Transmission Capacity of Wireless Ad Hoc Networks with Successive Interference Cancellation

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***Abstract*— The transmission capacity of a wireless ad hoc network can be defined as the maximum allowable area spectral efficiency such that the outage probability does not exceed some specified threshold. This work studies the improvement in transmission capacity obtainable with successive interference cancellation (SIC), an important receiver technique that has been shown to achieve the capacity of several classes of multiuser channels, but has not been carefully evaluated in the context of an ad hoc wireless network. This paper develops closed-form bounds for the transmission capacity of CDMA ad hoc networks with SIC receivers, for both perfect and imperfect interference cancellation. In addition to providing the first closed-form capacity results for SIC in ad hoc networks (or, to our knowledge, any type of multiuser detection), design-relevant insights are made possible. In particular, although the capacity gain from perfect SIC is very large, any imperfections in the interference cancellation rapidly degrade its usefulness. More encouragingly from a receiver complexity standpoint, due to the geographic properties of ad hoc networks, only a few – often just one – interfering nodes need to be cancelled in order to get the vast majority of the available performance gain.**

I. Introduction

Understanding the performance limits of decentralized ("ad hoc") wireless networks is a subject of much recent work. Due to the difficultly of directly analyzing the capacity of an unconstrained n node network [1], most recent work has followed the lead of the seminal work of Gupta and Kumar [2] and studied how the capacity scales with n under a variety of different modelling and implementation scenarios [3], [4], [5], [6], [7], [8].

In contrast, other recent work by Baccelli [9], [10] and the present authors [11] adopted a stochastic geometric approach for studying the performance of ad hoc networks, which has the merit that insights about the performance for fixed densities of nodes. In particular, [11] developed an analytical framework termed the *transmission capacity*, the maximum allowable spatial density λ of transmitters in an ad hoc network, as a function of various parameters like transmit distance and required signal to interference plus noise ratio (SINR), and such that a specified outage probability ϵ is met. Using this framework, transmission capacity bounds were found for frequency hopping (FH) and direct sequence (DS) CDMA (with a matched filter receiver) with the conclusion that in contrast to centralized networks, frequency hopping was significantly superior in an ad hoc network. In particular, our results predict a performance improvement on the order of $M^{1-\frac{2}{\alpha}}$, where M is the spreading factor and $\alpha > 2$ is the path loss exponent. The conclusion is that it is preferable to *avoid* interference by frequency hopping rather than trying to *suppress* it through random spreading. Intuitively, most outages are a result of an interfering node close to a receiver, and due to the strength of this interference these occurrences are better addressed by using different channels as opposed to receivers trying to suppress the interference.

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A. Successive interference cancellation

Since outages are predominantly caused by just a few nearby interfering users, an appealing alternative to interference avoidance (which consumes resources such as time or frequency slots) is interference cancellation. In fact, it is well-known that the matched filter receiver considered in [11], while dominant in commercial CDMA systems, is dramatically sub-optimal in theory relative to multiuser receivers, particularly in the presence of widely variant receiver powers [12], [13]. In general, multiuser receivers achieve a performance gain by exploiting the structure of the multiuser interference, rather than just treating it as wideband noise. A particularly interesting type of multiuser detection is successive interference cancellation (SIC), first suggested in [14], one form of which is shown in Fig. 1. The key idea of SIC is that users are decoded one after another, with the receiver cancelling interference after each user. For example, the decoded data for the first user is re-encoded and by using accurate channel knowledge, can be made to very closely resemble its *received* signal. Hence, it can be subtracted out of the composite received signal, and the second user to be decoded experiences less interference than it otherwise would have. The process can be continued for an arbitrary number of users.

In addition to its simplicity and amenability to implementation [15], SIC is well-justified from a theoretical point of view. Simple successive interference cancellation implementation with suboptimal coding was shown to nearly achieve the Shannon capacity of multiuser AWGN channels, assuming accurate channel estimation and a large spreading factor [16]. Other more recent work has proven that SIC with single-user decoding in fact achieves the Shannon capacity region boundaries for both the broadcast (downlink) and multiple access (uplink) multiuser channel scenarios [17], [18], as well-summarized in [19]. Quantifying SIC's benefit in ad hoc networks is naturally more problematic, but initial evidence for its promise is given in [20]. Since it is well-suited to asynchronous signals of unequal powers [21], and has much lower complexity than most other multiuser receivers, it appears to be a natural fit for a wireless ad hoc networks from the standpoint of both theory and practice.

Accurately modelling and analyzing SIC in ad hoc networks requires some nontrivial extensions from centralized networks. For instance, it has been shown that a particular (unequal) distribution of received powers is needed for SIC systems to perform well, especially when the interference cancellation is imperfect [22], [23]. Achieving such a distribution at each receiver in an ad hoc network is impossible due to the random spatial characteristics of the network. Related to this, to be realistic it should be assumed that only strong signals can be cancelled, hence at any given location in the network, only the nearby interferers are cancellable. In order to accurately quantify SIC's performance in ad hoc networks, Section II will develop a realistic (but analytically tractable) model in view of such considerations.

B. Main Results

The principal contribution of this paper are closed-form (and reasonably tight) bounds on the transmission capacity for wireless ad hoc networks for imperfect successive interference cancellation, where a residual fraction ζ of the interference is left after each stage. Results for perfect SIC are first derived for pedagogical purposes, and naturally are the special (and analytically simpler) case where $\zeta \to 0$. Prior results for wireless networks without SIC are also a special case where $\zeta \to 1$. The model and results are general enough that any multiuser receiver structure with residual cancellation error ζ on the K closest nodes would be covered by our analysis.

The following are key insights that can be gleaned from our theoretical analysis :

- Most of the performance improvement obtainable through SIC is gained by cancelling just the single transmitter with the largest interference level; cancelling additional transmitters may carry a negligible benefit.
- The performance improvement from SIC is very sensitive to how effectively the interference is cancelled. Even the residual interference of close-by interferers can often be the dominant interference source.
- The spectral efficiency, defined as the transmission capacity normalized by the spreading factor, of DS-CDMA with SIC may or may not be superior to FH-CDMA without SIC, again depending upon the cancellation effectiveness.

The rest of this paper is organized as follows. Section II will introduce the mathematical model and notation, and Section III will review the previous main results on transmission capacity from [11]. Sections IV and V will derive the transmission capacity of perfect and imperfect SIC ad hoc networks, respectively. Section VI presents numerical and simulation results demonstrating the performance improvement and provides interpretations of the main observed trends. The paper concludes in Section VII.

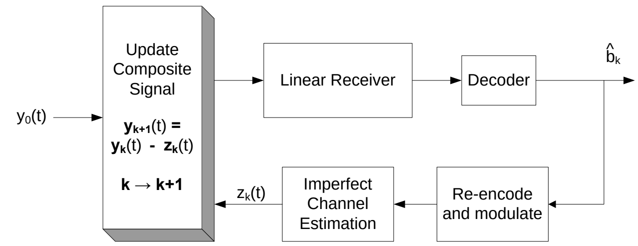

Fig. 1. Successive interference cancellation

II. Mathematical Models

A. Wireless ad hoc network

Our model employs a homogeneous Poisson point process (PPP) $\Pi = \{X_i, i \in \mathbb{N}\}$ on the plane $\mathbb{R}^2$ to represent the locations of all nodes transmitting at some time instant t. The PPP model for terminal positions has proven very accurate for CDMA cellular networks, and has produced analytical calculations for blocking probability that come within a few percent of actual blocking rates for service providers.

In prior work [11] we considered two models: $i)$ all transmitters use the same transmission power, ρ, and all transmission distances are over the same distance $r_{\rm tx}$, and $ii)$ transmitters vary their transmission power over variable distances to achieve a specified receive power. In that work we demonstrated the transmission capacity scales very similarly for both models, so for analytical simplicity we use the first model in this paper.

Our channel model considers only path loss attenuation effects and ignores additional channel effects such as shadowing and fast fading, which have been recently shown to not have an especially large effect on capacity scaling [7], [10]. In particular, if the (normalized to $d = 1$) transmit power is ρ and the path loss exponent is $\alpha > 2$ then the received power at a distance $d > 1$ from the transmitter is $\rho d^{-\alpha}$. We denote the SINR threshold required for successful transmission as β. One additional simplification employed in this paper relative to [11] is that here we will assume the ambient noise power is negligibly small, which is generally the case in any interference-limited wireless network. This assumption is made to simplify the analysis and resulting expressions, and is verified to be reasonable via simulation.

A beneficial consequence of the Poisson assumption is that, by Slivnyak's Theorem [24], the client-average outage probability may be found by evaluating the SINR seen by a receiver located at the origin. Intuitively, the distribution of the point process is unaffected by the addition of a receiver at the origin, and this receiver is "typical" in the sense that evaluating the performance seen at the origin gives the client-average performance over all receivers. Measuring the performance at the origin is often termed the Palm measure, and in keeping with standard notation we will denote probability and expectation of functionals of Π evaluated at the origin by $\mathbb{P}^0$ and $\mathbb{E}^0$ respectively. Also let $|X_i|$ denote the distance from node $i \in \Pi$ to the origin. With this in mind we may define the optimal contention density and the transmission capacity.

Definition 1: The optimal contention density, λ^ϵ, *is the maximum spatial density of nodes that can contend for the channel subject to the constraint that the typical outage probability is less than* ϵ *for some* $\epsilon \in (0,1)$*:*

$$\lambda^\epsilon = \sup\left\{\lambda \ : \ \mathbb{P}^0\left(\frac{\rho r_{\rm tx}^{-\alpha}}{\sum_{i\in\Pi}\rho|X_i|^{-\alpha}} \leq \beta\right) \leq \epsilon\right\}. \quad (1)$$

Definition 2: The transmission capacity, c^ϵ, *is the density of successful transmissions resulting from the optimal contention density, multiplied by the achievable data rate* b *of a typical (i.e. average) successful (not in outage) transmission:* $c^\epsilon = \lambda^\epsilon b(1-\epsilon)$.

Hence, the transmission capacity has units of bits per second per Hertz per area, or area spectral efficiency. As in previous work [11], for simplicity in this paper it is assumed henceforth that $b \equiv 1$, so the focus is on quantifying the number of successful transmissions, rather than on the data rate of those transmissions. This is appropriate in the context of this paper since the motivation for multiuser receivers is to increase the number of simultaneous users. The general definition allows other transmission or receiver schemes to increase the data rate for a fixed number of users and be credited appropriately in the transmission capacity framework.

B. Notation

For convenience we summarize here most of the notation used in the paper.

$a \vee b$	$\max\{a,b\}$
$a \wedge b$	$\min\{a,b\}$
$x \in A \setminus B$	$x \in A, x \notin B$
$b(O,r)$	$\{x : \lvert x\rvert \leq r\}$, i.e. a ball of radius r centered at origin
$a(O,r_1,r_2)$	$\{x : r_1 \leq \lvert x\rvert \leq r_2\}$, i.e. an annulus between r_1 and r_2
ρ	transmit power
$r_{\rm tx}$	transmit distance
β	target (required) SINR
ϵ	required outage probability, i.e. $P^0[SINR \leq \beta] \leq \epsilon$
α	path loss exponent
y	$y \doteq \frac{r_{\rm tx}^{-\alpha}}{\beta}$ is the normalized aggregate interference threshold
r_s	$r_s \doteq y^{-\frac{1}{\alpha}} = \beta^{\frac{1}{\alpha}} r_{\rm tx}$ is the "splitting radius"
K	maximum *and* expected number of cancelled users
$r_{\rm sic}$	radius around receiver that includes, on average, K interferers
ζ	residual interference after cancellation, we assume $\zeta \in (0,1)$
psic	specifies perfect SIC, i.e., $\zeta = 0$
nsic	specifies no SIC, i.e., $\zeta = 1$
$\Pi(\lambda)$	A Poisson Point Process Π with density λ
λ_u	Upper bound on λ, i.e. $\lambda \geq \lambda_u \Rightarrow P^0[SINR \leq \beta] \geq \epsilon$
λ_l	Lower bound on λ, i.e. $\lambda \leq \lambda_l \Rightarrow P^0[SINR \leq \beta] \leq \epsilon$
λ_C	A lower bound attained with Chebychev inequality
λ_M	A lower bound attained with Markov inequality
λ^ϵ	specifies that the resulting density is for some ϵ
λ_r	λ such that K users are $\in b(0,r_{\rm tx})$ on average.
λ_s	λ such that K users are $\in b(0,r_s)$ on average

III. TRANSMISSION CAPACITY WITHOUT SIC

First, we give upper and lower bounds on the transmission capacity without any interference cancellation, denoted as $\rm nsic$ to mean "no SIC". This is the special case of the results of [11] where the spreading factor $M = 1$, so this theorem is stated without proof and is given here for completeness and to compare with the bounds derived for the SIC cases.

