and transmitter 2 is transmitting to some other receiver, and therefore is causing interference for the receiver at the origin. We are interested in studying the probability that the receiver can successfully receive the transmission from transmitter 1 when the nodes use FH-CDMA and DS-CDMA.

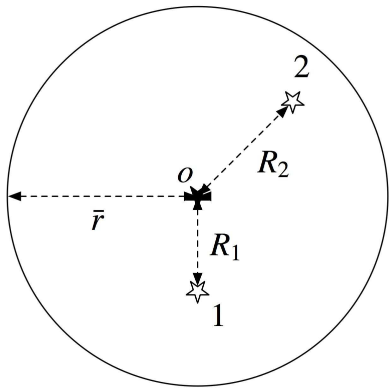

Fig. 1. Transmitter 1, at distance R_1 from the receiver at the origin, sends a signal to the receiver, while transmitter 2, at distance R_2, causes interference.

If we assume that transmitters 1 and 2 are uniformly and independently distributed in the ball $b(0, \bar{r})$, then it is easily seen that the CDF for their distance from the origin is $F_{R_1}(r) = F_{R_2}(r) = \left(\frac{r}{\bar{r}}\right)^2$ with corresponding PDF $f_{R_1}(r) = f_{R_2}(r) = \frac{2r}{\bar{r}^2}$. We define $R = \frac{R_2}{R_1}$ as the *ratio* of the distances. Straightforward analysis shows the CDF for R to be

$$F_R(r) = \begin{cases} \frac{r^2}{2}, & 0 \le r \le 1 \\ 1 - \frac{1}{2r^2}, & r \ge 1 \end{cases} \tag{1}$$

Our propagation model ignores shadowing and multi-path effects and focuses just on path loss. Specifically, we use a simplified path loss model where the received power $P_r = \rho r^{-\alpha}$, where $\alpha > 2$ is the path loss exponent and ρ is the transmitted power multiplied by some constant – we will simply refer to ρ as the transmit power. The actual value for α depends on the environment, but $\alpha \in [3, 5]$ is a fairly typical range [9]. We will assume for this model that both transmitters use a fixed power level ρ.

Consider first the case where the transmitters and receivers use DS-CDMA with a nominal SINR requirement of β and a spreading factor M. The outage probability for DS-CDMA, p_o^{DS}, is the probability the SINR is inadequate, i.e., $p_o^{DS} = \mathbb{P}\Big(\frac{\rho R_1^{-\alpha}}{\rho R_2^{-\alpha}} \le \frac{\beta}{M}\Big)$. Simple analysis shows

$$p_o^{DS} = \mathbb{P}\Big(\frac{R_2}{R_1} \le (\frac{\beta}{M})^{\frac{1}{\alpha}}\Big) = F_R\Big((\frac{\beta}{M})^{\frac{1}{\alpha}}\Big) = \frac{1}{2}\Big(\frac{\beta}{M}\Big)^{\frac{2}{\alpha}}. \tag{2}$$

Next consider the outage probability for the FH-CDMA case, p_o^{FH}. Here we assume there are M sub-channels available, and that each transmitter chooses a channel independently, so the outage probability is multiplied by $\frac{1}{M}$, i.e., the probability that the two transmitters choose the same sub-channel:

$$p_o^{FH} = \mathbb{P}\Big(\frac{\rho R_1^{-\alpha}}{\rho R_2^{-\alpha}} \le \beta\Big)\frac{1}{M} = F_R\Big(\beta^{\frac{1}{\alpha}}\Big)\frac{1}{M} = \frac{1}{2}\beta^{\frac{2}{\alpha}}\frac{1}{M} \tag{3}$$

Taking the ratio of the loss probability for DS over the loss probability for FH we obtain:

$$\frac{p_o^{DS}}{p_o^{FH}} = \frac{\frac{1}{2}\Big(\frac{\beta}{M}\Big)^{\frac{2}{\alpha}}}{\frac{1}{2}\beta^{\frac{2}{\alpha}}\frac{1}{M}} = M^{1-\frac{2}{\alpha}}. \tag{4}$$

Note the loss probability ratio is 1 for $\alpha = 2$ and monotonically increases in α for $\alpha > 2$. Thus the benefit of FH-CDMA over DS-CDMA is more pronounced in transmission areas with high attenuation.

This simple model illustrates that when an ad hoc network is interference constrained, avoiding interference by random hopping (FH-CDMA) is preferable to interference suppression (DS-CDMA). Notably, the gain offered by FH-CDMA is significant even when there are only a few interfering sources for a typical path loss exponent of $\alpha \in [3, 5]$ [9]. This is reminiscent of the well-known 'near-far' problem in wireless communications (especially CDMA), where an interfering transmitter in near proximity to a receiver causes such a high level of interference that successful reception is impossible at the receiver. This 'near-far' problem is inevitable for ad hoc networks and poses a particularly significant obstacle for regions with a large path loss exponent. For FH-CDMA, the outage probability is independent of neighboring interference power provided the interfering nodes are not simultaneously contending for the same sub-channel as the receiver. For DS-CDMA, however, the outage probability is very sensitive to the interference power level. This translates directly to an increased outage probability for DS-CDMA, for a given configuration of node positions. The two transmitter model is obviously unrealistic–this motivates us to consider a more general ad hoc network, described in the next section.

