Lecture #12

EE 313 Linear Systems and Signals

Professor Robert W. Heath Jr.
Announcements

- HW # 5 was due at the beginning of class
- Requests for midterm regrades were due with your homework
- HW #6 is due on Thursday at the beginning of class
- Please sign up for Piazza
Preview of today’s lecture

◆ Brief review

◆ Output of an LTI system for a periodic input

◆ When does the Fourier series work?
  ✦ Determine whether a signal satisfies the finite energy conditions
  ✦ Determine whether a given signal satisfy Dirichlet conditions

◆ Key properties of Fourier series Part I
  ✦ Connect time-domain & frequency domain structure
  ✦ Incorporate FS properties into your problem solving
Brief review

Key points

- Eigenfunctions of LTI systems are complex exponentials
- Determine a periodic signal from given Fourier series coefficients
- Determine the Fourier series coefficients of a periodic signal
Eigenfunctions of a LTI systems

\[ x(t) = e^{st} \quad \text{LTI System} \quad y(t) = H(s)e^{st} \]

CT complex exponential with complex s

DT complex exponential with complex z

\[ x[n] = z^n \quad \text{LTI System} \quad y[n] = H(z)z^n \]

Eigenvalue (a complex scalar)

Eigenfunction
Analysis and synthesis equations

- Finding the coefficients: Use the analysis equation

\[ a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jn\frac{2\pi}{T} t} dt \]

- Finding the signal: Use the synthesis equations

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T} t} \]

* You can integrate over any period
Analyzing a cosine

- Consider \( \cos(100\pi t) \), it follows that \( \omega_0 = 100\pi \) and \( T = \frac{1}{50} \)

\[
\begin{align*}
a_k &= \int_0^{\frac{1}{50}} \cos(100\pi t) e^{-j2\pi k \frac{1}{50} t} \, dt \\
&= \int_0^{\frac{1}{50}} \frac{1}{2} \left( e^{j100\pi t} + e^{-j100\pi t} \right) e^{-j2\pi k \frac{1}{50} t} \, dt \\
&= \int_0^{\frac{1}{50}} \frac{1}{2} \left( e^{j2\pi k \frac{1}{50} t} + e^{-j2\pi k \frac{1}{50} t} \right) e^{-j2\pi k \frac{1}{50} t} \, dt \\
&= \frac{1}{2} \int_0^{\frac{1}{50}} e^{j2\pi k \frac{1}{50} t} e^{-j2\pi k \frac{1}{50} t} \, dt + \frac{1}{2} \int_0^{\frac{1}{50}} e^{-j2\pi k \frac{1}{50} \pi t} e^{-j2\pi k \frac{1}{50} t} \, dt
\end{align*}
\]
Analyzing a cosine (cont.)

\[
\begin{align*}
&= \frac{1}{2} \int_{0}^{\frac{1}{50}} e^{j2\pi \frac{1}{50} t} e^{-j2\pi k \frac{1}{50} t} dt + \frac{1}{2} \int_{0}^{\frac{1}{50}} e^{-j2\pi \frac{1}{50} t} e^{-j2\pi k \frac{1}{50} t} dt \\
&= \frac{1}{2} \int_{0}^{\frac{1}{50}} e^{j2\pi \frac{1}{50} t(1-k)} dt + \frac{1}{2} \int_{0}^{\frac{1}{50}} e^{-j2\pi \frac{1}{50} t(1+k)} dt \\
&= \frac{1}{2} \delta[1-k] + \frac{1}{2} \delta[1+k] \quad \text{(orthogonality of complex exponentials)}
\end{align*}
\]

\[
\begin{align*}
a_1 &= \frac{1}{2} \\
a_{-1} &= \frac{1}{2}
\end{align*}
\]

\[
\cos(100\pi t) = \sum_{k} a_k e^{j2\pi \frac{1}{50} t} = \frac{1}{2} e^{j2\pi \frac{1}{50} t} + \frac{1}{2} e^{-j2\pi \frac{1}{50} t}
\]
Synthesizing a square wave

\[ x(t) = \sum_{n=0}^{\infty} \frac{1}{\pi(2n + 1)} \sin((2n + 1)\pi t) \]

(real cartesian form)
Synthesizing a square wave

\[ x(t) = \sum_{n=0}^{\infty} \frac{1}{\pi(2n + 1)} \sin((2n + 1)\pi t) \]

(real cartesian form)

4 terms

5 terms

500 terms
What happens to periodic signals when input to LTI systems?