Theorem 1: As $\epsilon \to 0$*, the lower and upper bounds on the transmission capacity subject to the outage constraint* ϵ *when transmitters employ a fixed transmission power* ρ *for receivers that are a fixed distance* $r_{\rm tx}$ *away are:*

$$c_l^{\epsilon,\rm nsic} = (1-\epsilon)\lambda_l^{\epsilon,\rm nsic}, \qquad c_u^{\epsilon,\rm nsic} = (1-\epsilon)\lambda_u^{\epsilon,\rm nsic}, \quad (2)$$

where the (Markov (M) and Chebychev (C)) lower and upper bounds on the optimal contention density are:

$$\begin{aligned}
\lambda_{l,M}^{\epsilon,\rm nsic} &= \left(1-\frac{2}{\alpha}\right)\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\rm tx}\right)^2} + O(\epsilon^2),\\
\lambda_{l,C}^{\epsilon,\rm nsic} &= \left(1-\frac{1}{\alpha}\right)\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\rm tx}\right)^2} + O(\epsilon^2),\\
\lambda_{u}^{\epsilon,\rm nsic} &= \frac{-\ln(1-\epsilon)}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\rm tx}\right)^2}
\end{aligned}$$

Comments on Theorem 1. Several points are noteworthy:

- In order to obtain bounds on the transmission capacity for FH and DS it suffices to $i)$ multiply $\lambda_l^{\epsilon,\rm nsic}$ and $\lambda_u^{\epsilon,\rm nsic}$ by M for FH and $ii)$ multiply $\lambda_l^{\epsilon,\rm nsic}$ and $\lambda_u^{\epsilon,\rm nsic}$ by $M^{\frac{2}{\alpha}}$ for DS [11]. In words, FH increases the transmission capacity linearly with M and uses M times more bandwidth, while DS increases it as $M^{\frac{2}{\alpha}}$ and uses M times more bandwidth.
- The quantity $r_s \doteq \beta^{\frac{1}{\alpha}} r_{\rm tx}$ is a minimum interference-free radius since a necessary condition for successful reception is that there be no transmitters in $b(O,r_s)$. We term this the "splitting radius" since it is useful to split the interferers into two groups: those inside r_s and those outside of r_s. The bounds illustrate that transmission capacity has a strong connection with sphere packing: πr_s^2 is the area of the disk corresponding to the interference-free radius. Note that reducing the required β, e.g. through spreading, reduces the interference-free radius, thereby permitting a larger number of spheres to be packed into the space (at the cost of lower spectral efficiency)
- For small ϵ the transmission capacity grows linearly in ϵ; thus relaxation of the QoS requirement from 1% outage to 2% outage should double the capacity.
- The above lower bounds are obtained through the use of the Markov and Chebychev inequalities, by bounding the probability of the "far-field" nodes (i.e., outside of r_s) generating sufficient aggregate interference to cause an outage at the origin. It is of interest to obtain the tightest possible bounds on the transmission capacity, i.e., the greatest lower bound and the smallest upper bound. We define the *bounds ratio* γ as the ratio of lower over the upper bound, i.e. $0 \leq \gamma \leq 1$.

Definition 3: The Markov and Chebychev bound ratios for the no SIC case, $\gamma_M^{\text{nsic}}, \gamma_C^{\text{nsic}}$, are defined as:

$$\gamma_M^{\text{nsic}} = \frac{\lambda_{l,M}^{\epsilon,\text{nsic}}}{\lambda_u^{\epsilon,\text{nsic}}}, \quad \gamma_C^{\text{nsic}} = \frac{\lambda_{l,C}^{\epsilon,\text{nsic}}}{\lambda_u^{\epsilon,\text{nsic}}}. \tag{3}$$

Applying these definitions to the bounds in Theorem 1 gives:

$$\gamma_M^{\text{nsic}} = 1 - \frac{2}{\alpha}, \quad \gamma_C^{\text{nsic}} = 1 - \frac{1}{\alpha}. \tag{4}$$

Thus the use of the Chebychev inequality improves the bound ratio by a factor of $\frac{\alpha-1}{\alpha-2}$. In general, the Chebychev inequality gives tighter bounds but is more complicated.

IV. Perfect Successive Interference Cancellation

A. SIC model

Successive interference cancellation allows users to be decoded one at a time, and then subtracted out from the composite received signal in order to improve the performance of subsequently decoded users. In practice, this corresponds to decoding the strongest user first, since it will experience the best SINR and hence be the most accurately decoded, which is a prerequisite for accurate interference cancellation. More generally, by similar reasoning, users should be decoded in order of their received powers [16], [25], even though this is not always the preferred order from an information theoretic viewpoint [19]. In an ad hoc network with a path loss channel model, this corresponds to cancelling the interference from nodes closer to the receiver than the desired transmitter.

An accurate characterization of the performance gains due to SIC should be based on a plausible interference cancellation scenario, otherwise the results can in fact be quite optimistic and misleading. Particularly, an accurate model would capture that a SIC-equipped receiver is able to reduce the interference power of up to K nearest interfering nodes by a factor $1-\zeta$ (i.e. residual interference power of ζ), assuming these nodes are closer than our desired transmitter. However, it is difficult to work with this exact model in a mathematical framework, since it requires a characterization of the joint distribution of the distances of the K nodes nearest to the origin, see [26].

Instead of pursuing this exact approach, we utilize a closely-related SIC model that is more amenable to analysis. In particular, define the *cancellation radius*, denoted r_{sic}, such that the receiver is capable of reducing the interference power by ζ of any and all transmitters located within distance r_{sic} of it. The cancellation radius is chosen so that there are K interfering nodes falling within the radius *on average*. Since the average number of points in a Poisson process of intensity λ falling in a circle of radius r is $\lambda\pi r^2$, we find $r_{\text{sic}} = \sqrt{\frac{K}{\pi\lambda}}$. It is normally only feasible to cancel the interference from those nodes whose interference power measured at the receiver exceeds the signal power. Thus we add the requirement that $r_{\text{sic}} \leq r_{\text{tx}}$, i.e., by requiring the cancellation radius not exceed the signal transmission radius we are ensuring the interference power of cancelled nodes exceeds the signal power.

Definition 4: A (K,ζ) SIC receiver *operating in a network with a transmission density of λ is capable of reducing the interference power by a factor $1-\zeta$ for all interfering nodes within distance $r_{\text{sic}} = r_{\text{tx}} \wedge \sqrt{\frac{K}{\pi\lambda}}$.*

Note that $r_{\text{sic}} = r_{\text{tx}}$ for $\lambda \leq \lambda_r = \frac{K}{\pi} r_{\text{tx}}^{-2}$, and is decreasing in λ for $\lambda > \lambda_r$. Put simply, for low densities all the nodes closer than the desired transmitter are cancellable. For higher densities we can only cancel the closest K nodes (on average), which are inside $r_{\text{sic}} \leq r_{\text{tx}}$.

Now, let $b(O,r) = \{x : |x| \leq r\}$ be the ball of radius r centered at the origin, and let $\bar{b}(O,r) = \mathbb{R}^2 \setminus b(O,r)$. The appropriate modification to Definition 1 that allows for SIC is as follows.

Definition 5: The optimal contention density *for a network of (K,ζ) receivers, $\lambda^{\epsilon,\text{sic}}$, is the maximum spatial density of nodes that can contend for the channel subject to the constraint that the typical outage probability is less than ϵ for some $\epsilon \in (0,1)$:*

$$\lambda^{\epsilon,\text{sic}} = \sup\left\{\lambda : \mathbb{P}^0\left(\frac{\rho r^{-\alpha}}{Y(\lambda)} \leq \beta\right) \leq \epsilon\right\}, \tag{5}$$

where

$$Y(\lambda) = \zeta \times \sum_{i \in \Pi \cap b(O,r_{\text{sic}})} \rho |X_i|^{-\alpha} + \sum_{i \in \Pi \cap \bar{b}(O,r_{\text{sic}})} \rho |X_i|^{-\alpha}. \tag{6}$$

The first term in (6) is the partially cancelled aggregate interference at the receiver from all nodes lying within the cancellation radius; the second term is the uncancelled interference from nodes lying outside that set.

We will break the analysis into two parts: $i)$ *perfect SIC*, denoted psic, where the interference from nodes within r_{sic} is cancelled entirely, i.e., $\zeta = 0$, and $ii)$ *imperfect SIC*, denoted sic, where the interference is partially cancelled, i.e., $\zeta \in (0,1)$. Note that the case $\zeta = 1$ corresponds to the case of no SIC (Theorem 1), and was analyzed in [11].

B. Main Result

The major result is a set of closed form expressions for lower and upper bounds on the transmission capacity.

Theorem 2: As $\epsilon \in (0,1) \to 0$, the lower and upper bounds on the transmission capacity when receivers are equipped with perfect SIC ($\zeta = 0$) are:

$$c_l^{\epsilon,\text{psic}} = (1-\epsilon)\lambda_l^{\epsilon,\text{psic}}, \qquad c_u^{\epsilon,\text{psic}} = (1-\epsilon)\lambda_u^{\epsilon,\text{psic}}. \tag{7}$$

An upper bound on the optimal contention density is:

$$\lambda_u^{\epsilon,\text{psic}} = \frac{-\ln(1-\epsilon) + K}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2}. \tag{8}$$

The Markov (M) lower bound on the optimal contention density is:

$$\lambda_{l,M}^{\epsilon,\text{psic}} = \begin{cases} \left(\frac{(\alpha-2)\epsilon}{2}\right) \frac{\beta^{\frac{2}{\alpha}-1}}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} & \epsilon \leq \frac{2K}{\alpha-2}\beta \\ \left(\frac{(\alpha-2)\epsilon}{2}\right)^{\frac{2}{\alpha}} \frac{K^{1-\frac{2}{\alpha}}}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} & \text{else} \\ \left(1 - \frac{2}{\alpha}\right) \frac{\epsilon + K}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} & \epsilon \geq \frac{2K}{\alpha-2} \end{cases} \tag{9}$$

C. Comments on Theorem 2.

Based on initial inspection of these results, several points are noteworthy:

- Compared with no SIC, perfect SIC improves the upper bound by
$$\frac{\lambda_u^{\epsilon,\mathrm{psic}}}{\lambda_u^{\epsilon,\mathrm{nsic}}} = 1 + \frac{K}{-\ln(1-\epsilon)} \approx 1 + \frac{K}{\epsilon} \text{ when } \epsilon \ll 1, \quad (10)$$
which is linear in K. This sensibly implies for maximum performance, as many users should be cancelled as possible.
- As shown in Appendix VII, the Chebychev bound can be computed, but not expressed in a fully closed-form. For brevity, only the Markov lower bound is given in Theorem 2 and its proof, but the Chebychev is tighter so is used in the numerical results section.
- Compared with no SIC, for $\epsilon < \epsilon_{c,M}^{\mathrm{psic}}$, perfect SIC improves the Markov lower bound by a factor
$$\frac{\lambda_{l,M}^{\epsilon,\mathrm{psic}}}{\lambda_{l,M}^{\epsilon,\mathrm{nsic}}} = \frac{\alpha}{2}\beta^{\frac{2}{\alpha}-1}, \quad (11)$$
which is independent of K. This implies that path loss is a more important aspect for the lower bound – it is preferable to have the far-field users attenuated by a hostile channel, since they cannot be cancelled anyway.
- The bounds for perfect SIC are very loose for small ϵ. It is straightforward to see that
$$\lambda_u^{\epsilon,\mathrm{psic}} = \frac{\epsilon + K}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\mathrm{tx}}\right)^2} + O(\epsilon^2), \quad (12)$$
and corresponding bound ratios:
$$\begin{aligned} \frac{\lambda_{l,M}^{\epsilon,\mathrm{psic}}}{\lambda_u^{\epsilon,\mathrm{psic}}} &= \frac{\alpha-2}{2}\beta^{\frac{2}{\alpha}-1}\left(\frac{\epsilon}{\epsilon+K}\right), \\ \frac{\lambda_{l,C}^{\epsilon,\mathrm{psic}}}{\lambda_u^{\epsilon,\mathrm{psic}}} &= (\alpha-1)\beta^{2(\frac{1}{\alpha}-1)}\left(\frac{\epsilon}{\epsilon+K}\right). \end{aligned}$$
Both ratios are arbitrarily close to 0 as $\epsilon \to 0$. The poor bound ratio is a consequence of the fact that there is no known "upper bound event" that provides a sufficient condition for outage and also results in a tight bound. The imperfect SIC model does not suffer from this problem.