III. Second model: Point process ad hoc network with fixed transmission distance

Our second model utilizes a homogeneous Poisson point process $\Pi = \{X_i\}$ on the plane $\mathbb{R}^2$ to represent the locations of all nodes transmitting at some time t. The transmission capacity of the network, as defined earlier, is the maximum intensity λ of the process Π such that outage probability is less than ϵ, for $0 < \epsilon \ll 1$. We will write $R_i = |X_i|$ for the distance from node i to the origin. Note that the transmission capacity may be improved through local scheduling or mobility (we assume nodes are fixed and always transmitting); these topics are left for future work.

To evaluate the outage probability we will condition on a typical transmitter at the origin giving what is known as the Palm distribution for transmitters on the plane [10]. It follows by Slivnyak's Theorem [10] that this conditional distribution corresponds to a homogenous Point process with the same intensity and an additional point at the origin. Now shifting this entire point process so that the receiver associated with the typical transmitter lies at the origin, we see that the conditional distribution of potential interferers is a homogenous Poisson point process with the same intensity. We will denote this process by Π and denote probability with respect to this distribution by $\mathbb{P}^0$.

For the FH-CDMA case we assume each transmitter chooses its sub-channel independently. We let Π_m denote the set of transmitters which select sub-channel m, for $m = 1, \ldots, M$.

Because of the independent sampling assumption, each process Π_m is a homogeneous Poisson point process with intensity $\frac{\lambda}{M}$.

The ambient noise density is denoted by N_o. For FH-CDMA the total ambient noise power is $N_o \frac{W}{M} \equiv \eta$, i.e., only the power from the frequency sub-band corresponding to the active sub-channel causes noise for the receiver. For DS-CDMA the total ambient noise power is $N_o W = M\eta$, i.e., power from the entire band, W, causes noise for the receiver.

We assume for simplicity that $i)$ all transmitters utilize the same transmission power, ρ, and $ii)$ all transmission distances are over the same distance d. These assumptions will be removed in the third model.

It is easily seen that the appropriate requirements on λ are given below:

$$FH \qquad \mathbb{P}^0\Big(\frac{\rho r^{-\alpha}}{\eta + \sum_{i \in \Pi_m} \rho R_i^{-\alpha}} \le \beta\Big) \le \epsilon, \tag{5}$$

$$DS \qquad \mathbb{P}^0\Big(\frac{\rho r^{-\alpha}}{M\eta + \sum_{i \in \Pi} \rho R_i^{-\alpha}} \le \frac{\beta}{M}\Big) \le \epsilon. \tag{6}$$

We will obtain upper and lower bounds on λ in the form $\lambda_* \le \lambda \le \lambda^*$. The lower bound λ_* is such that $\lambda < \lambda_*$ ensures $p_o < \epsilon$, i.e., the QoS requirement is definitely met, and the upper bound λ^* is such that $\lambda > \lambda^*$ ensures $p_o > \epsilon$, i.e., the QoS requirement is definitely violated. These bounds, and the transmission capacity ratio obtained of FH-CDMA over DS-CDMA, are given in the following theorem.

Theorem 3.1: For small ϵ, the lower and upper bounds on transmission capacity for FH-CDMA and DS-CDMA when transmitters employ a fixed transmission power ρ and a fixed transmission distance r are:

$$\frac{1}{2}\frac{\epsilon}{\pi}\Big(\kappa M\Big)^{\frac{2}{\alpha}} \quad \le \lambda_{DS} \le \quad \frac{\epsilon}{\pi}\Big(\kappa M\Big)^{\frac{2}{\alpha}} \tag{7}$$

$$\frac{1}{2}\frac{\epsilon M}{\pi}\kappa^{\frac{2}{\alpha}} \quad \le \lambda_{FH} \le \quad \frac{\epsilon M}{\pi}\kappa^{\frac{2}{\alpha}} \tag{8}$$

where $\kappa = \frac{r^{-\alpha}}{\beta} - \frac{\eta}{\rho}$.

The transmission capacity ratio is

$$\gamma_{fixed} = \frac{\lambda_*^{FH}}{\lambda_*^{DS}} = \frac{\lambda^{*,FH}}{\lambda^{*,DS}} = M^{1-\frac{2}{\alpha}}. \tag{9}$$

See appendix for proof.

