# Floating Point Arithmetic

## (The IEEE Standard)

# *Floating Point Arithmetic (and The IEEE Standard)*

* Floating Point Arithmetic

  - Representations
  - Issues
  - Normalized, Unnormalized, Subnormal
  - Precision
  - Wobble

* The IEEE Standard

  - Why
  - What it contains, what it doesn't contain
  - Formats
  - Rounding
  - Operations
  - Infinities, NANs
  - Exceptions
  - Traps

## Several Issues Come Up:

* How many bits for range, how many bits for precision?
* What to do with numbers too small to represent with this scheme?
* What to do with numbers that do not correspond to exact representations?
* What to do with numbers too large to be represented?
* Shall we distinguish numbers too large with true infinities?
* What about nonsense numbers?

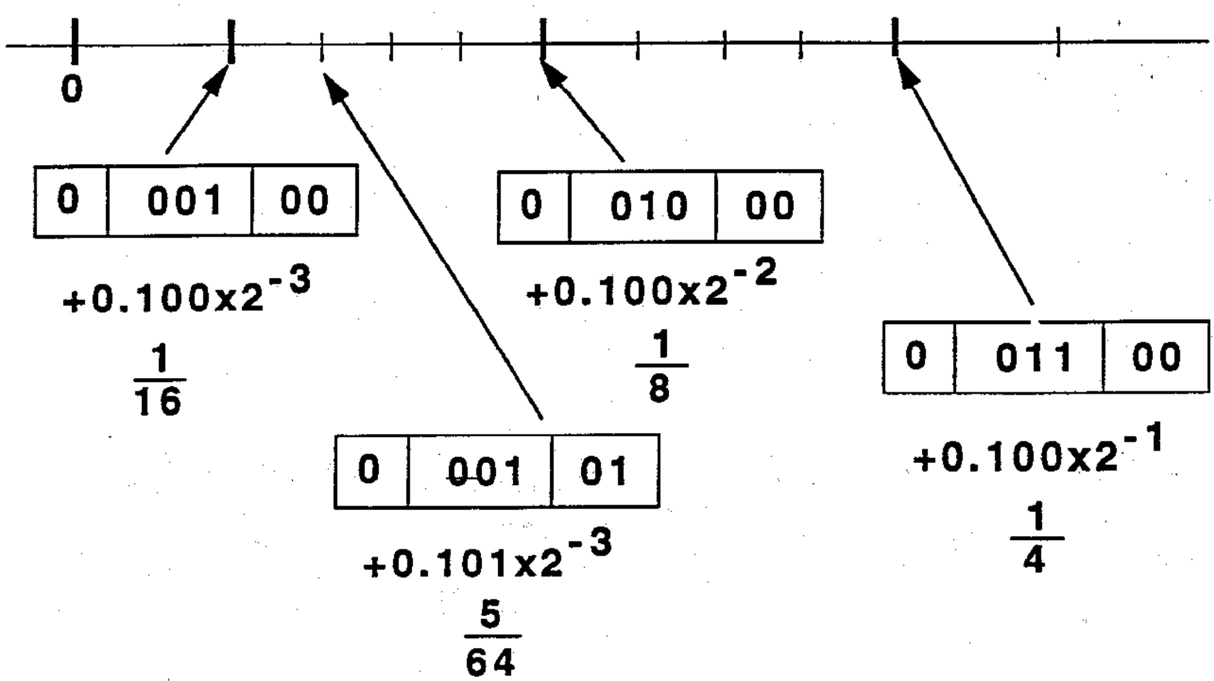
  (Examples:

  Arcsin 2, $\frac{0}{0}$, $\infty - \infty$)

