Brief Announcement: On the Expected Overpayment of VCG Mechanisms in Large Networks

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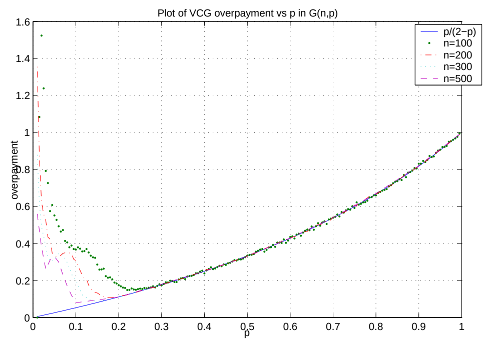
The VCG mechanism for buying a path from s to t in a network with selfish edges, selects the lowest-cost path (LCP) and pays each edge e on the path the edge's cost plus a bonus equal to the increase in cost of the lowest-cost path from s to t if edge e were omitted from the graph G, that is $payment(e, s, t) = cost(e) + LCP(G - e, s, t) - LCP(G, s, t)$. These payments are unacceptably high in some graphs containing two node-disjoint paths, and there are few positive results which prove small upper bounds on the expected payments in large and more realistic networks [4, 1].

In this paper, we give an improvement on the VCG overpayment bound of Mihail, Papadimitriou and Saberi [4] for random graphs. As a corollary, we show that the overpayment is not necessarily correlated with the competitiveness of a graph (measured, for example, by the number of node-disjoint paths between a source-destination pair [1]), and it is not necessarily correlated with the size of the graph, contrary to the conjecture in Mihail *et al.* [4] that the overpayment in random graphs $G_{n,p}$ is $\Theta(1/np)$. Our results show that the overpayment is low when there are two almost equally short paths between the source and destination; the presence of more edges or more competition does not necessarily reduce it and may even hurt it.

THEOREM 1. *For $G \in G_{n,p}$ (the Erdős-Renyi random graph model with n nodes and edge probability p) with unit edge costs and $np = \omega(\sqrt{n \log n})$, with probability $\Omega(1 - n^{-c})$ for some constant $c > 0$, the average VCG overpayment is $\frac{p}{2-p}$.*

The theorem shows that for a constant p the overpayment is $\Theta(1)$ rather than the previously conjectured $\Theta(1/np)$ [4]. Further, it is less than one and surprisingly increases with p, *i.e.,* it increases with the presence of more competition in the graph, from a threshold on. This counterintuitive phenomenon is explained by the fact that with the increase of competitiveness and number of edges in the graph, the lowest cost paths become shorter (and cheaper) so that the overpayment increases. To an extent, this unexpected increase of overpayment is an artifact of its definition as a measure of payment or bonus per unit cost. Despite this, the measure of payment per unit cost is still sensible in network design, where one would try to design a network topology with as little cost as possible, and aim to minimize payments per unit cost.

In addition, we can consider the average payments per path rather than per unit cost, by taking the ratio of total payment to the total number of node pairs, or number of paths. Similarly we can define average bonus per path, and average cost per path (the average distance in a graph with unit costs). The proof of the theorem extends



to show that the average path bonus is p, the average path payment is 2 and the average path cost is $2-p$, so again these do not change as n grows large. The path payment remains constant since for $p > \omega(\sqrt{\log n/n})$, with high probability the second cheapest path is of cost 2.

These theoretical results are supported by simulation; Figure shows the overpayment in random graphs with $n = 100, 200, 300$ and 500 nodes and varying p on the horizontal axis. The minimum overpayment occurs at about $p = \sqrt{\log n/n}$, which suggests a simple randomized algorithm for minimizing overpayment: if p is below or above this threshold value (equivalently if the average degree is below/above $\sqrt{n \log n}$), respectively add or remove edges at random until the threshold is reached.

Simulation on power-law graphs shows that the VCG overpayment steadily decreases from 0.49 down to 0.43 when the number of nodes in the graph grows from $n = 100$ to $n = 20,000$. This supports the conjecture of Mihail *et al.* [4] that the overpayment in power-law graphs is also bounded by constant. From simulation, the overpayment in power-law graphs with *uniformly random costs* also appears to be below 1.

Our theoretical results explain why the average overpayment is not necessarily correlated with the degree of competition or the number of distinct node disjoint paths in a graph. The overpayment is low when the difference between the cost of the two cheapest paths is low, and the presence of more paths does not necessarily affect it, and might even worsen it. Similarly, the property which makes the overpayment on the Internet AS level graph so low [2] is not the large number of available paths, but rather then presence of two paths of almost equal cost between every pair of nodes.

For proofs and details, see Karger and Nikolova [3].

1. REFERENCES

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