D. Proof of Theorem 2.

The idea behind the proof is to identify necessary and sufficient conditions for outage, calculate or bound the probabilities of the corresponding events, and then determine the spatial densities such that the probabilities of the necessary and sufficient events equal the specified QoS parameter ϵ. The sufficient condition event we employ is the set of realizations of the point process Π with one or more interfering nodes close enough to the receiver so that one such node alone is capable of causing outage. The necessary condition is more complex: if we have an outage it can be due to either a few nodes near the receiver or the combination of a large number of far away nodes. Let
$$F_u^{\mathrm{psic}}(\lambda), \quad F^{\mathrm{psic}}(\lambda), \quad F_l^{\mathrm{psic}}(\lambda) \quad (13)$$

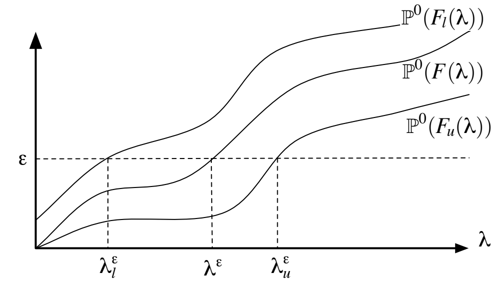

Fig. 2. Illustration of the technique to find lower and upper bounds on the contention density: $\lambda_l^\epsilon \le \lambda^\epsilon \le \lambda_u^\epsilon$ through the use of necessary and sufficient events for outage.

be events parameterized by the spatial density λ so that
$$F_u^{\mathrm{psic}}(\lambda) \subset F^{\mathrm{psic}}(\lambda) \subset F_l^{\mathrm{psic}}(\lambda). \quad (14)$$
The probability of all three events will be nondecreasing in λ. The event $F_u^{\mathrm{psic}}(\lambda)$ is the sufficient event, $F^{\mathrm{psic}}(\lambda)$ is the outage event, and $F_l^{\mathrm{psic}}(\lambda)$ is the necessary event. As illustrated in Figure 2, lower and upper bounds on the optimal contention density are obtainable from the probabilities of the necessary and sufficient events, provided we can solve
$$\begin{aligned} \lambda_u^{\epsilon,\mathrm{psic}} &= \left\{\lambda : \mathbb{P}^0(F_u^{\mathrm{psic}}(\lambda)) = \epsilon\right\}, \\ \lambda_l^{\epsilon,\mathrm{psic}} &= \left\{\lambda : \mathbb{P}^0(F_l^{\mathrm{psic}}(\lambda)) = \epsilon\right\} \end{aligned}$$
for λ. Since these equations are in general not solvable for λ, we define several different events that will help us attain bounds on λ.

Definition 6:
$$\begin{aligned} F^{\mathrm{psic}}(\lambda) &= \left\{Y(\lambda) > y\right\}, \\ F_u^{\mathrm{psic}}(\lambda) &= \left\{\Pi \cap a(O, r_{\mathrm{sic}}, r_s) \neq \emptyset\right\} \\ F_f^{\mathrm{psic}}(\lambda) &= \left\{Y(\lambda, r_s) > y\right\} \end{aligned}$$
where $Y(\lambda)$ is given by Definition 5 with $\zeta = 0$ and
$$Y(\lambda, r_s) = \sum_{i \in \Pi \cap \bar{b}(O, r_{\mathrm{sic}} \vee r_s)} |X_i|^{-\alpha} \quad (15)$$
is the normalized aggregate interference by all nodes outside of the the radius $r_{\mathrm{sic}} \vee r_s$.

In words, $F^{\mathrm{psic}}(\lambda)$ is the outage event, $F_u^{\mathrm{psic}}(\lambda)$ is the event that one or more nodes lie in the annulus with radii r_{sic} and r_s, and $F_f^{\mathrm{psic}}(\lambda)$ is the event that the aggregate interference generated by nodes outside the radius $r_{\mathrm{sic}} \vee r_s$ is sufficient to cause an outage. It is straightforward to establish that
$$F_u^{\mathrm{psic}}(\lambda) \subset F^{\mathrm{psic}}(\lambda) \subset F_l^{\mathrm{psic}}(\lambda) = F_u^{\mathrm{psic}}(\lambda) \cup F_f^{\mathrm{psic}}(\lambda). \quad (16)$$
It is helpful to think of $r_s \doteq y^{-\frac{1}{\alpha}}$ as the radius splitting the "near-field" interference, $b(O, r_s)$, from the "far-field" interference, $\mathbb{R}^2 \setminus b(O, r_s)$. A similar approach is employed in [11] for the proof of the transmission capacity without SIC, but with the additional degree of freedom that the near/far field boundary was optimized over all s. It is shown that $s = r_s$ is the optimal splitting radius. A similar optimization could be

performed here but the analysis becomes much more complex and the tractability of the model is lost. For that reason we use a fixed near/far field splitting radius r_s throughout this paper, where r_s is the maximum radius such that a single node at that distance from the receiver can by itself cause an outage at the receiver. Clearly

$$\mathbb{P}^0(F_u^{\rm psic}(\lambda)) \ \leq \ \mathbb{P}^0(F^{\rm psic}(\lambda)) \ \leq \ \mathbb{P}^0(F_l^{\rm psic}(\lambda)), \quad (17)$$

and

$$\mathbb{P}^0(F_l^{\rm psic}(\lambda)) \leq \mathbb{P}^0(F_u^{\rm psic}(\lambda)) + \mathbb{P}^0(F_f^{\rm psic}(\lambda)). \quad (18)$$

Define two spatial transmission density thresholds:

$$\lambda_r = \tfrac{K}{\pi r_{\rm tx}^2}, \quad \lambda_s = \tfrac{K}{\pi r_s^2}. \quad (19)$$

These correspond to the densities where there are on average K users inside of a radius $r_{\rm tx}$ and r_s, respectively.

It is straightforward to see then that

$$r_{\rm sic} = \begin{cases} r_{\rm tx}, & \lambda \leq \lambda_r \\ \sqrt{\frac{K}{\pi\lambda}}, & \text{else} \end{cases} \quad (20)$$

and that

$$r_{\rm sic} \begin{cases} \geq r_s, & \lambda \leq \lambda_s \\ \leq r_s, & \text{else} \end{cases}. \quad (21)$$

Upper Bound. We begin by finding the upper bound; this requires solving $\mathbb{P}^0(F_u^{\rm psic}) = \epsilon$ for λ. The probability of one or more nodes lying in the annulus with radii $r_{\rm sic}$ and r_s is simply one minus the void probability for the set:

$$\mathbb{P}^0(F_u^{\rm psic}(\lambda)) = \begin{cases} 1 - \exp\left\{-\lambda\pi(r_s^2 - r_{\rm sic}^2)\right\}, & \lambda > \lambda_s \\ 0, & \text{else} \end{cases}. \quad (22)$$

It is evident here why the upper bound is so weak for the perfect SIC case: the upper bound event is zero for all $\lambda \leq \lambda_s$. Unfortunately there is no other easily computable sufficient event available. Note that the map $\lambda \to \mathbb{P}^0(F_u^{\rm psic}(\lambda))$ is onto $[0,1)$ and monotone increasing in λ; hence a unique inverse exists for all $\epsilon > 0$. Setting this expression equal to ϵ and solving for λ yields:

$$\lambda_u^{\epsilon,{\rm psic}} = \frac{1}{\pi r_s^2}\Big(-\ln(1-\epsilon) + K\Big) \geq \left(1 + \frac{K}{\epsilon}\right)\frac{1}{\pi r_s^2}\epsilon + O(\epsilon^2). \quad (23)$$

Lower Bound. We turn now to the lower bound. The lower bound event $F_l^{\rm psic}(\lambda, s)$ is the union of two events, $F_u^{\rm psic}(\lambda)$ and $F_f^{\rm psic}(\lambda)$; the probability of both events is increasing in λ. Moreover, a consequence of the assumption that the node positions form a Poisson process is that the two events are independent seeing as they concern disjoint regions of $\mathbb{R}^2$. Fix ϵ and consider some pair (ϵ_u, ϵ_f) such that $\epsilon_u + \epsilon_f = \epsilon$. If we can identify a pair $(\lambda_u^{\epsilon_u,{\rm psic}}, \lambda_f^{\epsilon_f,{\rm psic}})$ satisfying:

$$\mathbb{P}^0(F_u(\lambda_u^{\epsilon_u,{\rm psic}})) \leq \epsilon_u, \ \mathbb{P}^0(F_f(\lambda_f^{\epsilon_f,{\rm psic}})) \leq \epsilon_f \quad (24)$$

then

$$\mathbb{P}^0(F_l(\lambda_u^{\epsilon_u,{\rm psic}} \wedge \lambda_f^{\epsilon_f,{\rm psic}})) \leq \epsilon_u + \epsilon_f = \epsilon. \quad (25)$$

Thus $\lambda_u^{\epsilon_u,{\rm psic}} \wedge \lambda_f^{\epsilon_f,{\rm psic}}$ is a valid lower bound since choosing $\lambda < \lambda_u^{\epsilon_u,{\rm psic}} \wedge \lambda_f^{\epsilon_f,{\rm psic}}$ ensures the outage probability is less than ϵ. This argument holds for all partitions (ϵ_u, ϵ_f) summing to ϵ. The greatest lower bound is obtained by maximizing $\lambda_u^{\epsilon_u,{\rm psic}} \wedge \lambda_f^{\epsilon_f,{\rm psic}}$ over all feasible partitions:

$$\lambda_l^{\epsilon,{\rm psic}} = \sup_{(\epsilon_u,\epsilon_f)\ :\ \epsilon_u+\epsilon_f=\epsilon} \left\{\lambda_u^{\epsilon_u,{\rm psic}} \wedge \lambda_f^{\epsilon_f,{\rm psic}}\right\}. \quad (26)$$

Note that the minimum of two functions is maximized by minimizing the distance between them. If that minimum distance is zero then the minimum of the two functions is their value at the point of intersection.

The probability $\mathbb{P}^0(F_f^{\rm psic}(\lambda)) = \mathbb{P}^0(Y(\lambda, r_s) > y)$ cannot be computed exactly; it must be bounded. We can obtain two bounds via the Markov and Chebychev inequalities. The former is weaker but the resulting equations are simpler, the latter is stronger but the equations are more complex, so its derivation is left to Appendix VII. To compute the Markov bound we need $\mathbb{E}^0[Y(\lambda, r_s)]$; to compute the Chebychev bound we need $\mathbb{E}^0[Y(\lambda, r_s)]$ and $Var(Y(\lambda, r_s))$. Both are obtainable via Campbell's Theorem [11], [24], which gives that $E[\sum_{x\in\Pi} f(x)] = \int_{\mathbb{R}^2} \lambda f(x)\nu_d(dx)$ where $\nu_d(\cdot)$ is the Lebesgue measure (area). The Markov bound states that

$$\mathbb{P}^0(F_f^{\rm psic}(\lambda)) \leq \frac{\mathbb{E}^0[Y(\lambda, r_s)]}{y}. \quad (27)$$

It is straightforward to compute:

$$\frac{\mathbb{E}^0[Y(\lambda, r_s)]}{y} = \begin{cases} \frac{2\pi}{\alpha-2}\beta r_{\rm tx}^2\lambda, & \lambda \leq \lambda_c \\ \frac{2}{\alpha-2}K^{1-\frac{\alpha}{2}}\frac{1}{y}(\pi\lambda)^{\frac{\alpha}{2}}, & \lambda_c < \lambda \leq \lambda_k \\ \frac{2\pi}{\alpha-2}r_s^2\lambda, & \lambda > \lambda_k \end{cases}. \quad (28)$$

The value of the bound at the critical points is

$$\epsilon_{c,M}^{\rm psic} = \frac{\mathbb{E}^0[Y(\lambda_c, r_s)]}{y} = \beta\frac{2K}{\alpha-2} \quad (29)$$

$$\epsilon_{k,M}^{\rm psic} = \frac{\mathbb{E}^0[Y(\lambda_k, r_s)]}{y} = \frac{2K}{\alpha-2}. \quad (30)$$

This function is monotone increasing in λ and is onto $[0,1]$; hence the inverse function exists. Setting the expression equal to ϵ and solving for λ gives:

$$\lambda_{f,M}^{\epsilon,{\rm psic}} = \begin{cases} \left(\frac{\alpha-2}{2}\right)\beta^{\frac{2}{\alpha}-1}\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\rm tx}\right)^2}, & \epsilon \leq \epsilon_{c,M}^{\rm psic} \\ \left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}}\left(\frac{K}{\epsilon}\right)^{1-\frac{2}{\alpha}}\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\rm tx}\right)^2}, & \epsilon_{c,M}^{\rm psic} < \epsilon \leq \epsilon_{k,M}^{\rm psic} \\ \left(\frac{\alpha-2}{2}\right)\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\rm tx}\right)^2}, & \epsilon \geq \epsilon_{k,M}^{\rm psic} \end{cases} \quad (31)$$