# *First, An Example*

**We Simplify:** | S | EXP | FRA |

**In DEC format:** $(-1)^S * 0.1\,\underline{fra} * 2^{EXP-4}$

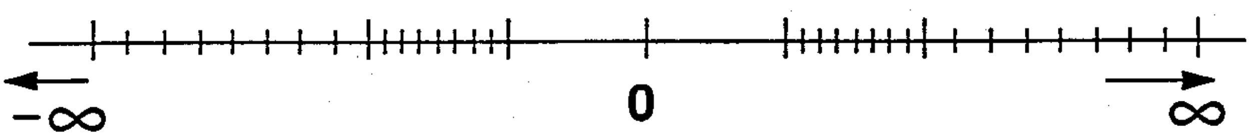


## *First, Some General Stuff:*

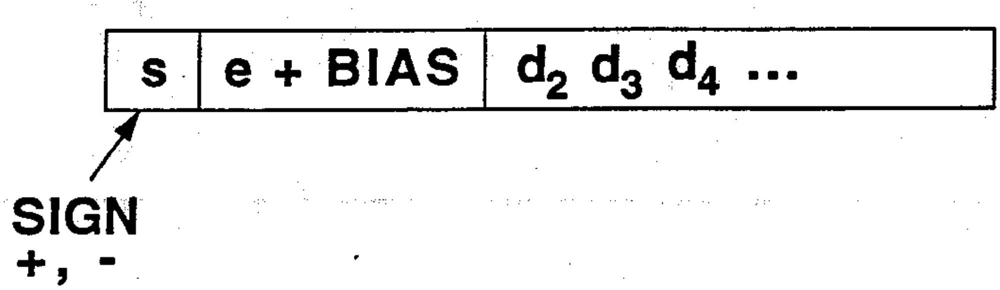
**A number can be represented as**

$$\pm\, d_0 \,.\, d_1 d_2 \ldots \beta^e$$

**These numbers correspond to points on the real line. If we insist that all representations be normalized, then the points are shown (normalized can mean: $d_0 = 0$, $d_1 = 1$)**




**(We can, incidentally, store the number in signed-magnitude format:)**



## *Normalized, Unnormalized, Subnormal*

Again, we are looking at $\pm\, d_0.d_1 d_2 \ldots * \beta^e$

1. If it is <u>normalized</u>, it is:

$$\pm\, 0.1\ d_2\, d_3\ \cdots * \beta^e$$

2. <u>Unnormalized</u> (after a subtract of like signs, for example)

$$\pm\, 0.0001\ d_2\, d_3\ \cdots * \beta^{e+3}$$

3. <u>Subnormal</u> means it can't be represented in the machine in normalized format

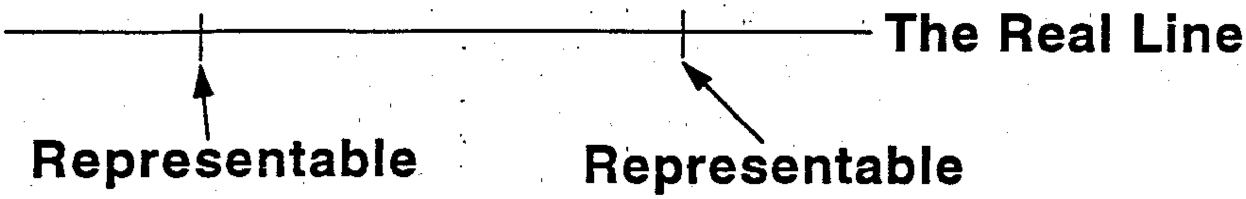
   - Recall the format

| | | |
|---|---|---|
| $\pm$ | e+BIAS | $d_2\ d_3\ \ldots$ |

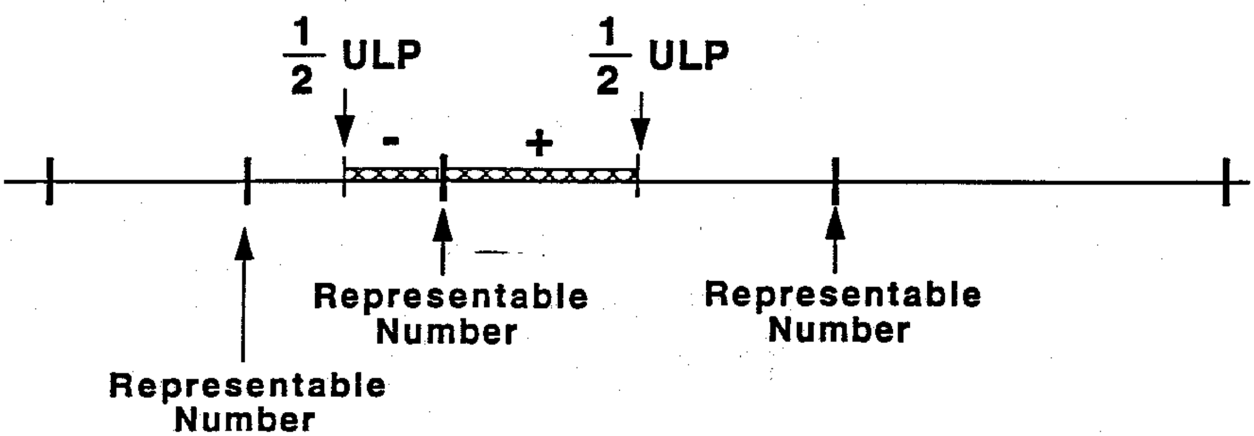
   Corresponds to $\pm\ 0.1\ d_2 d_3 \ldots * \beta^e$