Theorem 3.1 shows the capacity improvement of FH-CDMA over DS-CDMA equals the outage probability ratio obtained in Section II. The intuition follows from the same argument, i.e., that FH-CDMA is less sensitive to the near-far problem than DS-CDMA. In addition, we observe a capacity gain – the capacity improves linearly (FH-CDMA) and sub-linearly (DS-CDMA) in the spreading gain M (when $\alpha > 2$). Hence, if the traffic in an ad hoc network does not require a very high data rate, e.g., voice traffic, it is desirable to use a high spreading gain in order to achieve robust interference tolerance or high transmission capacity.

IV. Third model: point process ad hoc network with variable transmission distance

For our third model we remove the assumption that all transmitters use the same transmission power and have the same transmission distance. In real ad hoc networks transmission relay distances will be variable as will interference power levels. Both factors suggest transmitters use power control since too high of a signal power level causes unnecessary interference and too low of a signal power level will not be successfully received. Finding a system-wide optimal set of transmission power levels is the subject of recent work [11] and has shown that efficient distributed algorithms for global power control are difficult. For this work we take a simpler approach and assume that transmitters choose their transmission power as a function of their distance from their intended receiver and independent of the interference level of the receiver. We call this *pairwise power control* since each transmitter and receiver pair determines the transmission power independently of other pairs. Specifically, the transmitter chooses its transmission power such that the signal power at the receiver will be some fixed level ϱ. Thus if a transmitter and receiver are separated by a distance d then the transmitter will employ a transmission power ϱd^{α} so that the received signal power will be ϱ. We make no particular assumption on the value of ϱ, other than $\varrho > \frac{\eta}{\beta}$, which is required to keep the received signal power above the noise floor.

Formally, our third model consists of a *marked* homogeneous Poisson point process $\Phi = \{(X_i, D_i)\}$ where the points $\{X_i\}$ again denote transmitter locations and the marks $\{D_i\}$ denote the distance from transmitter i to its intended receiver. We assume the marks are independent and identically distributed with CDF $F_D(d)$, and that the marks are also independent of the points. We again use $R_i = |X_i|$ to denote the distance from node i to the origin. Similar to the second model, we evaluate the outage probability using the Palm distribution $\mathbb{P}^0$ which places a typical receiver at the origin. Also similar to the second model, we define the sampled sub-process Φ_m as a homogeneous marked Poisson point processes consisting of all transmitters on sub-channel m, for $m = 1, \ldots, M$.

We define the function $g(r,d)$ as giving the signal power level at a distance r from the transmitter when the transmitter's intended recipient is at a distance d. Thus, $g(r,d) = \varrho\Big(\frac{d}{r}\Big)^{\alpha}$. Note in particular that $g(d,d) = \varrho$, i.e., at the distance of the intended receiver the signal power is the desired level. Note that the transmission power is $g(1,d) = \varrho d^{\alpha}$.

Devices are assumed to have a maximum transmission power of $\bar{\varrho}$. Solving $\varrho d^{\alpha} \le \bar{\varrho}$ for d gives a maximum transmission distance of $\bar{d} = \big(\frac{\bar{\varrho}}{\varrho}\big)^{\frac{1}{\alpha}}$. We stated above that the transmission distances are assumed to be iid with CDF $F_D(d)$. We *assume* that a transmitter is equally likely to choose *any one* of the receivers within a circle of radius $\bar{d}$ of it. The probability a transmitter will have a transmission distance d should be proportional to d, i.e., $f_D(d) \propto d$. Normalizing this distribution with the constraint that $d \le \bar{d}$ gives the CDF and PDF as $F_D(d) = \Big(\frac{d}{\bar{d}}\Big)^2$ and $f_D(d) = \frac{2d}{\bar{d}^2}$.

It is easily seen that the appropriate requirements on λ are given below:

$$FH \qquad \mathbb{P}^0\Big(\frac{\varrho}{\eta + \sum_{i \in \Phi_m} g(R_i, D_i)} \le \beta\Big) \le \epsilon, \tag{10}$$

$$DS \qquad \mathbb{P}^0\Big(\frac{\varrho}{M\eta + \sum_{i \in \Phi} g(R_i, D_i)} \le \frac{\beta}{M}\Big) \le \epsilon. \tag{11}$$

We define similar bounds, λ_*, λ^* as for the second model. These bounds, and the transmission capacity ratio obtained of FH-CDMA over DS-CDMA, are given in the following theorem.

Theorem 4.1: For small ϵ, the lower and upper bounds on transmission capacity for FH-CDMA and DS-CDMA when transmitters employ variable transmission powers are

$$\frac{1}{2}\frac{\epsilon}{\pi\bar{d}^2}\Big(\delta M\Big)^{\frac{2}{\alpha}} \leq \lambda_{DS} \leq 4\frac{\epsilon}{\pi\bar{d}^2}\Big(\delta M\Big)^{\frac{2}{\alpha}} \quad (12)$$

$$\frac{1}{2}\frac{\epsilon M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}} \leq \lambda_{FH} \leq 4\frac{\epsilon M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}} \quad (13)$$

where $\delta = \frac{1}{\beta} - \frac{\eta}{\varrho}$.

