Power electronic circuits modeling

- **Switched model**
  Switching function $g(t)$:

  
  ![Switching function diagram](image)

  $g(t+1) = 0 \Rightarrow$ Switch is commanded to be off.

  $\log g(t+1) = 1 \Rightarrow$ Switch is commanded to be on.

- Consider a buck converter operating in continuous conduction mode (CCM) \( \Rightarrow \) i.e., \( i(t+1) > 0 \) \( \forall t \geq 0 \)

  ![Buck converter diagram](image)

  - When $g(t) = 1$
    
    
    
    \[
    \dot{x}_1 = E - x_2 \\
    C \dot{x}_2 = x_1 - x_2/R
    \]

  - When $g(t) = 0$
    
    
    
    \[
    \dot{x}_1 = -x_2 \\
    C \dot{x}_2 = x_1 - x_2/R
    \]
Hence

\[ \begin{align*}
    L \dot{x}_1 &= f(t) \cdot e - x_2 \\
    c \dot{x}_2 &= x_1 - \frac{x_2}{R}
\end{align*} \]

(a) Switched system
dynamic eqs.

Switching Function

Note that \( f(t) \) is non-linear. So power electronics circuits are non-linear circuits. Because of \( f(t) \) in (a) it cannot use Fourier, Laplace or identify impedances.

Steady state is a succession of transient states

That is \( x_1(t_0) \neq x_1(t_1) \) and \( x_1(t_1) \neq x_1(t_2) \) but \( x_1(t_0) = x_1(t_2) \)

Steady state

Equilibrium points \( \rightarrow \) are those points where \( \dot{x}_1 = 0 \) and \( \dot{x}_2 = 0 \) (e.g. "velocity" \( x_2 \) is zero)
For $g_{d}(t) = 1$ →
\[
\begin{align*}
\dot{O} &= E - x_2 \\ \dot{x}_{201} &= E \\
0 &= x_1 - x_2/R \quad \rightarrow \quad x_{101} &= E/R
\end{align*}
\]

For $\dot{g}(t) = 0$ →
\[
\begin{align*}
O &= -x_2 \\ \dot{x}_{202} &= 0 \\
0 &= x_1 - x_2/R \quad \rightarrow \quad x_{202} &= 0
\end{align*}
\]

In matrix form $(1)$ and $(2)$ can be written as:
\[
\begin{align*}
L \ddot{x}_1 &= f(t) E - x_2 \\
C \ddot{x}_2 &= x_1 - \frac{x_2}{R}
\end{align*}
\]
\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} &= A \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + B \begin{bmatrix}
f(t) E \\
0
\end{bmatrix}
\end{align*}
\]

Circuit structure:
\[
B = \begin{bmatrix}
f(t) E \\
0
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 \\
-1/R & 0
\end{bmatrix}
\]

Based on control input $\rightarrow$ 
\[
B = \begin{bmatrix}
f(t) E \\
0
\end{bmatrix}, \quad u = f(t)
\]

Based on power input $\rightarrow$ 
\[
B = \begin{bmatrix}
f(t) E \\
0
\end{bmatrix}, \quad u = E
\]

For a boost converter:

\[
\begin{align*}
\dot{g}(t) &= 1 - g(t) \\
L \ddot{x}_1 &= E \\
C \ddot{x}_2 &= -\frac{x_2}{R} \\
\end{align*}
\]

\[
\begin{align*}
0 &= x_1 - x_2/R \\
\end{align*}
\]

\[
\begin{align*}
\dot{g}(t) &= 1 \\
L \ddot{x}_1 &= E - \frac{g(t)}{R} x_2 \\
C \ddot{x}_2 &= \frac{g(t)}{R} x_1 - \frac{x_2}{R} \\
\end{align*}
\]

Switched model
Equilibrium points:

\[ f(x_1) = 1 \rightleftharpoons \begin{cases} 0 = \frac{E}{x} & \rightarrow \quad \text{There is no equilibrium} \\ 0 = -\frac{x_2}{R} \end{cases} \]

If I leave the switch closed \( x_1 \to \infty \):

\[ f(x_1) = \infty \rightleftharpoons \begin{cases} 0 = E - x_2 & \rightarrow \quad x_{2_{eq}} = \frac{E}{R} \\ 0 = x_1 - \frac{x_2}{R} \end{cases} \]