It remains to maximize $\lambda_{f,M}^{\epsilon_f,{\rm psic}} \wedge \lambda_u^{\epsilon_u,{\rm psic}}$ over all (ϵ_u, ϵ_f) such that $\epsilon_u + \epsilon_f = \epsilon$. Note that $\lambda_u^{\epsilon_u,{\rm psic}} \geq \lambda_s$ for all ϵ while $\lambda_{f,M}^{\epsilon_f,{\rm psic}} \leq \lambda_s$ for all $\epsilon \leq \epsilon_{k,M}^{\rm psic}$. Thus the optimum splitting pair is $\epsilon_f = \epsilon$ and $\epsilon_u = 0$, and the corresponding minimum of the two functions is $\lambda_{f,M}^{\epsilon,{\rm psic}}$. Note that $\epsilon_{k,M}^{\rm psic} \geq 1$ for all $K \geq 2$ and all $\alpha \leq 4$. We now find the optimal splitting pair when $\epsilon > \epsilon_{k,M}^{\rm psic}$. Note that $\lambda_u^{\epsilon_u,{\rm psic}}$ is non-linear in ϵ and hence finding the point of intersection with $\lambda_{f,M}^{\epsilon_f,{\rm psic}}$ is complicated. We find a lower bound on $\lambda_u^{\epsilon_u,{\rm psic}}$ by linearizing around $\epsilon = 0$; this leaves us with the problem of maximizing the minimum of

$$\begin{aligned} \lambda_u^{\epsilon_u,\text{psic}} &\geq \frac{K}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} + \frac{\epsilon_u}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} \\ \lambda_{f,M}^{\epsilon_f,\text{psic}} &= \left(\frac{\alpha-2}{2}\right) \frac{\epsilon_f}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2},\ \epsilon \geq \epsilon_{k,M}^{\text{psic}}. \end{aligned} \tag{32}$$

It is straightforward to establish the optimal splitting pair for the linearized $\lambda_u^{\epsilon_u,\text{psic}}$ is:

$$\epsilon_u = \left(\left(1-\tfrac{2}{\alpha}\right)\epsilon - \tfrac{2}{\alpha}K\right) \vee 0, \quad \epsilon_f = \left(\tfrac{2}{\alpha}\epsilon + \tfrac{2}{\alpha}K\right) \wedge 1, \tag{33}$$

where the two functions share a common value at this point of

$$\lambda_{l,M}^{\epsilon,\text{psic}} = \left(1-\frac{2}{\alpha}\right) \frac{\epsilon + K}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2},\ \epsilon > \epsilon_{k,M}^{\text{psic}}. \tag{34}$$

■

V. Imperfect Successive Interference Cancellation

A. Main Result

The major result is again a set of expressions for lower and upper bounds on the transmission capacity, now generalized for imperfect interference cancellation. Since the interference expressions are now more complicated (since $\zeta \neq 0$ in general), closed-form results are not attainable in all cases. The expressions for the lower bound will be given in terms of an easy optimization problem, the solution of which is trivial for a computer.

Theorem 3: Let $\epsilon \in (0,1)$. As $\epsilon \to 0$, the lower and upper bounds on the transmission capacity when receivers are equipped with imperfect SIC ($\zeta \in (0,1)$) are:

$$c_l^{\epsilon,\text{sic}} = (1-\epsilon)\lambda_l^{\epsilon,\text{sic}}, \qquad c_u^{\epsilon,\text{sic}} = (1-\epsilon)\lambda_u^{\epsilon,\text{sic}}. \tag{35}$$

The upper bound on the optimal contention density is:

$$\lambda_u^{\epsilon,\text{sic}} = \begin{cases} \frac{-\ln(1-\epsilon)}{\zeta^{\frac{2}{\alpha}}\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} & \epsilon \leq 1 - e^{-K\zeta^{\frac{2}{\alpha}}} \\ \frac{K - \ln(1-\epsilon)}{(1+\zeta^{\frac{2}{\alpha}})\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} & \text{else} \\ \frac{-\ln(1-\epsilon)}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2} & \epsilon \geq 1 - e^{-K\zeta^{-\frac{2}{\alpha}}} \end{cases} \tag{36}$$

The Markov (M) lower bound on the optimal contention density is:

$$\lambda_{l,M}^{\epsilon,\text{sic}} \geq \sup_{(\epsilon_u,\epsilon_f):\epsilon_u+\epsilon_f=\epsilon} \left\{\lambda_u^{\epsilon_u,\text{sic}} \wedge \lambda_{f,M}^{\epsilon_f,\text{sic}}\right\}, \tag{37}$$

where

$$\lambda_{f,M}^{\epsilon,\text{sic}} = \begin{cases} \frac{\alpha-2}{2} \frac{\beta^{\frac{2}{\alpha}}}{(1-\zeta)\beta + \zeta\beta^{\frac{2}{\alpha}}} \frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2}, & \epsilon \leq \epsilon_{c,M}^{\text{sic}} \\ \text{see below}, & \epsilon_{c,M}^{\text{sic}} \leq \epsilon < \epsilon_{k,M}^{\text{sic}} \\ \frac{\alpha-2}{2} \frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\text{tx}}\right)^2}, & \epsilon > \epsilon_{k,M}^{\text{sic}} \end{cases} \tag{38}$$

and $\lambda_{f,M}^{\epsilon,\text{sic}}$ for $\epsilon_{c,M}^{\text{sic}} \leq \epsilon < \epsilon_{k,M}^{\text{sic}}$ is the unique solution for λ satisfying equation:

$$\frac{2\pi\lambda\beta}{(\alpha-2) r_{\text{tx}}^{-\alpha}} \left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta r_{\text{tx}}^{2-\alpha} \beta^{\frac{2}{\alpha}-1}\right] = \epsilon. \tag{39}$$

The constants are given by:

$$\epsilon_{c,M}^{\text{sic}} = \left[(1-\zeta)\beta + \zeta\beta^{\frac{2}{\alpha}}\right] \tfrac{2K}{\alpha-2}, \quad \epsilon_{k,M}^{\text{sic}} = \tfrac{2K}{\alpha-2} \tag{40}$$

B. **Comments on Theorem 3.**

Several observations can be made from Theorem 3.

- First, we can see that this more general theorem is consistent with the special cases of no sic ($\zeta \to 1$) and perfect SIC ($\zeta \to 0$) given in Theorems 1 and 2, respectively. This can be confirmed by first noting that the three regimes converge to a single regime in both cases, i.e.

$$\lim_{\zeta\to 1} 1 - e^{-K\zeta^{\frac{2}{\alpha}}} = \lim_{\zeta \to 1} 1 - e^{-K\zeta^{-\frac{2}{\alpha}}} = 1 - e^{-K}, \tag{41}$$

  and then it is easy to see that $\lim_{\zeta\to 1} \lambda_u^{\epsilon,\text{sic}} = \lambda_u^{\epsilon,\text{nsic}}$ as expected. Next note that

$$\lim_{\zeta\to 0} 1 - e^{-K\zeta^{\frac{2}{\alpha}}} = 0, \quad \lim_{\zeta\to 0} 1 - e^{-K\zeta^{-\frac{2}{\alpha}}} = 1, \tag{42}$$

  and hence $\lim_{\zeta\to 0} \lambda_u^{\epsilon,\text{sic}} = \lambda_u^{\epsilon,\text{psic}}$ as expected.
- The theorem above gives the lower bound obtained through use of the Markov inequality. Again, corresponding results for the case of the Chebychev inequality are given Appendix VII. When $\zeta = 0$ it is straightforward to see the expressions for $\lambda_{l,M}^{\epsilon,\text{sic}}, \lambda_{l,C}^{\epsilon,\text{sic}}$ will reduce to those of $\lambda_{l,M}^{\epsilon,\text{psic}}, \lambda_{l,C}^{\epsilon,\text{psic}}$ after appropriate choice of the optimal splitting pair (ϵ_u, ϵ_f). Similarly, when $\zeta = 1$ it is straightforward to see the expressions for $\lambda_{l,M}^{\epsilon,\text{sic}}, \lambda_{l,C}^{\epsilon,\text{sic}}$ will reduce to those of $\lambda_{l,M}^{\epsilon,\text{nsic}}, \lambda_{l,C}^{\epsilon,\text{nsic}}$ after appropriate choice of the optimal splitting pair (ϵ_u, ϵ_f).
- The performance improvement due to SIC is very sensitive to the cancellation effectiveness parameter ζ as $\zeta \to 0$, especially for small ϵ. Looking at the upper bound, for example, we see that for small ϵ:

$$\frac{d}{d\zeta}\lambda_u^{\epsilon,\text{sic}} \propto -\zeta^{-(1+\frac{2}{\alpha})}, \tag{43}$$

  which means

$$\lim_{\zeta\to 0} \frac{d}{d\zeta}\lambda_u^{\epsilon,\text{sic}} = -\infty. \tag{44}$$

  Thus our model suggests that technology improvements which improve cancellation effectiveness may yield large increases in the transmission capacity.
- For small ϵ it is straightforward to show that the bounds are reasonably tight. In particular,

$$\begin{aligned} \lambda_{l,M}^{\epsilon,\text{sic}} &= \frac{(\alpha-2)\beta^{\frac{2}{\alpha}}}{2(1-\zeta)\beta + (2\zeta+\alpha-2)\beta^{\frac{2}{\alpha}}} \frac{\epsilon}{\pi r_s^2} + O(\epsilon^2), \\ \lambda_{l,C}^{\epsilon,\text{sic}} &= \frac{\alpha-1}{(1-\zeta)\beta^{2(1-\frac{1}{\alpha})} + \zeta + \alpha - 1} \frac{\epsilon}{\pi r_s^2} + O(\epsilon^2) \end{aligned}$$

  with corresponding bound ratios of

$$\begin{aligned} \frac{\lambda_{l,M}^{\epsilon,\text{sic}}}{\lambda_u^{\epsilon,\text{sic}}} &= \frac{(\alpha-2)(\beta\zeta)^{\frac{2}{\alpha}}}{2(1-\zeta)\beta + (2\zeta+\alpha-2)\beta^{\frac{2}{\alpha}}} + O(\epsilon^2), \\ \frac{\lambda_{l,C}^{\epsilon,\text{sic}}}{\lambda_u^{\epsilon,\text{sic}}} &= \frac{(\alpha-1)\zeta^{\frac{2}{\alpha}}}{(1-\zeta)\beta^{2(1-\frac{1}{\alpha})} + \zeta + \alpha - 1} + O(\epsilon^2). \end{aligned}$$

Note that these bound ratios are 0 for $\zeta = 0$, consistent with the poor bound ratios for perfect SIC, but $(\alpha-2)/\alpha$ and $(\alpha-1)/\alpha$ respectively for $\zeta = 1$, consistent with the ratios obtained for no SIC.

C. Proof of Theorem 3.

It was earlier shown that $r = r_s$ is a critical radius in the sense that it is the maximum distance a single interfering node can be from a receiver and still generate sufficient interference to cause an outage at that receiver. When the interference is partially cancelled through the use of imperfect SIC with parameter $\zeta \in (0,1)$ then the corresponding critical radius is improved (decreased) to $\zeta^{\frac{1}{\alpha}} r_s$: if the partially cancelled node is any further away it cannot by itself cause an outage. With this in mind, we define the following three spatial density thresholds and compute the corresponding value of $r_{\rm sic}$:

$$\begin{array}{lll} \lambda_r = \frac{K}{\pi} r_{\rm tx}^{-2}, & \lambda_s = \frac{K}{\pi} y^{\frac{2}{\alpha}}, & \lambda_l = \frac{K}{\pi} \zeta^{-\frac{2}{\alpha}} y^{\frac{2}{\alpha}} \\ r_{\rm sic} = r_{\rm tx}, & r_{\rm sic} = r_s, & r_{\rm sic} = \zeta^{\frac{1}{\alpha}} r_s. \end{array} \quad (45)$$

Looking at Figure 3, for each possible spatial density of transmissions λ we can identify the distances from the receiver where an interfering node at that distance will by itself generate sufficient interference to cause an outage. For example, for $\lambda \in (\lambda_s, \lambda_l)$ nodes at distances in the range $(0, \zeta^{\frac{1}{\alpha}} r_s)$ generate interference that is partially cancelled but even so sufficient to cause an outage, and nodes at distances in the range $(\sqrt{\frac{K}{\lambda}}, r_s)$ are outside the cancellation radius but are nonetheless sufficiently close to cause an outage.

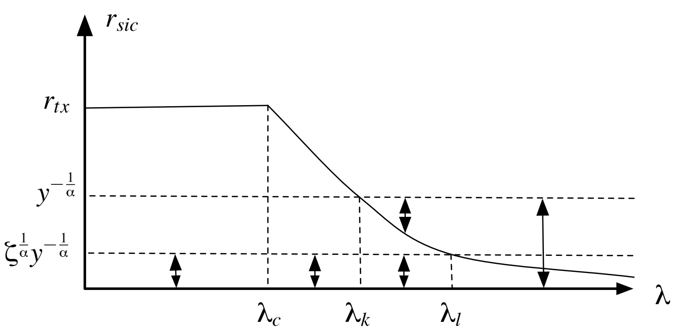

Fig. 3. The cancellation radius $r_{\rm sic}$ versus the spatial transmission density λ. The near/far field separation radius is r_s, this is also the farthest distance that an uncanceled node can be from the receiver and still cause an outage. The farthest distance that a canceled node can be from the receiver and still cause an outage is $\zeta^{\frac{1}{\alpha}} r_s$. The arrows denote the annular regions around the receiver where a single node could cause outage provided that node is in the near field.