Learning objectives
- Compute the output of an LTI system to a periodic input
Periodic inputs to LTI systems

\[ e^{st} \]

LTI System \[ \rightarrow \quad H(s)e^{st} \]

\[ \alpha e^{s_1 t} + \beta e^{s_2 t} \]

LTI System \[ \rightarrow \quad \alpha H(s_1)e^{s_1 t} + \beta H(s_2)e^{s_2 t} \]

\[ \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \]

LTI System \[ \rightarrow \quad \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k)e^{j\omega_0 kt} \]

Fourier series coefficients are modified by frequency response of the system
When does the Fourier series work?

Learning objectives
- Determine whether a signal satisfies the finite energy conditions
- Determine whether a given signal satisfy Dirichlet conditions
Complications with the Fourier series

Some technical issues with using Fourier series on certain signals
- Challenge arises from the discontinuous nature of the signal
- Ripple is known as the Gibbs phenomena

Where does the ripple go?

Convergence of the Fourier Series
**Error in the Fourier series**

- Consider the partial sum

\[
x_N(t) = \sum_{k=-N}^{N} a_k e^{jkw_0 t}
\]

- The error signal is given by

\[
e_N(t) = x(t) - x_N(t)
\]

- The squared error is a measure of the total error

\[
E_N = \int_T |e_N(t)|^2 dt
\]

**Squared error is minimized by the FS coefficients**
Finite energy signals

- Consider the class of signals that have finite energy
  \[ \int_T |x(t)|^2 dt < \infty \]

- For such signals the FS coefficients exist and
  \[ \lim_{N \to \infty} E_N = 0 \]

- This means that \( x(t) \) and \( \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \) are not equal at every \( t \)
  
  ✪ There is no energy in the difference

FS gives zero squared error for finite energy signals
The ripple

Gap shrinks

3 terms

5 terms

100 terms
Signals that satisfy the Dirichlet conditions

- A periodic signal $x(t)$ that satisfies the following conditions

  - (1) Absolute integrability
  - (2) Finite number of minima and maxima for a given time period
  - (3) Finite number of discontinuities for a period $T$

$x(t)$ and $\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$ are equal except at finite number of points
Absolute integrability

\[ \int_{-T}^{T} x(t) \, dt < \infty \]

- Example

\[ x(t) = \begin{cases} \frac{1}{|t|}, & t \in (-2, 2) \\ \text{repeat for all } T \end{cases} \]

Laplace function

\[ a_0 \text{ must exist...} \]

Bad signal
Finite number of min and max for a given time period

- Example

\[ x(t) = \begin{cases} 
\sin \left( \frac{1}{t} \right), & t \in (0, \frac{1}{\pi}) \\
\text{repeat every } \frac{1}{\pi} \text{ secs} & 
\end{cases} \]

Bad signal

infinite # of peaks
Finite number of discontinuities for a period $T$

Example

$$x(t) = \sum_{k=0}^{\infty} \left[ u(t - \frac{1}{2^k}) - u(t) \right] (-1)^k$$

![Diagram showing the function $x(t)$ and its discontinuities](image)

Bad signal

Infinite # of discontinuities
What does it all mean?

- For most signals, you can use the analysis equation

\[ a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt \]

Integral exists

- The equality in the synthesis equations varies with some signals

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

Satisfies finite energy, then equal in the sense that the squared error goes to zero

Satisfies Dirichlet, then equal everywhere except a finite number of points
What is meant by a finite number of points?

Signal differs from a cosine just at this one point.
Properties of Fourier series: Part I

Learning objectives

- Use Fourier series properties to simplify calculation & build intuition
- Analyze problems that include FS properties
Properties of the Fourier series

- The following notation is used to denote a signal and its FS coefficients

\[ x(t) \overset{FS}{\leftrightarrow} a_k \]

- Properties are used to figure out how transformations of the input signal lead to transformations of the FS coefficients

- Properties considered in this lecture (more in next lecture)
  - Linearity
  - Time shifting
  - Time reversal
  - Time scaling
  - Multiplication
**Linearity**

- If $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

  \[ x(t) \xrightarrow{FS} a_k \]

  \[ y(t) \xrightarrow{FS} b_k \]

  then

  \[ z(t) = A x(t) + B y(t) \]

  \[ z(t) \xrightarrow{FS} A a_k + B b_k \]

**FS of a sum of signals is the sum of their FS coefficients**
Time shifting

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \xrightarrow{FS} a_k$

- If $y(t) = x(t - t_0)$, $y(t)$ is periodic with the same period $y(t) \xrightarrow{FS} b_k$