- Buck-boost converter

\[ \begin{array}{c}
\text{Buck} \\
\text{Boost} \\
\end{array} \]

\[ \text{opposite effects} \quad \text{redundant switches} \]
\[
\begin{align*}
\dot{X}_1 &= \frac{1}{2}(\dot{V}_1 - V_0) X_2 \\
\dot{X}_2 &= \frac{1}{2}(\dot{V}_1 - V_0) X_1 - \frac{X_2}{R}
\end{align*}
\]

**Equilibrium points:**

\[
\begin{align*}
\dot{V}_1 + 1 &= 0 \quad \Rightarrow \quad 0 &= \dot{V}_1 (\text{?}) \\
X_{2,01} &= 0 \\
\dot{V}_1 + 1 &= 0 \quad \Rightarrow \quad 0 &= -X_2 \quad \Rightarrow X_{2,02} = 0 \\
0 &= X_1 - \frac{X_2}{R} \quad \Rightarrow X_{1,02} = \infty
\end{align*}
\]

*Fly back converter*

From the buck-boost converter let’s split the inductor in two coupled inductors

\[
E \quad \Rightarrow \quad \text{Not a transformer} \\
\text{they are 2 coupled inductors}
\]

\[
\frac{d\phi}{dt} = \frac{V_1}{N_1} = \frac{V_2}{N_2}
\]

\[
\phi = A_L (i_1 N_1 + i_2 N_2) \quad \text{general form}
\]

\[
\text{Flux linkage} \quad \Rightarrow \quad A_L = \frac{\phi}{L} \quad \text{Reduction} \\
\phi \phi = L
\]

\[
L = \text{Ohm's law for a magnetic circuit}
\]
\[ \frac{d\phi}{dt} = \frac{\xi E}{N_1} - \frac{\xi' V_c}{N_2} \]

\[ \frac{dV_c}{dt} = \frac{\xi' \phi}{L_N} - \frac{V_c}{R} \]

\[ i_2 = 0 \text{ when } i_1 \rightarrow \infty \]

This is why it can't be considered a transformer. Otherwise the general form should be valid.

\[ \Phi = A_L i_2 N_2 \rightarrow i_2 = \frac{\phi}{A_L N_2} \]

---

**Fast average model**

**Fast average operator**

\[ \bar{\phi}(t) = \frac{1}{T_{sw}} \int_{T_{sw}}^{t} \phi(t') dt' \]

An operator is a machine, what kind of machine is this?

If I apply a Laplace transform on both sides, I obtain that \( F(s) \) is proportional to \( \frac{E}{s} \)
Since \( \frac{T}{5} \) is indicative of a low-pass filter, the fast average operator acts as a low-pass filter.

So when I apply the fast average operator to the switching function \( f(t) \), I obtain the instantaneous duty cycle \( \overline{f}(t) \):

\[
\overline{f}(t) = \frac{1}{T_{SW}} \int_{t}^{t+T_{SW}} f(\tau) \, d\tau
\]

From (1)

\[
\begin{align*}
L \dot{x}_1 &= g(t) \xi - x_2 \\
C \dot{x}_2 &= x_1 - \frac{x_2}{12}
\end{align*}
\]

Fast average operator

\[
\begin{align*}
L \overline{x}_1 &= \overline{f}(t) \xi - \overline{x}_2 \\
C \overline{x}_2 &= \overline{x}_1 - \frac{\overline{x}_2}{12}
\end{align*}
\] (2)

---

**time domain**

Blue → Switched model  
Red → Fast average model

**state space** (phase portrait)
Simulations performed with simulink with a buck converter with $E = 48V$, $R = 0.50$, $C = 500\mu F$, $L = 100\mu F$.

Note that in order to realize the switching function we sample an instantaneous duty cycle signal with linear transitions that do not add distortion

Like earlier I can represent the fast average model in a matrix form.

$$L \dot{x}_1 = \bar{d}(t)E - x_2$$

$$C \dot{x}_2 = \bar{x}_1 - \frac{x_2}{R}$$

If $\bar{d}(t)$ is constant an equal to $D$, then

$$L \dot{x}_1 = DE - x_2$$

$$C \dot{x}_2 = \bar{x}_1 - \frac{x_2}{R}$$

Eq. point $x_0 = \left( \begin{array}{c} \frac{DE}{R} \\ V_0 \end{array} \right)$.
Limit cycle

(ii) does not lead to an equilibrium point

Equilibrium point only achieved in a weighted average sense

\[ \bar{\mathbf{x}}_e = \left( \frac{D E/R}{D E} \right) \mathbf{x}_{e_0} + (1-D) \mathbf{x}_{e_2} \]

What if we are not in con and \( i_{\text{SO}} \) for part of the period (we are in discontinuous conduction mode - DCM)