The upper bound event corresponding to Figure 3 is:

$$F_u^{\rm sic}(\lambda) = \left\{ \begin{array}{l} \left\{\Pi \cap b(O, \zeta^{\frac{1}{\alpha}} r_s)\right\} \\ \left\{\Pi \cap \left(b(O, \zeta^{\frac{1}{\alpha}} r_s) \cup a(O, \sqrt{\frac{K}{\pi\lambda}}, r_s)\right)\right\} \\ \left\{\Pi \cap b(O, r_s)\right\} \end{array} \right. \quad (46)$$

where the three cases apply for the intervals

$$\lambda \leq \lambda_s, \quad \lambda_s < \lambda \leq \lambda_l, \quad \lambda > \lambda_l \quad (47)$$

respectively. The probability of the event is:

$$\mathbb{P}^0(F_u^{\rm sic}(\lambda)) = \left\{ \begin{array}{l} 1 - \exp\left\{-\lambda\pi\zeta^{\frac{2}{\alpha}} r_s^2\right\} \\ 1 - \exp\left\{-\lambda\pi(1+\zeta^{\frac{2}{\alpha}}) r_s^2 + K\right\} \\ 1 - \exp\left\{-\lambda\pi r_s^2\right\} \end{array} \right. \quad (48)$$

The bound evaluated at the critical points λ_s, λ_l gives:

$$\begin{array}{rcl} \mathbb{P}^0(F_u^{\rm sic}(\lambda_s)) & = & 1 - \exp\left\{-K\zeta^{\frac{2}{\alpha}}\right\} \\ \mathbb{P}^0(F_u^{\rm sic}(\lambda_l)) & = & 1 - \exp\left\{-K\zeta^{-\frac{2}{\alpha}}\right\} \end{array} \quad (49)$$

The map $\lambda \to \mathbb{P}^0(F_u^{\rm sic}(\lambda))$ is onto $[0,1)$ and monotone increasing in λ; hence a unique inverse exists for all $\epsilon > 0$. Setting this expression equal to ϵ and solving for λ yields:

$$\lambda_u^{\epsilon,\rm sic} = \left\{ \begin{array}{ll} \frac{-\ln(1-\epsilon)}{\zeta^{\frac{2}{\alpha}}\pi r_s^2} & \epsilon \leq 1 - e^{-K\zeta^{\frac{2}{\alpha}}} \\ \frac{K - \ln(1-\epsilon)}{(1+\zeta^{\frac{2}{\alpha}})\pi r_s^2} & \text{else} \\ \frac{-\ln(1-\epsilon)}{\pi r_s^2} & \epsilon \geq 1 - e^{-K\zeta^{-\frac{2}{\alpha}}} \end{array} \right. \quad (50)$$

We turn now to the lower bound. Define the following events:

Definition 7:

$$\begin{array}{rcl} F^{\rm sic}(\lambda) & = & \left\{Y(\lambda) > y\right\}, \\ F_f^{\rm sic}(\lambda) & = & \left\{Y(\lambda, r_s, \zeta) > y\right\} \end{array}$$

where $Y(\lambda)$ is given by Definition 5 and

$$\begin{array}{rcl} Y(\lambda, r_s, \zeta) & = & \displaystyle\sum_{\Pi \cap a(O, r_s, r_{\rm sic})} \zeta |X_i|^{-\alpha} \\ & & + \displaystyle\sum_{\Pi \cap \bar{b}(O, r_{\rm sic} \vee r_s)} |X_i|^{-\alpha} \end{array}$$

is the normalized aggregate interference generated by all partially cancelled nodes in the annulus $a(O, r_s, r_{\rm sic})$ plus the interference generated by the uncanceled nodes outside the radius $r_{\rm sic} \vee r_s$.

We first compute the Markov bound on $\mathbb{P}^0(F_f^{\rm sic}(\lambda))$: $\mathbb{E}^0[Y(\lambda, r_s, \zeta)]$

$$= \left\{ \begin{array}{l} 2\pi\lambda\left[\zeta \int_{r_s}^{r_{\rm tx}} r^{-\alpha} r dr + \int_{r_{\rm tx}}^{\infty} r^{-\alpha} r dr\right] \\ 2\pi\lambda\left[\zeta \int_{r_s}^{\sqrt{\frac{K}{\pi\lambda}}} r^{-\alpha} r dr + \int_{\sqrt{\frac{K}{\pi\lambda}}}^{\infty} r^{-\alpha} r dr\right] \\ 2\pi\lambda \int_{r_s}^{\infty} r^{-\alpha} r dr \end{array} \right.$$

$$= \left\{ \begin{array}{l} \frac{2\pi\lambda}{\alpha-2}\left[(1-\zeta) r_{\rm tx}^{2-\alpha} + \zeta y^{1-\frac{2}{\alpha}}\right] \\ \frac{2\pi\lambda}{\alpha-2}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta y^{1-\frac{2}{\alpha}}\right] \\ \frac{2\pi\lambda}{\alpha-2} y^{1-\frac{2}{\alpha}} \end{array} \right.$$

where the three expressions hold for the intervals

$$\lambda \leq \lambda_r, \quad \lambda_r < \lambda \leq \lambda_s, \quad \lambda > \lambda_s \quad (51)$$

respectively. The Markov bound is:

$$\mathbb{E}^0[Y(\lambda, r_s, \zeta)]/y = \left\{ \begin{array}{l} \frac{2\pi\lambda}{(\alpha-2)y}\left[(1-\zeta) r_{\rm tx}^{2-\alpha} + \zeta y^{1-\frac{2}{\alpha}}\right] \\ \frac{2\pi\lambda}{(\alpha-2)y}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta y^{1-\frac{2}{\alpha}}\right] \\ \frac{2\pi\lambda}{\alpha-2} r_s^2 \end{array} \right. \quad (52)$$

The value of the bound at the critical points is

$$\begin{array}{rcl} \epsilon_{c,M}^{\rm sic} & = & \frac{\mathbb{E}[Y(\lambda_r, r_s, \zeta)]}{y} = \left[(1-\zeta)\beta + \zeta\beta^{\frac{2}{\alpha}}\right] \frac{2K}{\alpha-2} \\ \epsilon_{k,M}^{\rm sic} & = & \frac{\mathbb{E}[Y(\lambda_s, r_s, \zeta)]}{y} = \frac{2K}{\alpha-2} \end{array}$$

TABLE I
SIMULATION PARAMETERS (UNLESS OTHERWISE NOTED)

Symbol	Description	Value
α	Path loss exponent	4
M	Spreading factor	16
$\beta = \frac{3}{M}$	Target $SINR$ (DS-CDMA)	$\frac{3}{16}$
$r_{\rm tx}$	Transmission radius	10m
K	Max. no. cancelable nodes	10
ζ	Cancellation effectiveness	$\frac{1}{10}$
ϵ	Target outage probability	0.1

This function is monotone increasing in λ and is onto $[0,1]$; hence the inverse function exists. Setting the expression equal to ϵ and solving for λ gives:

$$\lambda_{f,M}^{\epsilon,{\rm sic}} = \begin{cases} \frac{\alpha-2}{2} \frac{\beta^{\frac{2}{\alpha}}}{(1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}} \frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\rm tx}\right)^2}, & \epsilon \leq \epsilon_{c,M} \\ \text{see below}, & \epsilon_{c,M} \leq \epsilon < \epsilon_{k,M} \\ \frac{\alpha-2}{2} \frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}} r_{\rm tx}\right)^2}, & \epsilon > \epsilon_{k,M} \end{cases} \quad (53)$$

It is not possible to obtain a closed form expression for $\lambda_{f,M}^{\epsilon,{\rm sic}}$ for $\epsilon_{c,M}^{\rm sic} \leq \epsilon < \epsilon_{k,M}^{\rm sic}$; it is the unique solution to the equation:

$$\frac{2\pi\lambda}{(\alpha-2)y}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta y^{1-\frac{2}{\alpha}}\right] = \epsilon. \quad (54)$$

The same comments made in the proof of Theorem 2 regarding finding the optimal splitting pair to maximize $\lambda_{f,M}^{\epsilon_f,{\rm sic}} \wedge \lambda_u^{\epsilon_u,{\rm sic}}$ over all (ϵ_u, ϵ_f) such that $\epsilon_u + \epsilon_f = \epsilon$ hold here.

VI. NUMERICAL AND SIMULATION RESULTS

In this section we present some numerical and simulation results of Theorems 1 – 5. Two types of simulations were performed: one where up to the first K nodes within $r_{\rm tx}$ are cancelled by a factor of $1-\zeta$ and one where all nodes within $r_{\rm sic}$ are cancelled by a factor of $1-\zeta$. The former aims to approximate the *actual* SIC system, while the latter is our stochastic geometric approximate *model* of the SIC system. The terms *Simulation (actual)* and *Simulation (model)* are used to differentiate the results from these two simulators. Both have 90% confidence intervals.

Table I lists the nominal values used for the numerical and simulation results, which are based as closely as possible on realistic parameters for a typical indoor wireless ad hoc network. The target SINR of $3 \approx 5$dB assumes the existence of error correction codes. For conciseness, we restrict our attention to comparing performance of three representative scenarios: $i)$ no SIC ($K = 0$ and $\zeta = 1$), $ii)$ perfect SIC with $K = 10$ ($\zeta = 0$), and $iii)$ imperfect SIC with $K = 10$ and $\zeta = \frac{1}{10}$.

A. Probability of outage versus transmission density

Figure 4 contains three plots of the outage probability $p_o(\lambda)$ versus the transmission density λ. The top plot shows the case of no SIC, the middle plot shows the case of imperfect SIC with $K = 10$ and $\zeta = \frac{1}{10}$, and the bottom plot shows the case of perfect SIC with $K = 10$. For each case we show the Markov and Chebychev lower bound, the upper bound, and simulation results. All plots confirm that the simulation results lie between the upper and lower bounds, and that the Chebychev bound is a tighter lower bound than the Markov bound. Naturally, the Markov and Chebychev "lower bounds" on transmission capacity are thus upper bounds on outage probability while the "upper bound" on capacity is actually a lower bound on outage probability [1]. We also observe that the simulation results for the *actual* and *model* simulators are quite close, granting validity to our stochastic geometric approximation of the actual behavior of the SIC receiver.

Perhaps the most striking trend in the plots is that the bounds for no SIC and imperfect SIC are fairly tight while the bounds for perfect SIC are quite poor. As mentioned earlier, the loose bounds for perfect SIC are a consequence of the fact that there is no tight and simple sufficient condition for outage. It is also apparent that the scaling of the outage probability for perfect SIC is fundamentally different than that of imperfect SIC, even though the perfect SIC parameters ($K = 10, \zeta = 0$) are quite similar to the imperfect SIC parameters ($K = 10, \zeta = \frac{1}{10}$).

B. Optimal contention density versus outage constraint

Figure 5 shows the optimal contention density λ^ϵ versus the outage constraint ϵ for the no SIC, imperfect SIC, and perfect SIC scenarios. For each scenario we show the Chebychev lower bound, the *actual* simulation results, and the upper bound. The dramatic difference between perfect SIC and imperfect SIC are apparent, again highlighting the sensitivity of the optimal contention density to the cancellation effectiveness parameter ζ. Also apparent is the fact that the no SIC and imperfect SIC bounds are tight while the perfect SIC bounds are loose. Finally, we see that the optimal contention density is linear in the outage constraint ϵ over a wide range of values of ϵ, thus validating our linear approximations for small ϵ.

C. Optimal contention density versus number of cancelable interferers

Figure 6 shows the optimal contention density λ^ϵ versus the number of cancelable nodes K for the no SIC, imperfect SIC, and perfect SIC scenarios. Of course the no SIC scenario is independent of K, but also apparent is the insensitivity for the imperfect SIC scenario. Recall that K is the *maximum* number of cancelable interferers; the insensitivity can be explained by the fact that fewer than K nodes typically lie in the disk $b(O, r_{\rm tx})$ around a receiver at the optimal contention density. Note that the perfect SIC results highlight how loose the bounds are for this scenario, and that the optimal contention density levels out first for $K \approx 5$. Finally, note that the imperfect SIC case demonstrates an improvement over no SIC by a factor of about 3.

D. Optimal contention density versus path loss exponent

Figure 7 shows the optimal contention density λ^ϵ versus the path loss exponent α for the case of imperfect SIC with

[1] This is due to the fact that transmission capacity is inversely proportional to outage probability.