   - Suppose we successively divide by $\beta$. We can do this <u>until</u> e+BIAS = 1. Below that we can't represent numbers (except 0). Why? Suppose we let e+BIAS = 0. How do we now represent 0?

# *Precision*




- ✱ Uncertainty is at Most: $\frac{1}{2}$ ULP
- ✱ Precision deals with worst unavoidable error
- ✱ Precision is a function of representation
  Accuracy is a function of your algorithm
- ✱ Relative uncertainty (the issue of wobble)




**One ULP just above a power of β is β times as large as one ULP just below.**

# *The IEEE Standard*

## Reasons:

1. Direct Support for:

   - Execution-time diagnosis of anomalies
   - Smoother handling of exceptions
   - Interval arithmetic at reasonable cost

2. Provide for development of:

   - Standard elementary functions
   - Very high precision arithmetic
   - Coupling of numeric & symbolic computation

# *The IEEE Standard (Continued)*

## What does it contain:

- Formats: single, double, extended
- Operations: +,-,*, ÷ ,√‾, REM,CMP
- Rounding modes
- Conversion: Int/Fl., Dec/Fl., Fl/Fl
- Exceptions: Underflow, Overflow, Div Ø, Inexact, Invalid

## What it does not contain:

- Requirements for implementation in HDWR or SFWR
- Interpretation of NaNs
- Formats for Integers, BCD
- Conversions other than above

## *The Formats*

**There are four; we start with one as an example.**

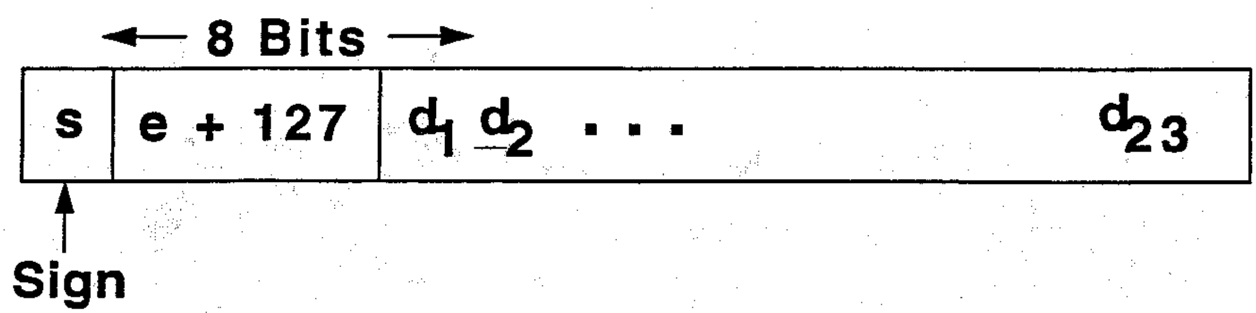
## Single

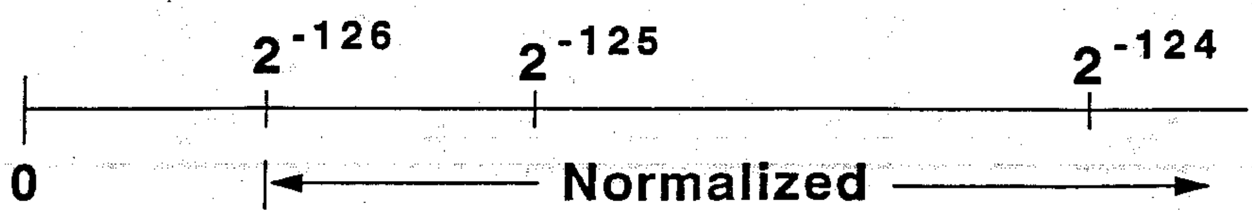
**Representable Numbers:**

- **Normalized**

$$1.d_1 d_2 d_3 \ldots d_{23} * 2^e$$

$$\text{where} \quad -126 \le e \le +127$$



Note: The range of exponents

$$-126 \leq e \leq +127$$

Coupled with the BIAS (127) which is added to the exponent yields an 8 bit string from 00000001