When \( Q_1 = \text{ON} \),

\[ V_c = V_{in} - V_{out} = L \frac{d i_c}{dt} = L \frac{d i_{\text{peak}}}{dt} \]

Then \( i_{\text{peak}} = \frac{D_i T}{L} (E - V_{out}) \) \( (**) \)

Also

\[ P_{in} = \frac{1}{T} \int_0^T V_{in} i_{in} dt = \]

\[ = \frac{E}{T} \int_0^T i_{in} dt \]

Now,

\[ \langle i_{in} \rangle \]

\[ \langle i_{in} \rangle = \frac{1}{T} D_i T i_{\text{peak}} = \frac{D_i i_{\text{peak}}}{2} \]

\[ = \frac{1}{2} \text{Area of Triangle} \]
And, since \( P_{in} = P_{out} \rightarrow E\langle \dot{i}_{in} \rangle = \frac{V_{out}^2}{R} \)

\[
\frac{D^2 T}{Dx^2} (E - V_{out}) \dot{V}_{in} = \frac{V_{out}^2}{R}
\]

\[
V_{out} = -\frac{D^2 E RT}{4L} + \sqrt{\frac{R_T}{2L} + \frac{E^2 T^2 D^2}{16L^2}}
\]

A complete average model for a buck converter is:

\[
L \dot{\bar{x}}_1 = \frac{dE}{dt} - 2 \bar{x}_1 \bar{x}_2 \frac{d}{dt} (E - \bar{x}_1)
\]

\[
C \dot{\bar{x}}_2 = \bar{x}_1 - \frac{\bar{x}_2}{R}
\]

For the boost converter:

\[
\begin{cases}
L \dot{\bar{x}}_2 = E - g'(\bar{x}_1) \bar{x}_2 \\
C \dot{\bar{x}}_2 = g''(\bar{x}_1) \bar{x}_1 - \frac{\bar{x}_2}{R}
\end{cases}
\]

But I cannot replace \( g'(t) \) by \( \dot{g}(t) \) as I did with the buck converter without some clarification:

Fast average issue:

\[
\frac{1}{T} \int_{-T/2}^{T/2} \left( \int_{t-T}^{t} x_1 \, dt \right) \, dt = \frac{1}{T} \int_{-T/2}^{T/2} \left( \int_{t-T}^{t} dt \right) \, dt x_1(t)
\]

That is, the fast average operator applied to \( g'(t) \) is not necessarily \( \dot{g}(t) \).
Since in the switched model the state variables follow linear transitions consider that

\[ x_i = a_i t \]

Then

\[ x_i g(t) = \frac{1}{T_{sw}} \int_{t_i}^{t + T_{sw}} A t + \frac{f(t) dt}{2} D \left( 2t + D T_{sw} \right) \]

This is the only difference.

For \( T_{sw} \) very small there is no problem.

So, for high switching frequency \( (f_{sw} = \frac{1}{T_{sw}}) \)

\[
\begin{align*}
    \dot{\bar{x}}_i &= \bar{e} - \bar{A} \bar{x} \\
    \dot{\bar{e}} &= \bar{A} \bar{x}, \quad t \in [0, T] \\
    \dot{\bar{x}}_i &= \bar{A} \bar{x}, \quad t \in [T, T_{sw}] \\
    \bar{e} &= \bar{x}_i - \bar{x}_2 \\
    \bar{A} &= \frac{1}{T_{sw}}
\end{align*}
\]

But how "high" is a "high" switching frequency?

Consider a switched linear system

\[
\dot{x} = \begin{cases} 
    A_1 g(t) + A_2 \left( 1 - g(t) \right) x, & t \in [0, T] \\
    f_1(t) A_1 x, & t \in [T, T_{sw}] \\
    f_2(t) A_2 x, & t \in [T_{sw}, t_i]
\end{cases}
\]
The exact solutions for \( x(t) \) and \( f_x(t) \) are

\[
x(t) = e^{A_2(t-t_0)} x(t_0)
\]

\[
x(t) = e^{A_2(T-t_0)} x(t_0) = e^{A_2(T-dT)} x(t_0)
\]

which is calculated as \( e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \)

Since \( x(t) \) is continuous at \( t_0 \) (\( x(t_0^-) = x(t_0^+) \)) then

\[
x(t) = e^{A_2 (t-dT)} x(t_0)
\]

\[
x(t_0) = e^{A_1 t_0} x(t_0)
\]