The transmission capacity ratio is

$$\gamma_{fixed} = \frac{\lambda_*^{FH}}{\lambda_*^{DS}} = \frac{\lambda^{*,FH}}{\lambda^{*,DS}} = M^{1-\frac{2}{\alpha}}. \quad (14)$$

See appendix for proof.

Power control in cellular networks solves the 'near-far' problem by equalizing receiving powers at the central base station. The pair-wise power control scheme in ad hoc networks can not fully solve the 'near-far' problem since transmitters have different intended receivers, but it offers a simple and distributed means by which to mitigate the interference across concurrent transmissions.

V. Conclusion

We have compared the performance of FH-CDMA and DS-CDMA in ad hoc networks in terms of transmission capacity. When capacity is constrained by neighboring interference, either because of high path loss or because of clustered interference, FH-CDMA is more appropriate for achieving high transmission capacity than DS-CDMA, especially when a large spreading gain is possible. The advantage of FH-CDMA also increases as the path loss exponent increases. This implies that in more severe propagation environments such as indoor wireless systems with walls and obstacles, frequency hopping enjoys an even larger advantage over direct sequence spreading.

Appendix

Please note that the appendices may be shortened for the camera-ready version. They are included in this submission for completeness.

Proof of Theorem 3.1

The QoS constraints can be written as

$$FH \qquad \mathbb{P}^0\Big(\sum_{\Pi_m} R_i^{-\alpha} \geq \kappa\Big) \leq \epsilon, \quad (15)$$

$$DS \qquad \mathbb{P}^0\Big(\sum_{\Pi} R_i^{-\alpha} \geq M\kappa\Big) \leq \epsilon. \quad (16)$$

for $\kappa = \frac{r^{-\alpha}}{\beta} - \frac{\eta}{\rho}$.

We first address the FH-CDMA case. Let $(\Omega, \mathcal{F}, \mathbb{P}^0)$ represent the underlying probability triple for the process Π, let $\omega \in \Omega$ represent outcomes, i.e., particular realization of the point process. Define the following events:

$$F = \Big\{\omega\Big|\sum_{\Pi_m(\omega)} R_i(\omega)^{-\alpha} \geq \kappa\Big\} \quad (17)$$

$$F_1 = \Big\{\omega\Big|\Pi_m(\omega) \cap b(0,s) \neq \emptyset\Big\} \quad (18)$$

$$F_2 = \Big\{\omega\Big|\sum_{\Pi_m(\omega)\cap\bar{b}(0,s)} R_i(\omega)^{-\alpha} \geq \kappa\Big\} \quad (19)$$

The event F consists of all outage outcomes, F_1 consists of all outcomes where there are one or more transmitters within s of the origin, and the event F_2 consists of all outcomes where the set of transmitters outside the ball $b(0,s)$ generate enough interference power to cause an outage at the origin.

Lemma 1.1: For $s = \kappa^{\frac{-1}{\alpha}}$ then $F_1 \subset F \subset (F_1 \cup F_2)$.

Proof For $s = \kappa^{\frac{-1}{\alpha}}$ even one transmitter within $b(0,s)$ can cause an outage, thus $F_1 \subset F$. Note we actually have $F = F_1 \cup F_2$.

Lemma 1.2: $\mathbb{P}^0(F_1) = 1 - e^{-\frac{\lambda}{M}\pi s^2}$.

Proof Note $1 - \mathbb{P}^0(F_1)$ is the probability there are no transmitters in $b(0,s)$, which is simply the void probability [10]. For a Poisson process in the plane with intensity λ the void probability for $b(0,s)$ is $e^{-\lambda\pi s^2}$.

Lemma 1.3: $\lambda^* = -\ln(1-\epsilon)\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$.

Proof Clearly $\mathbb{P}^0(F_1) \leq \mathbb{P}^0(F)$. If we can find a λ^* such that $\lambda > \lambda^* \Rightarrow \mathbb{P}^0(F_1) > \epsilon$ then it follows that $\mathbb{P}^0(F) > \epsilon$. We find such a λ^* by solving $\mathbb{P}^0(F_1) = \epsilon$ and substituting $s = \kappa^{\frac{-1}{\alpha}}$.