- Then $b_k = a_k e^{-jk\omega_0 t_0}$

Note $|b_k| = |a_k|$ since $|e^{jk}| = 1$

Shift in time results in a phase shift in frequency
Example 1

Let $x(t)$ be a periodic signal with a fundamental period $T$, and FS coefficients $a_k$. Derive the FS coefficients of the following signal:

$$x(t - t_0) + x(t + t_0)$$
Time reversal

- Let \( x(t) \) have period \( T = \frac{2\pi}{\omega_0} \), and \( x(t) \overset{FS}{\leftrightarrow} a_k \)

- Then \( y(t) = x(-t) \), \( y(t) \) is periodic with the same period

- and

\[
y(t) \overset{FS}{\leftrightarrow} a_{-k}
\]

Reverse in time results in reverse in frequency
Time reversal proof

- Suppose that
  \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk \omega_0 t} \]

- Then
  \[ y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk \omega_0 t} \]

- Changing variables
  \[ y(t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm \omega_0 t} \]
  \[ y(t) \xrightarrow{FS} a_{-k} \]
Implications of time reversal on even and odd

- Let $x(t)$ have period $T = \frac{2\pi}{\omega_0}$, and $x(t) \overset{FS}{\leftrightarrow} a_k$

- If $x(t)$ is even then $x(t) = x(-t)$ and it follows that
  \[ a_k = a_{-k} \]

- If $x(t)$ is odd then $x(-t) = -x(t)$ and it follows that
  \[ a_k = -a_{-k} \]

Symmetry in leads to structure in FS coefficients
**Time scaling**

- Let \( x(t) \) have period \( T = \frac{2\pi}{\omega_0} \), and \( x(t) \xrightarrow{\text{FS}} a_k \)

- If \( y(t) = x(\alpha t) \), \( \alpha > 0 \)
  - \( \alpha < 1 \rightarrow \text{compression} \)
  - \( \alpha > 1 \rightarrow \text{stretching} \)

- Then \( y(t) = x(\alpha t) \) is periodic with period \( \alpha T \) and
  \[
  x(\alpha t) \xrightarrow{\text{FS}} a_k
  \]

**Scale in time does not change the FS coefficients**
Visualizing time scaling

Example: \( y(t) = x(\alpha t) \)

\( \alpha = \frac{1}{4} \)

Stretched signal has same structure
**Time scaling proof**

- Since \( x(t) \overset{FS}{\longleftrightarrow} a_k \) it follows that
  \[
  x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}
  \]

- Then
  \[
  x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 \alpha t}
  \]

  \[
  = \sum_{k=-\infty}^{\infty} a_k e^{j k (\alpha \omega_0) t}
  \]

  \[
  x(\alpha t) \overset{FS}{\longleftrightarrow} a_k
  \]
Multiplication

- If $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) \leftrightarrow_{FS} a_k$$

$$y(t) \leftrightarrow_{FS} b_k$$

- Then for $z(t) = x(t)y(t)$

$$z(t) = x(t)y(t) \leftrightarrow_{FS} h_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

Product in time leads to convolution in frequency
Conjugate symmetry

- If \( x(t) \) is periodic with period \( T = \frac{2\pi}{\omega_0} \) and \( x(t) \leftrightarrow\text{FS} \ a_k \)

- Then \( x^*(t) \leftrightarrow\text{FS} \ a_{-k}^* \)

- Implications
  - If \( x(t) \) is real, then the FS coefficients are conjugate symmetric
    \[ a_{-k}^* = a_k \]
  - If \( x(t) \) is real and even, then the FS coefficients are real and even
    \[ a_k = a_k^* \]
  - If \( x(t) \) is real and odd, then the FS coefficients are imaginary and odd
Example

Let \( x(t) \) be a periodic signal with a fundamental period \( T \), and FS coefficients \( a_k \). Derive the FS coefficients of the following signal:

- \( Even\{x(t)\} \)
Summary of lecture

- Outputs of LTI systems are periodic
  - Output can be found using Eigenfunction principle
  - Fourier series coefficients depend on the system frequency response

- Existence of the Fourier Series for a given period signal
  - Provides a zero-energy representation of finite energy signals
  - Provides an exact characterization of signals that satisfy the Dirichlet conditions (except at a finite number of points)

- Fourier series properties
  - Time-domain structure leads to frequency domain structure
  - Incorporate FS properties into your problem solving