Now let's call \( A_2 (1-d)T = A_{21} \) (it's a matrix) and

\[
A_1 dT = A_{01} \quad \text{(another matrix)}
\]

So, \( x(t) = e^{A_{21}} e^{A_{01}} \) \((3)\)

Before continuing, let's see some useful properties of the function of the exponential of a matrix:

\[
e^0 = I
\]

\[
e^{a+b} = e^a e^b
\]

\[
e^b e^{-b} = I
\]

If \( B \) is invertible then \( e^{B^{-1}} = \beta e^b \beta^{-1} \)
\[ \text{det}(e^B) = e^{\text{Tr}(B)} \]

L: Invariance of \( B \to x(B) = \sum_{j=1}^{n} a_{jj} \)

\[ e^{B^T} = (e^B)^T \]

If \( A \) and \( B \) commute (i.e., \( AB = BA \)) then \( e^{A+B} = e^A e^B \)

If \( A \) and \( B \) do not commute we can use Baker-Campbell-Hausdorff formula

\[ \exp \left( \sum_{n=1}^{\infty} \frac{1}{n!} [X, [X, \ldots [X, Y] \ldots ]] \right) = \exp(X) \exp(Y) \]

\[ \text{commutator} \left[ [X, Y] = X Y - Y X \right] \]

So if they commute \( [X, Y] = 0 \)

and \( \exp(X) \exp(Y) = \exp(X + Y) \)

So let's go back to (3) \( \chi(T) = e^{B_2} e^{B_1} \)

Since \( A_1 \) and \( A_2 \) do not necessarily commute then from (2)

\[ A_0 = A T = A_1 + A_2 + \frac{1}{2} \left[ A_1 , A_2 \right] = \]

\[ d' = 1-d = (d' A_1 + d' A_2) T + d A_1 (A_1 A_2 - A_2 A_1) T^2 + \ldots \]

Now, if \( T \) is small (i.e., for large) then \( T^2 \ll CT \) and

\[ A T = (d' A_1 + d' A_2) T \]

and \( \chi(T) = e^{AT} \chi_0 = e^{(d A_1 + d' A_2) T} \chi_0 \) \( (4) \)

solution \( \dot{\chi} = (d A_1 + d' A_2) \chi \)

\[ \chi(T) = \frac{1}{T^2} \left[ \chi(T) \right] \]

weighted average of \( \sum \left\{ \dot{\chi} = A_1 \chi \right\} \)

So the fast average model is a good approximation for the switched model if \( T \) is small (or \( f \) is large) so the following approximation is valid.
So let's go back to the buck converter for a quick example.

**Switched System**

\[
\begin{align*}
L\dot{x}_1 &= f(I^{\dagger})x_2 - x_2, \\
C\dot{x}_2 &= x_1 - \frac{x_2}{R} 
\end{align*}
\]

*For \( f(I^{\dagger}) = 0 \):

\[x_2 = 0, \quad x_1 = 0\]

*For \( f(I^{\dagger}) = 1 \):

\[x_2 = E, \quad x_1 = \frac{E}{R}\]

\[T_{sw} = 2 \times 10^{-2}\]

\[T = 2 \times 10^{-4}\]

\[T = 5 \times 10^{-4}\]
Small signal model

Let's consider once again the buck converter.

From (2)

\[
\begin{align*}
\frac{\dot{x}_1}{L} &= \dot{u} \left( u - \overline{x}_2 ight) \\
C \frac{\dot{x}_2}{R} &= \overline{x}_1 - \overline{x}_2
\end{align*}
\]
Consider the linear operator $\Delta$ that is defined as

$$\Delta(f) = f - f_0$$

so it just calculates the difference with respect to a point $f_0$.

So, $\delta_x = \Delta(x_1) = x_1 - x_1^0 \rightarrow (\delta x_1) = \dot{x}_1$. Coordinate in $x_1$ of the equilibrium point.

Let $\gamma_1 = x_1$, $\gamma_2 = x_2$ then

$$L \delta x_1 = E \delta \gamma_1 - \delta x_1$$

In the same way $L \delta x_2 = \delta x_2 - \delta x_2^R$.

Thus,

$$L \delta x_1 = E \delta \gamma_1 - \delta x_1$$

$$L \delta x_2 = \delta x_2 - \delta x_2^R$$

→ Model valid in a small neighborhood around the equilibrium point $x_0$.

If the input voltage $E$ is allowed to vary then:

$$L \delta x_1 = E \delta \gamma_1 + \delta E \delta \gamma_0 - \delta x_1$$

$$L \delta x_2 = \delta x_2 - \delta x_2^R$$

→ The product of two variables $d$ and $e$ becomes two terms.