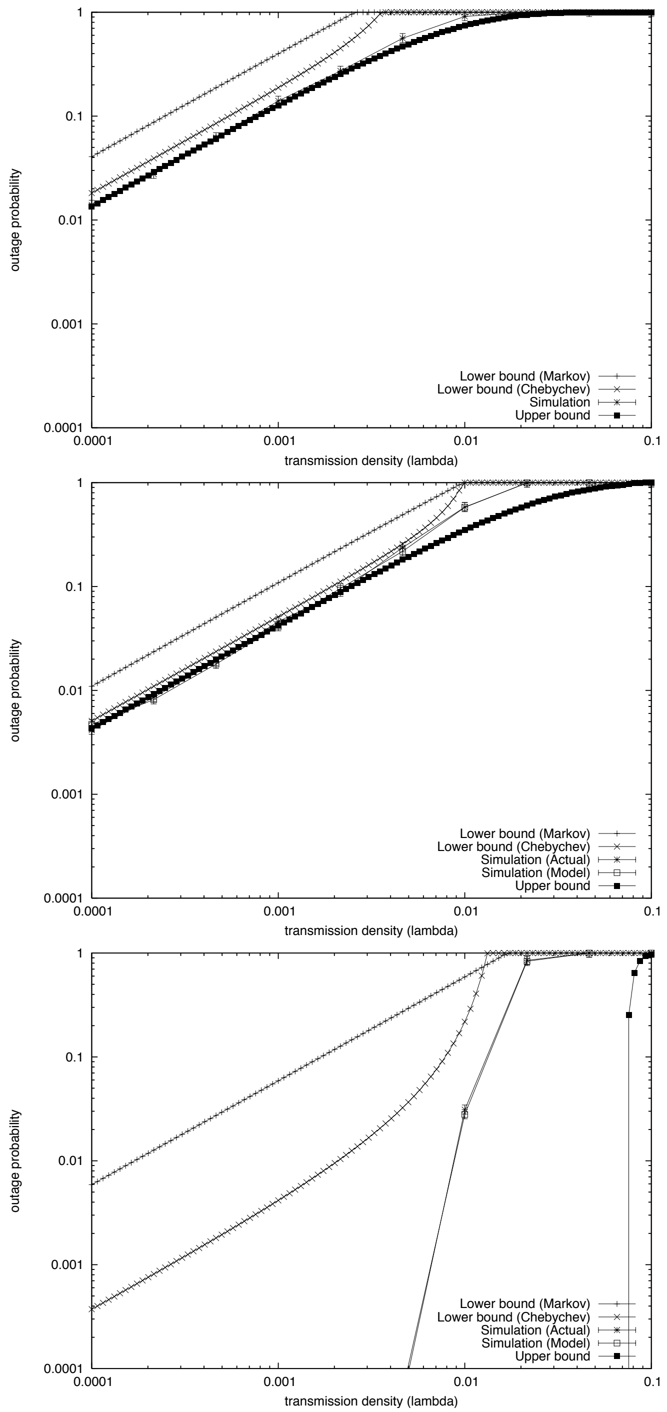

Fig. 4. Outage probability $p_o(\lambda)$ versus the transmission density λ for the cases of no SIC (top), imperfect SIC with $K = 10$ and $\zeta = \frac{1}{10}$ (middle), and perfect SIC with $K = 10$ (bottom). All three plots show both the Markov and Chebychev "lower" bounds and the "upper" bound, as well as simulation results with confidence intervals.

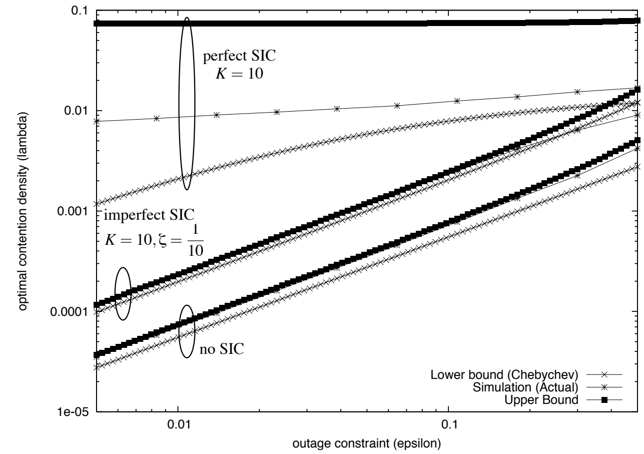

Fig. 5. Optimal contention density λ^{ϵ} versus the outage constraint ϵ for the no SIC, imperfect SIC, and perfect SIC scenarios.

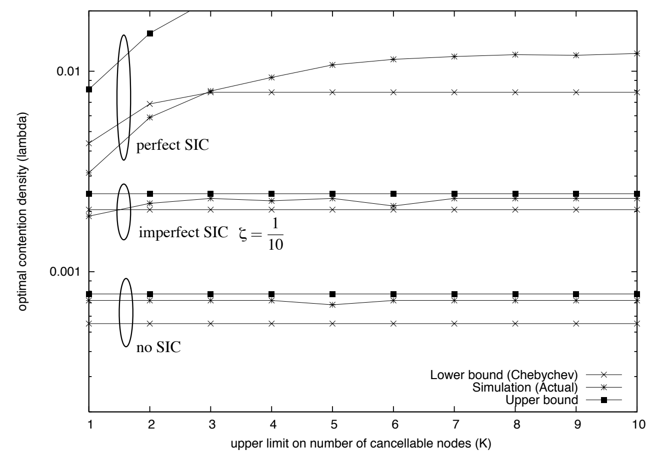

Fig. 6. Optimal contention density λ^{ϵ} versus the number of cancelable interferers K for the no SIC, imperfect SIC, and perfect SIC scenarios.

$K = 10$ and $\zeta = \frac{1}{10}$. The tightness of the bounds is increasing in α. Also of interest is the fact that the Markov lower bound is monotone increasing while the Chebychev lower bound and the upper bound are monotone decreasing. The *actual* simulation results are not monotone in α, and in fact the Chebychev lower bound lies above the simulation results for $\alpha \leq 3$; this illustrates a regime where the rough equivalence of the two models does not hold.

E. Optimal contention density versus cancellation effectiveness

Figure 8 shows the optimal contention density λ^{ϵ} versus the cancellation effectiveness parameter ζ for the no SIC, imperfect SIC, and perfect SIC scenarios. Of course the no SIC and perfect SIC results are independent of ζ: they are shown to confirm that these results are in fact special cases of the imperfect SIC model for $\zeta = 1$ and $\zeta = 0$ respectively. The plot is significant because it demonstrates the great sensitivity of the optimal contention density to ζ for small ζ. This sensitivity is why there is such a difference between the

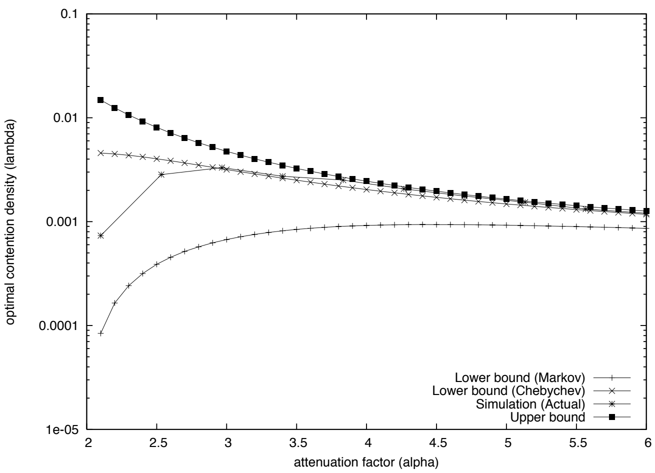

Fig. 7. Optimal contention density λ^{ϵ} versus the path loss exponent α for the imperfect SIC scenario.

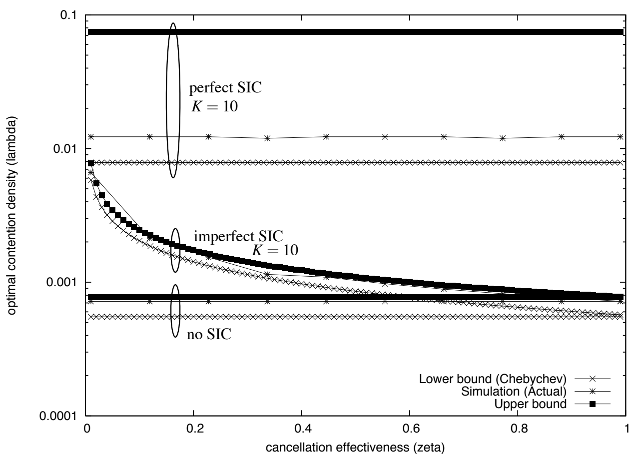

Fig. 8. Optimal contention density λ^{ϵ} versus the cancellation effectiveness parameter ζ for the no SIC, imperfect SIC, and perfect SIC scenarios.

perfect SIC ($K = 10, \zeta = 0$) results and the imperfect SIC results with similar parameters ($K = 10, \zeta = \frac{1}{10}$). This sensitivity suggests that SIC receiver designers might find significant performance improvements by focusing their efforts on improving the cancellation effectiveness.

F. *Spectral efficiency versus spreading factor*

Figure 9 shows a plot of the spectral efficiency λ^{ϵ}/M versus the spreading factor M. Note that the optimal contention density λ^{ϵ} is increasing in M but this increase comes at the cost of increased resource (spectrum) utilization, hence normalizing by M gives an indication of the efficiency measured in terms of the spatial density per Hz. Four scenarios are shown: the three DS-CDMA scenarios used above, i.e., no SIC, imperfect SIC, and perfect SIC, and a FH-CDMA scenario. Note that, by Theorem 1, FH-CDMA is linear in M and hence the spectral efficiency is constant in M. Also, DS-CDMA with no SIC is sub-linear in M and hence the spectral efficiency is decreasing in M. These results are discussed at more length in [11]. As expected the use of imperfect or perfect SIC increases the

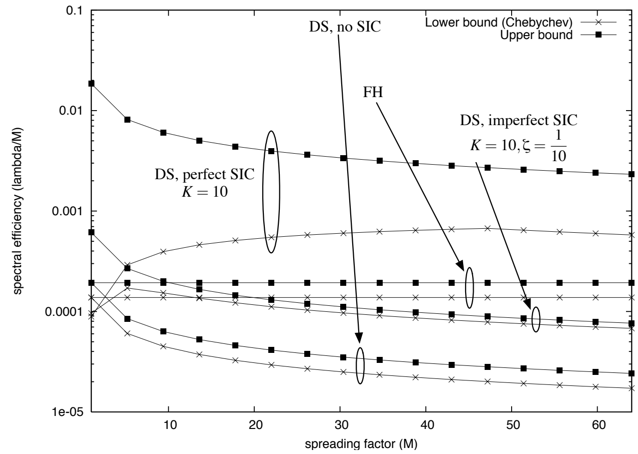

Fig. 9. Spectral efficiency λ^{ϵ}/M versus the spreading factor M for four scenarios: DS-CDMA with no SIC, DS-CDMA with imperfect SIC, DS-CDMA with perfect SIC, and FH-CDMA.

optimal contention density, and hence the spectral efficiency above that of DS-CDMA with no SIC. Perhaps surprising is the fact that imperfect SIC offers improvements in spectral efficiency above FH-CDMA only for small M, in this case $M \leq 10$. The perfect SIC DS-CDMA offers improvement above FH-CDMA for all values of M shown. The plots indicate that the cancellation effectiveness parameter can be very significant in determining whether DS-CDMA with SIC will over or under perform FH-CDMA.

VII. Conclusion

The primary contribution of this work is a tractable framework for analyzing the performance improvement obtainable through the use of successive interference cancellation in wireless ad hoc networks. Through the use of stochastic geometric models and analysis we are able to obtain (in most cases) reasonably tight closed form expressions for the transmission capacity in terms of the fundamental SIC parameters, i.e., the number of cancelable nodes K and the cancellation effectiveness ζ. Our analysis and simulation results support the claims that $i)$ performance is highly sensitive to the cancellation effectiveness parameter but less sensitive to the number of cancelable nodes, and $ii)$ the spectral efficiency of DS-CDMA with SIC is always higher than DS-CDMA without SIC, but may not always exceed that of FH-CDMA.

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APPENDIX A: ADDENDUM TO THEOREM 2

In this section, the Chebychev bound for the case of perfect SIC is derived.