Lemma 1.4: If $\mathbb{P}^0(F_1) < \sqrt{1+\epsilon} - 1$ and $\mathbb{P}^0(F_2) < \sqrt{1+\epsilon} - 1$ then $\mathbb{P}^0(F) < \epsilon$

Proof Clearly $\mathbb{P}^0(F) \leq \mathbb{P}^0(F_1 \cup F_2)$. Also, by definition, $\mathbb{P}^0(F_1 \cup F_2) = \mathbb{P}^0(\bar{F}_1 \cap F_2) + \mathbb{P}^0(F_1 \cap \bar{F}_2) + \mathbb{P}^0(F_1 \cap F_2)$. Next note that F_1, F_2 are independent since they concern disjoint regions on the plane, implying $\mathbb{P}^0(F_1 \cup F_2) = \mathbb{P}^0(\bar{F}_1)\mathbb{P}^0(F_2) + \mathbb{P}^0(F_1)\mathbb{P}^0(\bar{F}_2) + \mathbb{P}^0(F_1)\mathbb{P}^0(F_2)$. Using the weak bound of $\mathbb{P}^0(\bar{F}_1) \leq 1$ and $\mathbb{P}^0(\bar{F}_2) \leq 1$, we get

$$\mathbb{P}^0(F_1 \cup F_2) \leq 2(\sqrt{1+\epsilon}-1) + (\sqrt{1+\epsilon}-1)(\sqrt{1+\epsilon}-1) = \epsilon. \quad (20)$$

Lemma 1.5: $\mathbb{P}^0(F_1) < \sqrt{1+\epsilon} - 1$ if $\lambda < -\ln(2 - \sqrt{1+\epsilon})\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$

Proof Similar to Lemma 1.3, we solve $\mathbb{P}^0(F_1) = \sqrt{1+\epsilon} - 1$ for λ.

Lemma 1.6: Consider $0 \leq \epsilon \ll 1$, $\kappa > 0$, a homogeneous Poisson point process $\Pi = \{X_i\}$ on the plane of intensity λ, and $Y = \sum_{\Pi} g(X_i)$ for $g(\cdot)$ a non-negative continuous function such that the integrals $\int_{\mathbb{R}^2} g(x)dx$ and $\int_{\mathbb{R}^2} g(x)^2 dx$ exist. If $\lambda < \frac{\kappa^2}{\sigma^2}\epsilon$ then $\mathbb{P}^0\Big(\sum_{\Pi} g(X_i) \geq \kappa\Big) \leq \epsilon$, where $\sigma^2 = \frac{1}{\lambda}Var\Big(\sum_{\Pi} g(X_i)\Big)$.

Proof Let $\mu = \frac{\mathbb{E}[Y]}{\lambda}$. By Chebychev's inequality,

$$\mathbb{P}^0(Y \geq \kappa) \leq \mathbb{P}^0(|Y - \mu\lambda| \geq \kappa - \mu\lambda) \quad (21)$$
$$\leq \frac{\sigma^2 \lambda}{(\kappa - \mu\lambda))^2} \quad (22)$$

for $\lambda\mu \leq \kappa$. We can rearrange $\frac{\sigma^2\lambda}{(\kappa-\mu\lambda))^2} = \epsilon$ as a quadratic equation for λ:

$$\mu^2\lambda^2 - (2\mu\kappa + \frac{\sigma^2}{\epsilon})\lambda + \kappa^2 = 0. \quad (23)$$

Solving for λ gives

$$\lambda = \frac{\kappa}{\mu} + \frac{\sigma^2}{2\mu^2\epsilon}\Big(1 - \sqrt{1 + \frac{4\kappa\mu\epsilon}{\sigma^2}}\Big). \quad (24)$$

We find the first three terms in the MacLaurin expansion of $\sqrt{1+a\epsilon}$ as $\sqrt{1+a\epsilon} \approx 1 + \frac{a}{2}\epsilon - \frac{a^2}{8}\epsilon^2$. Applying this for $a = \frac{4\kappa\mu}{\sigma^2}$ and rearranging gives $\lambda \approx \frac{\kappa^2}{\sigma^2}\epsilon$.

Lemma 1.7: $\mathbb{P}^0(F_2) < \sqrt{1+\epsilon} - 1$ if $\lambda < (\sqrt{1+\epsilon}-1)(\alpha - 1)\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$

Proof Define $\sigma^2 = \frac{1}{\lambda}Var\Big(\sum_{\Pi_m \cap \bar{b}(0,s)} R_i^{-\alpha}\Big)$. In words, $\sigma^2\lambda$ is the variance of a function of a homogeneous Poisson point process on the plane with intensity λ, where the function is the normalized aggregate interference power seen at the origin caused by all transmitters outside the circle $b(0,s)$. By the previous lemma, $\mathbb{P}^0(F_2) \leq \sqrt{1+\epsilon} - 1$ if $\lambda < \frac{\kappa^2}{\sigma^2}(\sqrt{1+\epsilon} - 1)$. Straightforward application of Campbell's Theorem [10] yields $\sigma^2 = \int_{\mathbb{R}^2 \cap \bar{b}(0,s)} \big(|x|^{-\alpha}\big)^2 dx = \frac{\pi s^{2(1-\alpha)}}{\alpha - 1}$. Note that Π_m has intensity $\frac{\lambda}{M}$. Thus we require $\frac{\lambda}{M} \leq \frac{\kappa^2}{\sigma^2}(\sqrt{1+\epsilon} - 1)$. Substituting σ^2 we get $\lambda \leq \frac{M\kappa^2(\alpha-1)}{\pi s^{2(1-\alpha)}}(\sqrt{1+\epsilon} - 1)$. Substituting $s = \kappa^{-\frac{1}{\alpha}}$ and rearranging yields the lemma.

