Review

\[ x_1' + a_1 x_1 + 5 x_2 = c u \]
\[ x_1 = x_1, \quad x_2 = x_1' = \dot{x}_1 \quad \rightarrow \quad \dot{x}_1 = x_2 \]
\[ x_1' + a_1 x_1 + 5 x_2 = c u \quad \rightarrow \quad \dot{x}_2 = -5 x_1 - a x_2 + c u \]
\[ \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \]

State space representation of a linear system:

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*} \]

\[ x \rightarrow \text{vector} \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \rightarrow \text{state variables} \]

For nonlinear systems: \( \dot{x} = f(x, u) \)

Equilibrium point(s) - (only one or many connected in linear systems)

Recall that \( \dot{x} \) is like a velocity, so equilibrium means \( \dot{x} = 0 \) (no velocity)

Other ways of representing linear systems is with a transfer function \( G(s) = \frac{Y(s)}{U(s)} \)

\[ X_S = \Delta X + B u \quad \rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (sI - A) = B u \]
\[ T = C x + D u \quad \rightarrow \quad x = (sI - A)^{-1} B u \]
\[ Y = C (sI - A)^{-1} B u + D u \]
\[ y = C (sI - A)^{-1} B u + D u \]
\[ y(t) = \sum_{j=1}^{n} x_j e^{\lambda_j t} + X (u, t) \]

\[ \lambda_j \rightarrow \text{eigenvalues} \quad \rightarrow \text{obtained from solving} \]

\[ \det(\lambda I - A) = 0 \]

\[ (A - \lambda I)X = 0 \]

Characteristic polynomial: \[ P(\lambda) = \det(\lambda I - A) = 0 \]

Example of eigenvalue use → Stability

\[ \text{Re}(\lambda) < 0 \rightarrow \text{stable} \]

\[ \text{Re}(\lambda) \geq 0 \rightarrow \text{unstable} \]
So let's go back to power electronics as an application. Consider first the buck converter.

Buck converter:
Switched system

\[
\begin{align*}
L\dot{x}_1 &= f(t)E - x_2 \\
C\dot{x}_2 &= x_1 - \frac{x_2}{R}
\end{align*}
\]

\[\dot{x}_{eq, lib} = \frac{E}{R}\]

For \( f(t) = 0 \):
\[x_2 = 0, x_1 = 0\]

For \( f(t) = 1 \):
\[x_2 = E, x_1 = \frac{E}{R}\]
Fast average
\[ \begin{align*}
L \dot{x}_1 &= dE - x_2 \\
C \dot{x}_2 &= x_1 - \frac{x_2}{R}
\end{align*} \]

\[ \Rightarrow \dot{z} = Ax + Bu \]

\[ A = \begin{pmatrix} 0 & 1 \\ \frac{1}{L} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

Based on control input \[ u = \left( \begin{array}{c} d \\ 0 \end{array} \right), \quad \mu = d \]

Based on power input \[ u = \left( \begin{array}{c} d \\ 0 \end{array} \right), \quad \mu = \frac{dE}{C} \]

\[ \dot{x}_1 = \frac{1}{L} \left( dE - x_2 \right) \]
\[ \dot{x}_2 = \frac{1}{C} \left( x_1 - \frac{x_2}{R} \right) \]

Equilibrium point:
\[ x_2 = dE \]
\[ x_{10} = \frac{x_2}{R} = \frac{dE}{R} \]

\[ \text{Since we want } x_2 \text{ constant, then } dH = 0 \]
Regulation vs. Tracking

Follow some trajectory

with respect to an operation point

Linear controller

\( e = V_{\text{ref}} - x_2 \)

\( d = ku \int e \, dt = ku \int (V_{\text{ref}} - x_2) \, dt \)

If \( x_2 < V_{\text{ref}} \rightarrow d \) increases, \( e > 0 \)

If \( x_2 > V_{\text{ref}} \rightarrow d \) decreases, \( e < 0 \)

If \( x_2 = V_{\text{ref}} \rightarrow d \) constant, \( e = 0 \)

Thus \( e = 0 \) and \( d \) constant at the equilibrium point

Some example, \( ku = 0 \)
From
\[ \begin{align*}
Lx_1 &= DE - x_2 \\
C \dot{x}_2 &= x_1 - \frac{x_2}{R} \Rightarrow x_1 &= \frac{x_2}{R} \quad \text{(Line)}
\end{align*} \]

More on feedback
Consider

\[ \begin{align*}
J^* + e &= J - J^* \\
\text{Controller} \quad \text{Plant} \quad \text{Disturbance}
\end{align*} \]

Feedback gain
Closed loop gain $T = \frac{G}{1 + GH}$

Bandwidth $\rightarrow$ The frequency at which the gain is enough to compensate it, where it is the frequency at which $T(s)$ dropped 3dB with respect to its low frequency value.