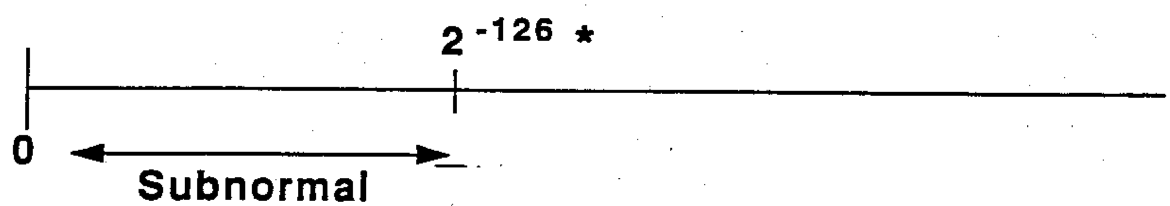
⋮

11111110

Two strings remain: 00000000, 11111111

* Subnormal numbers (Exp field = 00000000)

$$0.d_1 d_2 \ldots d_{23} \quad 2^{-126}$$




* Infinities (Exp field = 11111111)

| | | |
|---|---|---|
| s | 11111111 | 000 ... 0 |

## *Formats (Continued)*

**That still leaves those strings characterized:**

| s | 11111111 | Not Zero |
|---|---|---|

**These are defined as NaNs.**

**They result from invalid operations**

**(Like,: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$)**

**Generalizing to the other formats**

| | Single | Single-X | Double | Double-X |
|---|---|---|---|---|
| Precision | 24 bits | ≥32 | 53 | ≥64 |
| Exponent | 8 bits | ≥11 | 11 | ≥15 |
| Word Length | 32 bits | ≥43 | 64 | ≥79 |
| Exp BIAS | +127 | -- | +1023 | -- |
| $e_{max}$ | +127 | ≥1023 | +1023 | ≥16382 |
| $e_{min}$ | -126 | ≤-1022 | -1022 | ≤-16382 |

## *Rounding*

1st We perform the <u>operation</u> & produce the infinitely precise result

2nd We round to fit it into the destination format

<u>Four</u> Rounding Modes

1. <u>Default:</u> To nearest. If equally near, then to the one having A Ø in LSB

2. Directed roundings

   - Toward + ∞
   - Toward - ∞
   - Toward Ø (Chop)

## *Operations*

* **Arithmetic:** +, - , *, ÷, REM

When $y \neq \emptyset$, $r = x$ REM $y$, is defined:

$r = x - y * n$, where $n$ is the integer nearest $\frac{x}{y}$

whenever $\left| n - \frac{x}{y} \right| = \frac{1}{2}$, then $n$ is EVEN

∴ Remainder is always exact

* Square root: Result defined if ARG $\geq \emptyset$.

* Conversion from one format to another

  - To fewer bits: rounded
  - To more bits: exact

# *Operations (Continued)*

* Conversion Fl. Pt. <----> Integers
  Binary <----> Decimal

* Comparison

  - Always exact
  - Never underflow, overflow
  - Four relations are possible
    {>, =, <, unordered}

Note: Invalid is signaled if unordered operands are compared and unordered is not the basis but > or < is the basis.

Examples:

| Predicate | > | < | = | ? | Invalid if unordered |
|---|---|---|---|---|---|
| = | F | F | T | F | No |
| ? ≠ | T | T | F | T | No |
| > | T | F | F | F | Yes |
| ?<≠ | F | T | T | T | No |

# *Infinities, NaNs, ±0*

$\infty$:

* $-\infty <$ (finite) $< +\infty$
* Arithmetic on $\infty$ is <u>exact</u>
* $\infty$ is created by
  - Overflow
  - "Divide by zero"

NaN:

* Signaling & Quiet

  <u>Signaling</u> - Reserved operand that signals the invalid Op. Exception for all operations in the standard. If no trap occurs, a quiet NaN is delivered

  <u>Quiet</u> - Operations on quiet NaNs produce quiet NaNs. They provide hooks to retrospective diagnostic information.