Lemma 1.8: $\lambda_* = -\ln(2 - \sqrt{1+\epsilon})\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$.

Proof Combining Lemmas 1.4, 1.5 and 1.7 we know that if $\lambda < \min\Big\{-\ln(2 - \sqrt{1+\epsilon}), (\sqrt{1+\epsilon} - 1)(\alpha - 1)\Big\}\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$ then $\mathbb{P}(F) < \epsilon$. We take the MacLaurin expansion of the two functions inside the minimum and find $-\ln(2 - \sqrt{1+\epsilon}) \approx \frac{1}{2}\epsilon$ and $(\sqrt{1+\epsilon} - 1)(\alpha - 1) \approx \frac{\alpha-1}{2}\epsilon$. Since $\alpha > 2$ this means that $-\ln(2 - \sqrt{1+\epsilon})$ is the smaller function for small ϵ. This yields the lemma.

Lemma's 1.3 and 1.8 give us $\lambda^*_{FH} = -\ln(1-\epsilon)\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$ and $\lambda_{*,FH} = -\ln(2 - \sqrt{1+\epsilon})\frac{M}{\pi}\kappa^{\frac{2}{\alpha}}$ respectively. Taking the MacLaurin expansion of $-\ln(1-\epsilon)$ and $-\ln(2-\sqrt{1+\epsilon})$ gives $\lambda^*_{FH} \approx \frac{\epsilon M}{\pi}\kappa^{\frac{2}{\alpha}}$ and $\lambda_{*,FH} \approx \frac{1}{2}\frac{\epsilon M}{\pi}\kappa^{\frac{2}{\alpha}}$ for small ϵ. Thus the spread of our bound is $\frac{\lambda^*_{FH}}{\lambda^*_{FH}} = 2$, so we know the transmission capacity of the FH-CDMA system within a factor of 2.

Looking at equations 15 and 16, it's clear that the exact same analysis for DS-CDMA holds provided we replace λ with $M\lambda$ and κ with $M\kappa$. Thus, if $\frac{1}{2}\frac{\epsilon M}{\pi}\kappa^{\frac{2}{\alpha}} \leq \lambda_{FH} \leq \frac{\epsilon M}{\pi}\kappa^{\frac{2}{\alpha}}$ holds for FH, then $\frac{1}{2}\frac{\epsilon M}{\pi}(M\kappa)^{\frac{2}{\alpha}} \leq M\lambda_{DS} \leq \frac{\epsilon M}{\pi}(M\kappa)^{\frac{2}{\alpha}}$ holds for DS.

It's clear that the transmission capacity ratio of FH-CDMA over DS-CDMA is $\gamma_{fixed} = M^{1-\frac{2}{\alpha}}$. ■

PROOF OF THEOREM 4.1

The QoS constraints can be written as

$$FH \qquad \mathbb{P}^0\Big(\sum_{\Phi_m}\big(\frac{D_i}{R_i}\big)^\alpha \geq \delta\Big) \leq \epsilon, \quad (25)$$

$$DS \qquad \mathbb{P}^0\Big(\sum_{\Phi}\big(\frac{D_i}{R_i}\big)^\alpha \geq \frac{\delta}{M}\Big) \leq \epsilon. \quad (26)$$

for $\delta = \frac{1}{\beta} - \frac{\eta}{\rho}$.

We first address the FH-CDMA case. Let $(\Omega, \mathcal{F}, \mathbb{P}^0)$ represent the underlying probability triple for the process Π, let $\omega \in \Omega$ represent outcomes, i.e., particular realization of the point process. Define the following events:

$$F = \Big\{\omega\Big|\sum_{\Phi_m(\omega)}\big(\frac{D_i(\omega)}{R_i(\omega)}\big)^\alpha > \delta\Big\} \quad (27)$$

$$F_1 = \Big\{\omega\Big|\Phi_m(\omega) \cap (b(0,s) \times [s\delta^{\frac{1}{\alpha}}, \bar{d}) \neq \emptyset\Big\} \quad (28)$$

$$F_2 = \Big\{\omega\Big|\Phi_m(\omega) \cap (b(0,s) \times \mathbb{R}^+) \neq \emptyset\Big\} \quad (29)$$

$$F_3 = \Big\{\omega\Big|\sum_{\Phi_m(\omega)\cap\bar{b}(0,s)}\big(\frac{D_i(\omega)}{R_i(\omega)}\big)^\alpha > \delta\Big\} \quad (30)$$