So I can now do standard linear analysis. For example, I can calculate the transfer functions from
\[
\begin{pmatrix}
\delta x_1 \\
\delta x_2 \\
\delta x_3 \\
\delta x_4 \\
\delta x_5 \\
\delta x_6
\end{pmatrix} =
\begin{pmatrix}
0 & 1/C & 0 & -1/L & 0 & 0 \\
1/C & 0 & -1/RC & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\delta e \\
\delta d \\
\delta e \\
\delta d \\
\delta e \\
\delta d
\end{pmatrix}
\]

Note that \( \delta x_2 \) is controlled through \( \delta x_1 \)

\[
\delta y = \delta x_2
\]

Using Laplace:

\[
L \left( \delta x_4 \right) = S \Delta x_1 (s) - \ldots
\]

\[
\begin{cases}
L S \Delta x_1 (s) = - \Delta x_2 (s) + \delta D (s) + \delta e (s) \\
L S \Delta x_2 (s) = \Delta x_1 (s) - \frac{\Delta x_2 (s)}{R}
\end{cases}
\]

\[
G(s) = \frac{\Delta x_2 (s)}{\Delta D (s)} \rightarrow \text{output} \div \text{control input}
\]

Transfer function

\[
L S \Delta x_1 (s) = L S \left( L S \Delta x_2 (s) + \Delta x_2 (s) \right) = - \Delta x_2 (s) + \delta D (s) + \delta e (s)
\]

\[
\Delta x_2 (s) \left( L C S^2 + L S + R \right) = \delta D (s)
\]

When calculating the transfer function with respect to the control input, it is considered that the power input is fixed, \( \delta e (s) = 0 \)

For the boost converter:
Linearization:

$$\Delta f = \frac{\partial f}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial x_2} \Delta x_2$$

If $f = \frac{\partial}{\partial x_2}$

Then $\delta f = x_{i_0} \delta x_1 + D_0 \delta x_2$

Thus,

$$\dot{x}_1 = x_{i_0} \delta x_1 - (1-D_0) \delta x_2$$

$$\dot{x}_2 = -x_{i_0} \delta x_1 + (1-D_0) \delta x_2 - \frac{\delta x_2}{R}$$

Equivalent circuits based on the fast energy well

Buck converter

\[
\begin{align*}
\dot{x}_1 &= T(t) e - x_1 \\
C \dot{x}_2 &= x_1 - \frac{x_2}{R}
\end{align*}
\]

![Buck converter diagram]
Boost converter

\[
\begin{align*}
L \frac{\dot{x}_1}{x_1} &= e^{\dot{d}'t}x_0 \\
C \frac{\dot{x}_2}{x_2} &= \frac{\dot{d}'}{N_2} x_1 - \frac{x_2^2}{\sqrt{R}} \\
(1-d) \cdot 1
\end{align*}
\]

Fly Back

\[
\begin{align*}
\dot{\phi} &= \frac{J E}{N_1} - \frac{J'}{N_2} x_2 \\
C \frac{\dot{x}_4}{x_4} &= \frac{\dot{d}'}{L_2} x_0 - \frac{x_2^2}{\sqrt{R}} \\
\frac{\Delta \phi}{\Delta c} N_1 &= \frac{\phi N_2}{L_2} = \frac{N_1}{N_2} \frac{\dot{x}_4}{\phi}
\end{align*}
\]

Some good papers for reference:

**Small-Signal Modeling of Pulse-Width Modulated Switched-Mode Power Converters**

R. D. MIDDLEBROOK, FELLOW, IEEE

**On the Use of Averaging for the Analysis of Power Electronic Systems**

PHILIP T. KREIN, MEMBER, IEEE, JOSEPH BENTSMAN, MEMBER, IEEE, RICHARD M. BASS, STUDENT MEMBER, IEEE, AND BERNARD L. LESEJUETRE
LARGE-SIGNAL DESIGN ALTERNATIVES FOR SWITCHING POWER CONVERTER CONTROL

Richard M. Bass¹ and Philip T. Krein²

Modeling of PWM Converters in Discontinuous Conduction Mode - A Reexamination

Jian Sun, Daniel M. Mitchell Matthew F. Greuel, Philip T. Krein and Richard M. Bass

GENERATION, CLASSIFICATION AND ANALYSIS OF SWITCHED-MODE DC-TO-DC CONVERTERS BY THE USE OF CONVERTER CELLS

Richard Tymerski and Vatche Vorperian