Consider that open loop I have $G = \frac{A_0}{1 + s/\omega_0}$

Closed loop $\rightarrow$ $T = \frac{G}{1 + GH} = \frac{A_0}{1 + A_0 + s/\omega_0}$

Assume $H=1$ ($\omega_c = \omega_0(1+K)$)

$\omega_c > \omega_0$

When I close the loop, I gain bandwidth but I loose gain.

Stability definitions (linear approach)

Cross over frequency $\omega_c \rightarrow$ 1st frequency at which $\left| G(s)H(s) \right| = 0 \mathrm{dB}$

$\omega_{180} \rightarrow \arg \left( G(s)H(s) \right) = 180^\circ$

$\left| G(j\omega_{180})H(j\omega_{180}) \right| = 1 \rightarrow \mathrm{stable}$

$\left| G(j\omega_{180})H(j\omega_{180}) \right| > 1 \rightarrow \mathrm{unstable}$

$\left| G(j\omega_{180})H(j\omega_{180}) \right| < 1 \rightarrow \mathrm{stable}$
Phase margin → $PM = \arg \left( \frac{G(j\omega_0)}{H(j\omega_0)} \right) + 180^\circ$

Gain margin → Represents how much I can increment the system's gain without losing stability

$GM = -\left| \frac{G(j\omega_0)}{H(j\omega_0)} \right|$

→ If $GM > 0$ → stable

**Linear controllers**

**$P$ control**

$$e = \left[ \begin{array}{c} 0 \end{array} \right]$$

$$\frac{d}{dt} \left[ \begin{array}{c} e \end{array} \right] = \left[ \begin{array}{c} \frac{1}{c_p} \end{array} \right] \left[ \begin{array}{c} s \end{array} \right] \left[ \begin{array}{c} e \end{array} \right]$$

$$C(s) \cdot \frac{1}{s} = A(s)D$$

$$d = k_p e = k_p (y^* - y)$$

Now I close the loop with $H=1$

$$\frac{y}{y^*} = \frac{G}{1+G^H} = \frac{A(s)}{1 + A(s)}$$

Notice that as $k_p \to \infty$ $\frac{y}{y^*} \to 1 \Rightarrow y = y^*$

$H y_{\text{output}}$ equals my desired goal.
So a high kp yields a good tracking (here regulated).

But → If I have noise then

(little noise) x (big kp) = lots of noise

\[ d = kp(y^* - y) \] → Large kp yield d > 1

Since d can only take values between 0 and 1 in reality,
a large kp is ineffective.

Large ds are bad for boost, such boost an other type of converters.

If e = 0 then d = 0 → So when I am meeting my control goal I have no control signal.

Large kp improves convergence to goal but worsens PM.

There are 2 important issues in proportional controls:

\[ y = y^* \text{ only when kp is } \infty \text{, but } \infty \text{ in practice doesn't exist} \]

When e = 0, d = 0

In proportional controllers I end up with a steady state error.
Another problem of P-control is that large k_p's lead to lower

$$P_m$$

Let's assume that

$$\Delta(s) = \frac{1}{(s+2)(s+6)}$$

$$\Delta_H = \frac{k_p}{(s+2)(s+6)}$$

**I-control**

$$e \int_0^t \frac{d}{k_p} \left( \text{plant} \right) \frac{d}{A(s)} \frac{d}{2}$$

$$d = d_0 + \int [e^2] dt = \int [y^2 - y] dt + d_0$$

$$\int \frac{e(s)}{s} \frac{1}{s} = \text{desired output}$$
\[ \frac{y}{y^*} = \frac{A(s) \text{li}(s)}{1 + A(s) \text{li}(s)} = \frac{A(s) \text{li}}{S + A(s) \text{li}}. \]

In the same way that \( T \to \infty \) implies \( u \to 0 \) then \( t \to \infty \) implies \( S \to 0 \). To steady state, \( S \to 0 \).

So with PI control at dc \( \frac{y}{y^*} \big|_{dc} = 1 \)

And as \( t \to \infty \) (regardless of dc or ac signal)

\[ y(t^+) \xrightarrow{t \to \infty} y(t) \text{ so } \lim_{t \to \infty} e = 0. \]

Since \( d = a_0 + ke \int e \, dt \to 0 \) when \( e = 0 \) even if \( d \neq 0 \) because the integral is a "sum" so the result at \( t \to \infty \) when \( e = 0 \) is not necessarily 0 because of all the contributions to the sum from \( t = 0 \) to \( t = \infty \).

\[ \text{If } A(s) = \frac{a_0}{s + p} \text{ then } T = \frac{ke A_0}{s^2 + sp + ke A_0}. \]

\[ \lambda_{1,2} = -p \pm \sqrt{p^2 - 4ke A_0}. \]
Hence, 

1) If $h_i$ is very large we can get complex solutions (in the frequency domain) which yield oscillatory behavior and overshoots in time domain.

2) Provided that $h_i > 0$, if $p > 0$ ( i.e. stable) then the closed loop equilibrium point is also stable.