The event F consists of all outage outcomes. The event F_1 consists of all outcomes where there are one or more transmitters within s of the origin with transmission distances exceeding $s\delta^{\frac{1}{\alpha}}$. This threshold is the smallest transmission distance such that even one transmitter in $b(0,s)$ with such a transmission distance will cause an outage at the origin. The event F_2 consists of all outcomes with one or more transmitters in $b(0,s)$; but note that not all outcomes in F_2 will cause an outage. Finally, the event F_3 consists of all outcomes where the interference power at the origin caused by all the transmitters outside $b(0,s)$ is adequate to cause an outage at the origin.

Lemma 1.9: For all s, $F_1 \subset F \subset (F_2 \cup F_3)$

Proof The proof is straightforward and is omitted. Note, contrary to Lemma 1.1, we don't have $F = (F_2 \cup F_3)$.

Lemma 1.10: For arbitrary s,

$$\mathbb{P}^0(F_1) = \Big(1 - e^{-\frac{\lambda}{M}\pi s^2}\Big)\Big(1 - \big(\frac{s\delta^{\frac{1}{\alpha}}}{\bar{d}}\big)^2\Big). \quad (31)$$

For $s = \frac{\bar{d}}{\sqrt{2}\delta^{\frac{1}{\alpha}}}$,

$$\mathbb{P}^0(F_1) = \frac{1}{2}\Big(1 - e^{-\frac{\lambda}{M}\pi s^2}\Big). \quad (32)$$

Proof Since the marks are independent of the point locations, we can decompose the probability into the product of the probabilities that there aren't any points in $b(0,s)$ times the probability that a given point has a mark in $[s\delta^{\frac{1}{\alpha}}, \bar{d})$. The probability of the former is $1 - e^{-\frac{\lambda}{m}\pi s^2}$ and the probability of the latter is $1 - \big(\frac{s\delta^{\frac{1}{\alpha}}}{\bar{d}}\big)^2$. This yields the first equation in the Lemma. The second equation is immediate for the specified s.

Lemma 1.11: $\lambda^* = \frac{-2\ln(1-2\epsilon)M\delta^{\frac{2}{\alpha}}}{\pi\bar{d}^2}$

Proof Clearly $\mathbb{P}^0(F_1) \leq \mathbb{P}^0(F)$. If we can find a λ^* such that $\lambda > \lambda^* \Rightarrow \mathbb{P}^0(F_1) > \epsilon$ then it follows that $\mathbb{P}^0(F) > \epsilon$.

We find such a λ^* by solving $\mathbb{P}^0(F_1) = \epsilon$ and substituting $s = \frac{\bar{d}}{\sqrt{2}\delta^{\frac{1}{\alpha}}}$.

Lemma 1.12: If $\mathbb{P}^0(F_1) < \sqrt{1+\epsilon} - 1$ and $\mathbb{P}^0(F_2) < \sqrt{1+\epsilon} - 1$ then $\mathbb{P}^0(F) < \epsilon$.

Proof Same as Lemma 1.4.

Lemma 1.13: $\mathbb{P}^0(F_2) < \sqrt{1+\epsilon} - 1$ if $\lambda < \frac{-2\ln(2-\sqrt{1+\epsilon})M\delta^{\frac{2}{\alpha}}}{\pi\bar{d}^2}$.

Proof Similar to Lemma 1.2, we find $\mathbb{P}^0(F_2) = 1 - e^{-\frac{\lambda}{M}\pi s^2}$, set this equal to $\sqrt{1+\epsilon} - 1$, solve for λ, and substitute $s = \frac{\bar{d}}{\sqrt{2}\delta^{\frac{1}{\alpha}}}$.

Lemma 1.14: Consider $0 \leq \epsilon \ll 1$, $\delta > 0$, a homogeneous marked Poisson point process $\Phi = \{(X_i, D_i)\}$ on the plane of intensity λ, and $Y = \sum_\Phi g(X_i, D_i)$ for $g(\cdot,\cdot)$ a non-negative continuous function such that the integrals $\int_{\mathbb{R}^2}\int_0^{\bar{d}} g(x,d) \; dx \; dd$ and $\int_{\mathbb{R}^2}\int_0^{\bar{d}} g(x,d)^2 \; dx \; dd$ exist. If $\lambda < \frac{\delta^2}{\sigma^2}\epsilon$ then $\mathbb{P}^0\left(\sum_\Phi g(X_i, D_i) \geq \delta\right) \leq \epsilon$, where $\sigma^2 = \frac{1}{\lambda}Var\left(\sum_\Phi g(X_i, D_i)\right)$.