Addendum to Theorem 2: As $\epsilon \in (0,1) \to 0$, the Chebychev lower bound on the optimal contention density when receivers are equipped with perfect SIC ($\zeta = 0$) is:

$$\lambda_{l,C}^{\epsilon,\mathrm{psic}} \geq \sup_{(\epsilon_u,\epsilon_f):\epsilon_u+\epsilon_f=\epsilon} \left\{ \lambda_u^{\epsilon_u,\mathrm{psic}} \wedge \lambda_{f,C}^{\epsilon_f,\mathrm{psic}} \right\}, \tag{A.1}$$

where

$$\lambda_{f,C}^{\epsilon,\mathrm{psic}} = \begin{cases} \frac{\alpha-2}{2\beta}\frac{1}{\pi r_{\mathrm{tx}}^2} + \frac{(\alpha-2)^2}{8(\alpha-1)}\frac{1}{\pi r_{\mathrm{tx}}^2}\frac{1}{\epsilon}\left(1-\sqrt{1+\frac{8(\alpha-1)}{(\alpha-2)\beta}\epsilon}\right), & (a) \\ \frac{y^{\frac{2}{\alpha}}}{\left(\frac{2}{\alpha-2}K^{1-\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}} + \frac{K^{\frac{1-\alpha}{2}}\pi^{\frac{\alpha}{2}}}{\sqrt{\alpha-1}\sqrt{\epsilon}}\right)^{\frac{2}{\alpha}}}, & (b) \\ \frac{\alpha-2}{2}\frac{1}{\pi\epsilon}y^{\frac{2}{\alpha}} + \frac{(\alpha-2)^2}{8(\alpha-1)}\frac{1}{\pi}y^{\frac{2}{\alpha}}\left(1-\sqrt{1+\frac{8(\alpha-1)}{\alpha-2}\epsilon}\right), & (c) \end{cases} \tag{A.2}$$

where the three expressions (a), (b), (c) hold for the regimes

$$\begin{array}{ll} (a) & \left(\lambda_r < \lambda_m^{\mathrm{psic}} \text{ and } \epsilon \leq \epsilon_{c,C}^{\mathrm{psic}}\right) \text{ or } \lambda_r > \lambda_m^{\mathrm{psic}} \\ (b) & \left(\lambda_s < \lambda_m^{\mathrm{psic}} \text{ and } \epsilon_{c,C}^{\mathrm{psic}} < \epsilon \leq \epsilon_{k,C}^{\mathrm{psic}}\right) \\ & \text{or } \left(\lambda_r \leq \lambda_m^{\mathrm{psic}} < \lambda_s \text{ and } \epsilon > \epsilon_{c,C}^{\mathrm{psic}}\right) \\ (c) & \lambda_s < \lambda_m^{\mathrm{psic}} \text{ and } \epsilon > \epsilon_{k,C}^{\mathrm{psic}} \end{array}$$

respectively. The constants are given by:

$$\lambda_m^{\mathrm{psic}} = \begin{cases} \frac{\alpha-2}{2\pi}r_{\mathrm{tx}}^{-2}\beta^{-1}, & \alpha-2-2\beta K \leq 0 \\ \frac{1}{\pi}\left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}}K^{1-\frac{2}{\alpha}}y^{\frac{2}{\alpha}}, & \text{else} \\ \frac{\alpha-2}{2\pi}r_{\mathrm{tx}}^{-2}\beta^{-\frac{2}{\alpha}}, & \alpha-2-2\beta K r^{\alpha} > 0 \end{cases} \tag{A.3}$$

and

$$\lambda_r = \frac{K}{\pi}r_{\mathrm{tx}}^{-2}, \quad \lambda_s = \frac{K}{\pi}y^{\frac{2}{\alpha}} \tag{A.4}$$

and

$$\begin{aligned} \epsilon_{c,C}^{\mathrm{psic}} &= \frac{K(\alpha-2)^2\beta^2}{(\alpha-1)(\alpha-2-2\beta K)^2} \\ \epsilon_{k,C}^{\mathrm{psic}} &= \frac{K(\alpha-2)^2}{(\alpha-1)(\alpha-2-2K)^2}. \end{aligned}$$

Proof of Addendum to Theorem 2: The variance of the far-field interference is again found by Campbell's Theorem:

$$Var(Y(\lambda,r_s)) = \begin{cases} \frac{\pi}{\alpha-1}r_{\mathrm{tx}}^{2(1-\alpha)}\lambda, & \lambda \leq \lambda_r \\ \frac{1}{\alpha-1}K^{1-\alpha}(\pi\lambda)^{\alpha}, & \lambda_r < \lambda \leq \lambda_s \\ \frac{\pi}{\alpha-1}y^{2(1-\frac{1}{\alpha})}\lambda, & \lambda > \lambda_s \end{cases} \tag{A.5}$$

Chebychev's inequality yields: for $y > \mathbb{E}^0[Y(\lambda,r_s)]$:

$$\mathbb{P}^0(Y(\lambda,r_s) > y) \leq \frac{Var(Y(\lambda,r_s))}{(y-\mathbb{E}^0[Y(\lambda,r_s)])^2}. \tag{A.6}$$

Substituting the expressions for the mean and variance:

$$\frac{Var(Y(\lambda,r_s))}{(y-\mathbb{E}^0[Y(\lambda,r_s)])^2} = \begin{cases} \frac{\frac{\pi}{\alpha-1}r_{\mathrm{tx}}^{2(1-\alpha)}\lambda}{\left(y-\frac{2\pi}{\alpha-2}r_{\mathrm{tx}}^{2-\alpha}\lambda\right)^2} \\ \frac{\frac{1}{\alpha-1}K^{1-\alpha}(\pi\lambda)^{\alpha}}{\left(y-\frac{2}{\alpha-2}K^{1-\frac{\alpha}{2}}(\pi\lambda)^{\frac{\alpha}{2}}\right)^2} \\ \frac{\frac{\pi}{\alpha-1}y^{2(1-\frac{1}{\alpha})}\lambda}{\left(y-\frac{2\pi}{\alpha-2}y^{1-\frac{2}{\alpha}}\lambda\right)^2} \end{cases} \tag{A.7}$$

where the three expressions above hold for

$$\lambda \leq \lambda_r, \quad \lambda_r < \lambda \leq \lambda_s, \quad \lambda > \lambda_s. \tag{A.8}$$

The value of the above bound at the critical points λ_r, λ_s is

$$\begin{aligned} \epsilon_{c,C}^{\mathrm{psic}} &= \frac{K\beta^2(\alpha-2)^2}{(\alpha-1)(\alpha-2-2\beta K)^2} \\ \epsilon_{k,C}^{\mathrm{psic}} &= \frac{K(\alpha-2)^2}{(\alpha-1)(\alpha-2-2K)^2}. \end{aligned}$$

Note that the first and third expressions in the Chebychev bound have the form

$$\frac{\sigma^2\lambda}{(y-\mu\lambda)^2}, \tag{A.9}$$

for constants μ and σ^2 independent of λ. Setting this equation equal to ϵ and solving for λ yields, for $y > \mu\lambda$:

$$\lambda = \frac{y}{\mu} + \frac{\sigma^2}{2\mu^2\epsilon}\left(1-\sqrt{1+\frac{4\mu y}{\sigma^2}\epsilon}\right) = \frac{y^2}{\sigma^2}\epsilon + O(\epsilon^2). \tag{A.10}$$

Setting the three expressions in the Chebychev bound equal to ϵ and solving for λ gives:

$$\lambda_{f,C}^{\epsilon,\mathrm{psic}} = \begin{cases} \frac{\alpha-2}{2\beta}\frac{1}{\pi}r_{\mathrm{tx}}^{-2} + \frac{(\alpha-2)^2}{8(\alpha-1)}\frac{1}{\pi}r_{\mathrm{tx}}^{-2}\frac{1}{\epsilon}\left(1-\sqrt{1+\frac{8(\alpha-1)}{(\alpha-2)\beta}\epsilon}\right) \\ \frac{y^{\frac{2}{\alpha}}}{\left(\frac{2}{\alpha-2}K^{1-\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}} + \frac{K^{\frac{1-\alpha}{2}}\pi^{\frac{\alpha}{2}}}{\sqrt{\alpha-1}\sqrt{\epsilon}}\right)^{\frac{2}{\alpha}}} \\ \frac{\alpha-2}{2}\frac{1}{\pi}y^{\frac{2}{\alpha}} + \frac{(\alpha-2)^2}{8(\alpha-1)}\frac{1}{\pi}y^{\frac{2}{\alpha}}\frac{1}{\epsilon}\left(1-\sqrt{1+\frac{8(\alpha-1)}{\alpha-2}\epsilon}\right) \end{cases} \tag{A.11}$$

The condition that $y > \mathbb{E}^0[Y(\lambda, r_s)]$ can be expressed as $\lambda < \lambda_m^{\mathrm{psic}}$ where $\lambda_m^{\mathrm{psic}}$ is the unique density such that $\mathbb{E}^0[Y(\lambda_m^{\mathrm{psic}}, r_s)] = y$. Straightforward algebra yields:

$$\lambda_m^{\mathrm{psic}} = \begin{cases} \frac{\alpha-2}{2\pi}r_{\mathrm{tx}}^{-2}\beta^{-1}, & \alpha-2-2\beta K \le 0 \\ \frac{1}{\pi}\left(\frac{\alpha-2}{2}\right)^{\frac{2}{\alpha}}K^{1-\frac{2}{\alpha}}y^{\frac{2}{\alpha}}, & \text{else} \\ \frac{\alpha-2}{2\pi}r_{\mathrm{tx}}^{-2}\beta^{-\frac{2}{\alpha}}, & \alpha-2-2\beta K r^{\alpha} > 0 \end{cases} \tag{A.12}$$

Note that the Chebychev bound is monotone increasing for $\lambda < \lambda_m^{\mathrm{psic}}$, monotone decreasing for $\lambda > \lambda_m^{\mathrm{psic}}$ and has a singularity at $\lambda_m^{\mathrm{psic}}$. Inverting the bound requires a careful analysis of when each of the three conditions occur:

$$\lambda_r \le \lambda_s \le \lambda_m^{\mathrm{psic}}, \quad \lambda_r \le \lambda_m^{\mathrm{psic}} \le \lambda_s, \quad \lambda_m^{\mathrm{psic}} \le \lambda_r \le \lambda_s. \tag{A.13}$$

Looking at Figure 10, it is apparent that the appropriate expression for the inverse depends on both ϵ's position relative to $\epsilon_{c,C}^{\mathrm{psic}}$ and $\epsilon_{k,C}^{\mathrm{psic}}$ as well as $\lambda_m^{\mathrm{psic}}$'s position relative to λ_r and λ_s. In particular, the three expressions above hold for

(a) $\left(\lambda_r < \lambda_m^{\mathrm{psic}} \text{ and } \epsilon \le \epsilon_{c,C}^{\mathrm{psic}}\right)$ or $\lambda_r > \lambda_m^{\mathrm{psic}}$

(b) $\left(\lambda_s < \lambda_m^{\mathrm{psic}} \text{ and } \epsilon_{c,C}^{\mathrm{psic}} < \epsilon \le \epsilon_{k,C}^{\mathrm{psic}}\right)$ or $\left(\lambda_r \le \lambda_m^{\mathrm{psic}} < \lambda_s \text{ and } \epsilon > \epsilon_{c,C}^{\mathrm{psic}}\right)$

(c) $\lambda_s < \lambda_m^{\mathrm{psic}}$ and $\epsilon > \epsilon_{k,C}^{\mathrm{psic}}$

respectively. ■

It remains to maximize $\lambda_{f,M}^{\epsilon_f,\mathrm{psic}} \wedge \lambda_u^{\epsilon_u,\mathrm{psic}}$ over all (ϵ_u, ϵ_f) such that $\epsilon_u + \epsilon_f = \epsilon$. Note that although the expressions are quite messy the actual algorithm to find the optimal pair is quite simple: find the splitting (ϵ_u, ϵ_f) summing to ϵ that minimizes the distance

$$\left|\lambda_{f,M}^{\epsilon_f,\mathrm{psic}} - \lambda_u^{\epsilon_u,\mathrm{psic}}\right|, \tag{A.14}$$

where the optimal splitting pair is found by finding the intersection of the two functions for the cases when the minimum distance is zero. For those cases where the two functions do not intersect then the optimal pair is trivial: the smaller function gets ϵ and the larger function gets 0. This can be easily done on a computer; we will study this algorithm in the numerical results section. ■

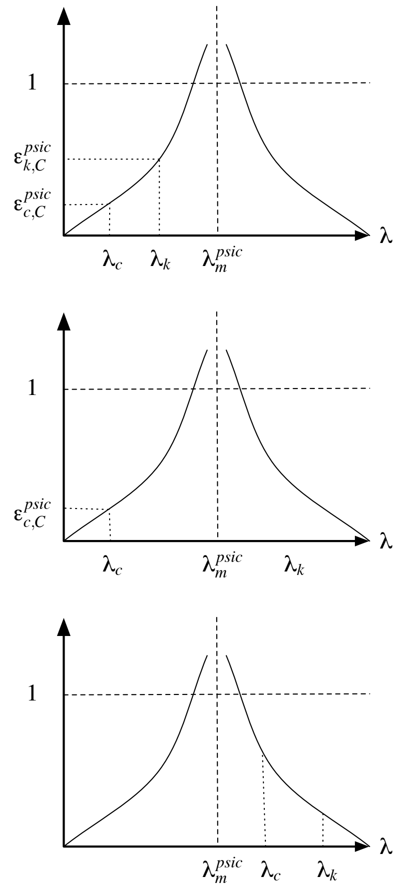

Fig. 10. Three possibilities for the three transmission densities: $\lambda_r, \lambda_s, \lambda_m^{\mathrm{psic}}$. The Chebychev bound is a convex increasing function for $\lambda < \lambda_m^{\mathrm{psic}}$; inversion of the function requires a careful analysis of the cases when each of the three inverse expressions for the bound are appropriate.