**PI control**

\[
\begin{align*}
\frac{\Delta y}{\Delta u} &= \frac{1(\Delta u)}{\Delta y} = \frac{1}{\Delta u} \\
\Delta y &= e - \frac{1}{\Delta u} \left( \frac{1}{s^2 + \frac{K_p}{K_i} s + 1} \right) \Delta u
\end{align*}
\]

\[
\frac{\Delta y}{\Delta u} = \frac{s(\Delta u s) + \Delta u s}{s(\Delta u s) + \Delta u s + 1}
\]

So as $t \to \infty$ the \( \frac{\Delta y}{\Delta u} \to 0 \)

PI controllers can achieve faster convergence without having large $h_i$'s that may create overshoots or oscillation, but without ending up with steady state error.

**Boost converter**

\[
\begin{cases}
\dot{x}_1 = e - \frac{v}{V} x_2 \\
\dot{x}_2 = \frac{v}{V} x_1 - \frac{x_2}{R}
\end{cases}
\]

For $g'(4) = 0$

\[
\begin{align*}
\dot{x}_1 &= e \\
\dot{x}_2 &= \frac{x_2}{R}
\end{align*}
\]

By point, \( x_1 = ? \)

\( x_2 = ? \)
If \( \dot{y}(4) = 0 \quad \Rightarrow \quad \dot{y}(4) = 1 \)

\[
L \dot{x}_1 = E \quad \Rightarrow \quad \text{No \ equilibrium}
\]

\[
x_1(4) = \frac{E}{L} + \frac{E}{L} x_1(4)
\]

Linear increase tending to infinity

Now assume that the inductor has some resistance \( r \)

\[
\begin{align*}
L \dot{x}_1 &= r x_1 - i'(t) x_2 + \dot{E} \\
C \dot{x}_2 &= i'(t) x_1 - \frac{x_2}{R}
\end{align*}
\]

For \( \dot{y}(4) = 0 \quad \Rightarrow \quad \dot{y}(4) = 1 \)

\[
\begin{align*}
L \dot{x}_1 &= r x_1 + \dot{E} \\
C \dot{x}_2 &= -\frac{x_2}{R}
\end{align*}
\]

\[
x_{ce} = \frac{E}{r} \quad \Rightarrow \quad x_{ce} = 0
\]

\[
L \dot{x}_1 < 0 \\
x_1 > 0
\]

For \( \dot{y}(4) = 1 \quad \Rightarrow \quad \dot{y}(4) = 0 \)

\[
\begin{align*}
L \dot{x}_1 &= -r x_1 - x_2 + \dot{E} \\
C \dot{x}_2 &= x_1 - \frac{x_2}{R}
\end{align*}
\]

\[
x_{ce} = \frac{x_{ce}}{R} - \frac{E}{R} \\
x_{ce} = \frac{E}{1 + \frac{r}{R}}
\]
(c) \( x_{e_0} = E \) because it is.

Let's attempt a geometric control to regulate output voltage.

Consider \( x(0) = 0 \).

(a) Consider that the first control action is \( g(t) = 0 \).

(b) Now consider that the first control action is \( g(t) = 1 \).

5. trajectories not converging to desired equilibrium.

Desired equilibrium.
Now, let's attend to regulate inductor current

Now, we achieved the desired goal.

Rules for hysteresis control:

1) The switching surfaces (curves or lines) must separate the equilibrium points.

2) The switches must act in opposition to the natural evolution toward an equilibrium point.

3) The switch action must have a direct action on the regulated state variable.

4) The desired operating point must be between switching surfaces.

5) A dead band should be provided to avoid chattering.

**PID cont.**

\[ d = k_c \text{sat}(e) + k_i e \]

→ Provides a similar action but faster than Integral term.
average

\[
\begin{align*}
L \dot{x}_1 & = r x_1 - d' x_2 + \varepsilon \\
L \dot{x}_2 & = d' x_1 - \frac{x_2}{r}
\end{align*}
\]

Suppose \( d' = 1 \)

\[
\begin{align*}
x_1 & = \frac{1}{r} \left( e - D' x_2 \right) \\
x_1 & = \frac{1}{D'} \frac{x_2}{R} \quad \Rightarrow \quad D' = \frac{x_2}{x_1 R} \\
x_2 & = \sqrt{-x_1^2 R^2 + e R x_1}
\end{align*}
\]

\[
\begin{align*}
\frac{e}{2} & \leq \frac{R}{R} \\
D' & \leq \sqrt{\frac{r}{R}} \\
x_1 & \rightarrow \\
x_2 & \rightarrow
\end{align*}
\]

2 eq. points

**PI controller for boost converter**

\[
\begin{align*}
L \dot{x}_1 & = r x_1 - d' x_2 + \varepsilon \\
L \dot{x}_2 & = d' x_1 - \frac{x_2}{r}
\end{align*}
\]
PT controller \[ d = k_i \int e dt + k_p e \]

\[ k_i \text{ and } k_p \text{ have to be small (slow controller) or } d \text{ should be limited to a max value of } \approx 0.05 - 0.9 \]

Otherwise the trajectory goes to the equilibrium point with highly unstable oscillations.

Next: A more challenging case: constant power load