Proof Same as Lemma 1.6.

Lemma 1.15: $\mathbb{P}^0(F_3) < \sqrt{1+\epsilon} - 1$ if $\lambda < \frac{(\sqrt{1+\epsilon}-1)M\delta^{\frac{2}{\alpha}}}{\pi\bar{d}^2}$.

Proof Define $\sigma^2 = \frac{1}{\lambda}Var\left(\sum_{\Phi_m \cap \bar{b}(0,s)}\left(\frac{D_i}{R_i}\right)^\alpha\right)$. In words, $\sigma^2\lambda$ is the variance of a function of a homogeneous marked Poisson point process on the plane with intensity λ, where the function is the normalized aggregate interference power seen at the origin caused by all transmitters outside the circle $b(0,s)$. By the previous lemma, $\mathbb{P}^0(F_3) \leq \sqrt{1+\epsilon} - 1$ if $\lambda < \frac{\delta^2}{\sigma^2}(\sqrt{1+\epsilon} - 1)$. Straightforward application of Campbell's Theorem [10] yields $\sigma^2 = \int_{\mathbb{R}^2 \cap \bar{b}(0,s)}\int_0^{\bar{d}}\left(\left(\frac{d}{|x|}\right)^\alpha\right)^2 f_D(d) \; dx \; dd = \frac{\pi s^{2(1-\alpha)}\bar{d}^{2\alpha}}{(\alpha-1)(\alpha+1)}$. Note that Φ_m has intensity $\frac{\lambda}{M}$. Thus we require $\frac{\lambda}{M} \leq \frac{\delta^2}{\sigma^2}(\sqrt{1+\epsilon} - 1)$. Substituting σ^2 we get $\lambda \leq \frac{(\sqrt{1+\epsilon}-1)(\alpha-1)(\alpha+1)M\delta^2}{\pi s^{2(1-\alpha)}\bar{d}^{2\alpha}}$. Substituting $s = \frac{\bar{d}}{\sqrt{2}\delta^{\frac{1}{\alpha}}}$ gives $\lambda < \frac{2^{1-\alpha}(\alpha+1)(\alpha-1)(\sqrt{1+\epsilon}-1)M\delta^{\frac{2}{\alpha}}}{\pi\bar{d}^2}$. For $2 \leq \alpha \leq 6$, a reasonable range for α, we have $2^{1-\alpha}(\alpha+1)(\alpha-1) > 1$, which we substitute in to simplify the bound.

Lemma 1.16: $\lambda_* = -(\sqrt{1+\epsilon} - 1)\frac{M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$.

Proof Combining Lemmas 1.12, 1.13 and 1.15 we know that if $\lambda < \min\left\{-2\ln(2-\sqrt{1+\epsilon}), (\sqrt{1+\epsilon}-1)\right\}\frac{M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$ then $\mathbb{P}(F) < \epsilon$. We take the MacLaurin expansion of the two functions inside the minimum and find $-2\ln(2-\sqrt{1+\epsilon}) \approx \epsilon$ and $(\sqrt{1+\epsilon}-1) \approx \frac{\epsilon}{2}$, so that $(\sqrt{1+\epsilon}-1)$ is the smaller function for small ϵ.

Lemma's 1.11 and 1.16 give us $\lambda^*_{FH} = -2\ln(1-2\epsilon)\frac{M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$ and $\lambda_{*,FH} = (\sqrt{1+\epsilon}-1)\frac{M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$ respectively. Taking the MacLaurin expansion of $-2\ln(1-2\epsilon)$ and $(\sqrt{1+\epsilon}-1)$ gives $\lambda^*_{FH} \approx 4\frac{\epsilon M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$ and $\lambda_{*,FH} \approx \frac{1}{2}\frac{\epsilon M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$ for small ϵ. Thus the spread of our bound is $\frac{\lambda^*_{FH}}{\lambda^*_{FH}} = 8$, so we know the transmission capacity of the FH-CDMA system within a factor of 8.

Looking at equations 25 and 26, it's clear that the exact same analysis for DS-CDMA holds provided we replace λ with $M\lambda$ and δ with $M\delta$. Thus, if $\frac{1}{2}\frac{\epsilon M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}} \leq \lambda_{FH} \leq 4\frac{\epsilon M}{\pi\bar{d}^2}\delta^{\frac{2}{\alpha}}$ holds for FH, then $\frac{1}{2}\frac{\epsilon M}{\pi\bar{d}^2}(M\delta)^{\frac{2}{\alpha}} \leq M\lambda_{DS} \leq 4\frac{\epsilon M}{\pi\bar{d}^2}(M\delta)^{\frac{2}{\alpha}}$ holds for DS.

It's clear that the transmission capacity ratio of FH-CDMA over DS-CDMA is $\gamma_{variable} = M^{1-\frac{2}{\alpha}}$. ■

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