APPENDIX B: ADDENDUM TO THEOREM 3

In this appendix, the Chebychev bound for the general case of imperfect interference cancellation is derived.

Addendum to Theorem 3: Let $\epsilon \in (0,1)$ and $y = \frac{r_{\mathrm{tx}}^{-\alpha}}{\beta}$. As $\epsilon \to 0$, the Chebychev lower bound on the optimal contention density when receivers are equipped with imperfect SIC ($\zeta \in (0,1)$) is:

$$\lambda_{l,C}^{\epsilon,\mathrm{sic}} \ge \sup_{(\epsilon_u,\epsilon_f):\epsilon_u+\epsilon_f=\epsilon} \left\{\lambda_u^{\epsilon_u,\mathrm{sic}} \wedge \lambda_{f,C}^{\epsilon_f,\mathrm{sic}}\right\}. \tag{B.1}$$

There are three expressions for $\lambda_{f,C}^{\epsilon,\mathrm{sic}}$, which we label (a), (b), and (c): given by

$$\lambda_{f,C}^{\epsilon,\mathrm{sic}} = \begin{cases} \frac{\alpha-1}{(1-\zeta)\beta^{2(1-\frac{1}{\alpha})}+\zeta}\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\mathrm{tx}}\right)^2} + O(\epsilon^2), & (a) \\ \text{see below}, & (b) \\ (\alpha-1)\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\mathrm{tx}}\right)^2} + O(\epsilon^2), & (c) \end{cases} \tag{B.2}$$

For (b) $\lambda_{f,C}^{\epsilon,\mathrm{sic}}$ is the unique solution to the equation:

$$\frac{\frac{\pi}{\alpha-1}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\alpha} + \zeta y^{2(1-\frac{1}{\alpha})}\right]\lambda}{\left(y - \frac{2\pi}{\alpha-2}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta y^{1-\frac{2}{\alpha}}\right]\lambda\right)^2} = \epsilon. \tag{B.3}$$

These three expressions (a), (b), (c) hold for the regimes

$$\begin{aligned}&(a) && \left(\lambda_r < \lambda_m^{\text{sic}} \text{ and } \epsilon \le \epsilon_{c,C}^{\text{sic}}\right) \text{ or } \lambda_r > \lambda_m^{\text{sic}}\\ &(b) && \left(\lambda_s < \lambda_m^{\text{sic}} \text{ and } \epsilon_{c,C}^{\text{sic}} < \epsilon \le \epsilon_{k,C}^{\text{sic}}\right)\\ & && \text{or } \left(\lambda_r \le \lambda_m^{\text{sic}} < \lambda_s \text{ and } \epsilon > \epsilon_{c,C}^{\text{sic}}\right)\\ &(c) && \lambda_s < \lambda_m^{\text{sic}} \text{ and } \epsilon > \epsilon_{k,C}^{\text{sic}}\end{aligned}$$

respectively. The constants are given by:

$$\lambda_m^{\text{sic}} = \begin{cases} \frac{\alpha-2}{2\pi r_{\text{tx}}^2\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right)}, & \alpha-2-2K\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right) < 0\\ \text{see below}, & \text{else}\\ \frac{\alpha-2}{2\pi r_{\text{tx}}^2\beta^{\frac{2}{\alpha}}}, & \alpha-2-2K > 0 \end{cases} \tag{B.4}$$

where λ_m^{sic} *is given by the unique solution of the equation*

$$\frac{2\pi\lambda}{\alpha-2}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta y^{1-\frac{2}{\alpha}}\right] = y \tag{B.5}$$

when $\alpha-2-2K\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right) \ge 0$ *and* $\alpha-2-2K \le 0$, *and*

$$\lambda_r = \frac{K}{\pi}r_{\text{tx}}^{-2}, \quad \lambda_s = \frac{K}{\pi}y^{\frac{2}{\alpha}} \tag{B.6}$$

and

$$\begin{aligned}\epsilon_{c,C}^{\text{sic}} &= \frac{K(\alpha-2)^2\left((1-\zeta)\beta^2+\zeta\beta^{\frac{2}{\alpha}}\right)}{(\alpha-1)\left((\alpha-2)-2K\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right)\right)^2}\\ \epsilon_{k,C}^{\text{sic}} &= \frac{K(\alpha-2)^2}{(\alpha-1)(\alpha-2-2K)^2}.\end{aligned}$$

Proof of Addendum to Theorem 3: The variance of the far-field interference is again found by Campbell's Theorem: $Var(Y(\lambda,r_s,\zeta))$

$$\begin{aligned}&= \begin{cases} 2\pi\lambda\left[\zeta\int_{r_s}^{r_{\text{tx}}} r^{-2\alpha}rdr + \int_{r_{\text{tx}}}^{\infty} r^{-2\alpha}rdr\right]\\ 2\pi\lambda\left[\zeta\int_{r_s}^{\sqrt{\frac{K}{\pi\lambda}}} r^{-2\alpha}rdr + \int_{\sqrt{\frac{K}{\pi\lambda}}}^{\infty} r^{-2\alpha}rdr\right]\\ 2\pi\lambda\int_{r_s}^{\infty} r^{-2\alpha}rdr \end{cases}\\ &= \begin{cases} \frac{\pi\lambda}{\alpha-1}\left[(1-\zeta)r_{\text{tx}}^{2-2\alpha} + \zeta y^{2(1-\frac{1}{\alpha})}\right]\\ \frac{\pi\lambda}{\alpha-1}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\alpha} + \zeta y^{2(1-\frac{1}{\alpha})}\right]\\ \frac{\pi\lambda}{\alpha-1}y^{2(1-\frac{1}{\alpha})} \end{cases}\end{aligned}$$

where the three expressions hold for the intervals

$$\lambda \le \lambda_r, \quad \lambda_r < \lambda \le \lambda_s, \quad \lambda > \lambda_s \tag{B.7}$$

respectively. The Chebychev bound is: for $y > \mathbb{E}^0[Y(\lambda,r_s,\zeta)]$:

$$\begin{aligned}\mathbb{P}^0(F_f^{\text{sic}}(\lambda)) &\le \frac{Var(Y(\lambda,r_s,\zeta))}{\left(y-\mathbb{E}^0[Y(\lambda,r_s,\zeta)]\right)^2}\\ &= \begin{cases} \frac{\frac{\pi}{\alpha-1}\left[(1-\zeta)r_{\text{tx}}^{2-2\alpha}+\zeta y^{2(1-\frac{1}{\alpha})}\right]\lambda}{\left(y-\frac{2\pi}{\alpha-2}\left[(1-\zeta)r_{\text{tx}}^{2-\alpha}+\zeta y^{1-\frac{2}{\alpha}}\right]\lambda\right)^2}\\ \frac{\frac{\pi}{\alpha-1}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\alpha}+\zeta y^{2(1-\frac{1}{\alpha})}\right]\lambda}{\left(y-\frac{2\pi}{\alpha-2}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}}+\zeta y^{1-\frac{2}{\alpha}}\right]\lambda\right)^2}\\ \frac{\frac{\pi}{\alpha-1}y^{2(1-\frac{1}{\alpha})}\lambda}{\left(y-\frac{2\pi}{\alpha-2}y^{1-\frac{2}{\alpha}}\lambda\right)^2} \end{cases}\end{aligned}$$

where the three expressions hold for the same three intervals as above. The value of the bound at the critical points is

$$\begin{aligned}\epsilon_{c,C}^{\text{sic}} &= \frac{Var(Y(\lambda_r,r_s,\zeta))}{\left(y-\mathbb{E}^0[Y(\lambda_r,r_s,\zeta)]\right)^2}\\ &= \frac{K(\alpha-2)^2\left((1-\zeta)\beta^2+\zeta\beta^{\frac{2}{\alpha}}\right)}{(\alpha-1)\left((\alpha-2)-2K\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right)\right)^2}\\ \epsilon_{k,C}^{\text{sic}} &= \frac{Var(Y(\lambda_s,r_s,\zeta))}{\left(y-\mathbb{E}^0[Y(\lambda_s,r_s,\zeta)]\right)^2}\\ &= \frac{K(\alpha-2)^2}{(\alpha-1)(\alpha-2-2K)^2}.\end{aligned}$$

Setting the three expressions in the Chebychev bound equal to ϵ and solving for λ yields rather unwieldy expressions. The first and third cases are in the form of (A.9); for simplicity we employ the solution given in the right side of (A.10) which is obtained by linearizing in ϵ around $\epsilon = 0$. Applying this result to the first and third cases above we find:

$$\lambda_{f,C}^{\epsilon,\text{sic}} = \begin{cases} \frac{\alpha-1}{(1-\zeta)\beta^{2(1-\frac{1}{\alpha})}+\zeta}\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\text{tx}}\right)^2} + O(\epsilon^2)\\ \text{see below}\\ (\alpha-1)\frac{\epsilon}{\pi\left(\beta^{\frac{1}{\alpha}}r_{\text{tx}}\right)^2} + O(\epsilon^2) \end{cases} \tag{B.8}$$

The second case, as was also true for the Markov bound, cannot be put in closed form. It is given by the unique solution of the equation:

$$\frac{\frac{\pi}{\alpha-1}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\alpha}+\zeta y^{2(1-\frac{1}{\alpha})}\right]\lambda}{\left(y-\frac{2\pi}{\alpha-2}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}}+\zeta y^{1-\frac{2}{\alpha}}\right]\lambda\right)^2} = \epsilon. \tag{B.9}$$

As was discussed in the proof of Theorem 2, the condition that $y > \mathbb{E}^0[Y(\lambda,r_s,\zeta)]$ can be expressed as $\lambda < \lambda_m^{\text{sic}}$ where λ_m^{sic} is the unique density such that $\mathbb{E}^0[Y(\lambda_m^{\text{sic}},r_s,\zeta)] = y$. Straightforward algebra yields:

$$\lambda_m^{\text{sic}} = \begin{cases} \frac{\alpha-2}{2\pi r_{\text{tx}}^2\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right)}, & \alpha-2-2K\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right) < 0\\ \text{see below}, & \text{else}\\ \frac{\alpha-2}{2\pi r_{\text{tx}}^2\beta^{\frac{2}{\alpha}}}, & \alpha-2-2K > 0 \end{cases} \tag{B.10}$$

where λ_m^{sic} is given by the unique solution of the equation

$$\frac{2\pi\lambda}{\alpha-2}\left[(1-\zeta)\left(\frac{K}{\pi\lambda}\right)^{1-\frac{\alpha}{2}} + \zeta y^{1-\frac{2}{\alpha}}\right] = y \tag{B.11}$$

when $\alpha-2-2K\left((1-\zeta)\beta+\zeta\beta^{\frac{2}{\alpha}}\right) \ge 0$ and $\alpha-2-2K \le 0$. Again referring to Figure 10, it is apparent that the appropriate expression for the inverse depends on both ϵ's position relative to $\epsilon_{c,C}^{\text{sic}}$ and $\epsilon_{k,C}^{\text{sic}}$ as well as λ_m^{sic}'s position relative to λ_r and λ_s. In particular, the three expressions above hold for

$$\begin{aligned}&(a) && \left(\lambda_r < \lambda_m^{\text{sic}} \text{ and } \epsilon \le \epsilon_{c,C}^{\text{sic}}\right) \text{ or } \lambda_r > \lambda_m^{\text{sic}}\\ &(b) && \left(\lambda_s < \lambda_m^{\text{sic}} \text{ and } \epsilon_{c,C}^{\text{sic}} < \epsilon \le \epsilon_{k,C}^{\text{sic}}\right) && \text{(B.12)}\\ & && \text{or } \left(\lambda_r \le \lambda_m^{\text{sic}} < \lambda_s \text{ and } \epsilon > \epsilon_{c,C}^{\text{sic}}\right)\\ &(c) && \lambda_s < \lambda_m^{\text{sic}} \text{ and } \epsilon > \epsilon_{k,C}^{\text{sic}} && \text{(B.13)}\end{aligned}$$

respectively. ∎

It remains to maximize $\lambda_{f,C}^{\epsilon_f,\mathrm{sic}} \wedge \lambda_u^{\epsilon_u,\mathrm{sic}}$ over all (ϵ_u, ϵ_f) such that $\epsilon_u + \epsilon_f = \epsilon$. The same comments apply here that were made for selecting the optimal splitting pair for the Markov bound: finding the optimal pair is trivial for a computer, whereas the corresponding expressions for the optimal are both messy and don't necessarily provide any insight.