## *Exceptions*

**When detected: Take Trap, or**
**Set Flag, or**
**Both**

**Flag can be reset <u>only</u> under program control**

- **Invalid**
  - Operation on a signaling NaN.
  - $\infty - \infty$ 0/0
  - $0 * \infty$ $\infty / \infty$
  - x REM y, where y=0 or x= $\infty$
  - $\sqrt{NEG}$
  - Conversion from Fl. to int. or decimal, when overflow, infinity, or NaN prevents the conversion
  - Comparison via predicates involving > or <, and Not?, when the operands are unordered

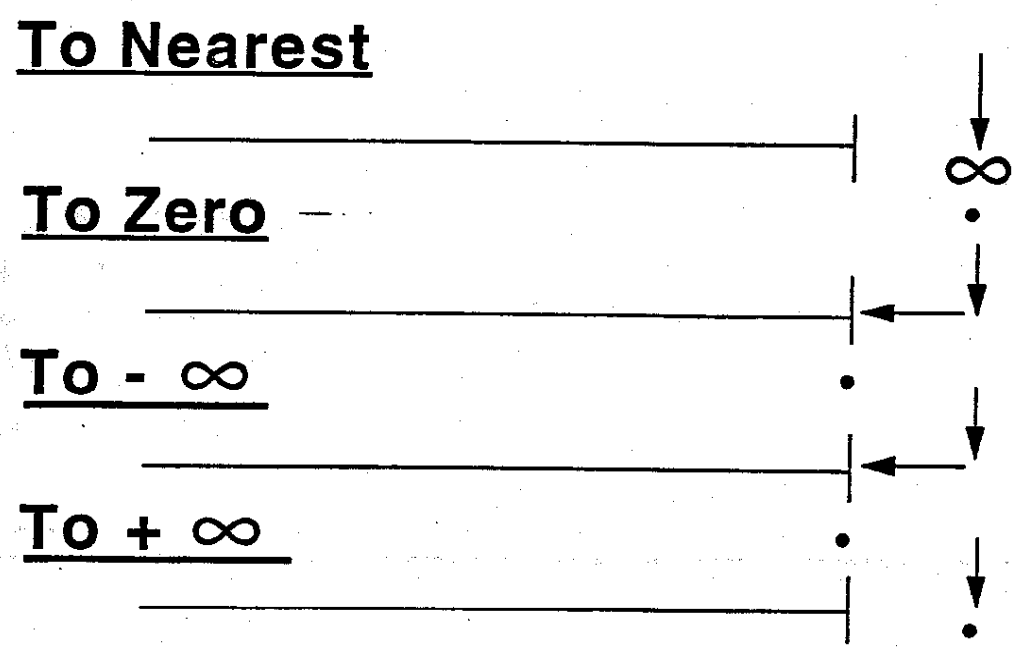
## *Exceptions (Continued)*

* <u>**Divide by zero**</u>

  **When f(finite) --> Infinite and exact**

* <u>**Overflow**</u>

  **When the destinations largest finite number is exceeded by what would have been the <u>rounded</u> floating point result if the exponent range were unbounded**



## *Exceptions(Continued)*

* Overflow (Continued)

  Trapped overflows! [Except for conversions]

  1 st, Divide infinitely precise Result by $2^a$

| | Single | Double | Extended |
|---|---|---|---|
| a = | 192 | 1536 | $3 * 2^{n-2}$ |

n = | exponent bits |

Why?

* Underflow

  - Tiny value (which could cause subsequent overflow)

  - Loss of precision

Delivered result may be zero, subnormal No., or $\pm 2^{\text{min-exp}}$

## *Exceptions (Continued)*

* **Underflow (continued)**

  **Trapped underflows!**

  **[All operations except conversions]**

  **1 st, Multiply infinitely precise Result by $2^a$**

* **Inexact**

  **When the result of an operation is not exact, or on non-trapped overflow.**

## *Traps*

For any of the five exceptions, a user should be able to:

* Specify a handler
* Request that an existing handler be disabled, saved, restored.

When a system traps, the trap handler should be able to determine:

* Which exception occurred on this operation
* The kind of operation being performed
* The destination format
* In overflow, underflow, & inexact, the correctly rounded result
* In invalid & divide by zero, the operand values