Lecture #15

EE 313 Linear Systems and Signals

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Announcements

◆ HW #7 is now past due

◆ HW #8 has been posted

◆ Quiz #6 is at the end of class
Preview of today’s lecture

◆ Introduction to the Fourier transform

◆ Fourier transform synthesis and analysis equations

◆ Existence of the Fourier transform
  ✦ Dirichlet conditions apply to the Fourier transform
  ✦ Delta functions allow “definitions” of transforms of certain signals

◆ Calculating the Fourier transform and its inverse
  ✦ Apply the analysis and synthesis equations
  ✦ Derive and use the transforms of special functions
Introduction to the Fourier transform

Key points
- Explain the connection between Fourier series and Fourier transform
Fourier series review

- Periodic signals (that are well behaved i.e. satisfy Dirichlet conditions) can be represented as a sum of complex sinusoids

\[ x(t) = \sum_{k=\infty}^{\infty} a_k e^{j k \omega_0 t} \]

- All of the information in \( x(t) \) is contained in \( \omega_0 \) and \( \{a_k\}_{k=-\infty}^{\infty} \)
What about aperiodic signals?

- Most interesting real-world signals are not periodic
  - The Fourier series for a non-periodic signal does not have any meaning

- Is it possible to represent an aperiodic signal as a sum of sinusoids?

Yes using the Fourier transform
Fourier transform

Key points

- Define Fourier transform
- Determine the Fourier transforms of CT and DT signals
Fourier transform

- For a signal $x(t)$, the Fourier transform (FT) $X(j\omega)$ is

$$X(j\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

- The notation $\mathcal{F}\{\cdot\}$ means to take the Fourier transform of the function inside the brackets

- In some books, $X(\omega)$ is used here instead of $X(j\omega)$
  - Sometimes even $X(j\Omega)$ or $X(\Omega)$ is used (e.g. in EE 351M)
Interpreting the Fourier transform

- The Fourier transform can be written

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \]

Inner product

\[ = \langle x(t), e^{j\omega t} \rangle \]

How much does \( x(t) \) look like complex sinusoid \( e^{j\omega t} \) with frequency \( \omega \)?

Measures how much the vectors "line up" with each other
Why do we care about complex sinusoids anyways?

- For an LTI system, with input \( e^{j\omega_0 t} \) it is easy to compute the output

\[
e^{j\omega_0 t} \rightarrow H(j\omega) \rightarrow H(j\omega_0)e^{j\omega_0 t}
\]

- If a signal can be represented as a sum of sinusoids, then it is possible to compute the output of an LTI system without convolution.

- Further notice that

\[
H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} \, dt
\]
Example – Unit impulse

- Consider the signal
  \[ x(t) = \delta(t) \]

- Its FT is given by
  \[ X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} \, dt \]
  \[ = \int_{-\infty}^{\infty} \delta(t) e^0 \, dt \]
  \[ = 1 \]
Inverse Fourier transform

- Given the frequency response $X(j\omega)$, $x(t)$ is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

  $$= \mathcal{F}^{-1}\{X(j\omega)\}$$

- Transform looks similar to direct transform except
  - Sign of the exponential is different
  - Scaling factor in front (results from using radians and not Hertz)

Similarity will lead to the concept of duality
**Example – Unit impulse in frequency**

- Consider the signal $X(j\omega) = \delta(\omega)$

- Its FT is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}$$

Notice the **duality** here in each domain.
Example – Shifted delta in frequency

- Consider an impulse in the frequency domain \( X(j\omega) = \delta(\omega - \omega_0) \)

- Its inverse FT is given by

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega
\]

\[
= \frac{e^{j\omega_0 t}}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega
\]

\[
= \frac{e^{j\omega_0 t}}{2\pi} \left[ 1 \right]
\]
Summarizing the Fourier transform and its inverse

Fourier transform (analysis)

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt \]

Inverse Fourier transform (synthesis)

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} \, d\omega \]

\[ x(t) \leftrightarrow X(j\omega) \]
Note: An alternative formulation

Fourier transform (analysis)

\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \quad f \text{ in Hz} \]

Inverse Fourier transform (synthesis)

\[ x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df \]

In EE 313 we will use \( X(j\omega) \) (in EE 471C we use above notation)
Existence of the Fourier transform

Key points
- Define Fourier transform
- Determine the Fourier transforms of CT and DT signals
Sufficient condition: Finite energy

- Consider the class of signals that have finite energy
  \[ \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \]

- For such signals the Fourier transform exists (is finite)

- Further there is zero-energy in the error in the sense that
  \[ \int_{-\infty}^{\infty} \left| x(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right|^2 = 0 \]

FT gives zero squared error for finite energy signals
Sufficient condition: satisfies Dirichlet

- An aperiodic signal $x(t)$ that satisfies the following conditions

  1. **Absolute integrability** $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

  2. Finite number of minima and maxima **over a finite interval**

  3. Finite number of discontinuities **over a finite interval**

$x(t)$ and $\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$ are equal except at a discontinuity.
What about periodic signals?

- Periodic signals do not satisfy either sufficient condition

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |x(t)| dt = \infty \]

- If Dirac delta is acceptable \( \delta(t) \) then can define the FT from FS

- Consider a periodic signal \( x(t) \leftrightarrow \{a_k\} \) then the FT of \( x(t) \) is

\[
X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)
\]

Fourier transform

FS coefficients
Periodic signals have a “line” spectrum
Example – Shifted delta in frequency (again)

- Consider the signal

\[ x(t) = \frac{1}{2\pi} e^{j\omega_0 t} \]

- The signal is periodic with fundamental frequency \( \omega_0 \)

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

\[ = \frac{1}{2\pi} \underbrace{e^{j\omega_0 t}}_{a_1}, \quad k = 1 \text{ only} \]

\[ e^{j\omega_0 t} \overset{\mathcal{F}}{\leftrightarrow} 2\pi \delta(\omega - \omega_0) \]
Fourier transform examples

Key points

- Use the Fourier transform synthesis and analysis equations
- Learn and use the transforms for common signals
Fourier transform of a causal exponential

- Consider the signal
  \[ x(t) = e^{-at}u(t), \quad \text{Re}\{a\} > 0 \]

- Its FT is given by
  \[
  X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt
  \]
  \[
  = \int_0^\infty e^{-(a+j\omega)t} dt
  \]
  \[
  = \left. \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right|_0^\infty
  \]
  \[
  = \frac{1}{a+j\omega}
  \]
Fourier transform of a cosine

- Consider the signal \( x(t) = \cos \omega_0 t \)

- This is a periodic signal, can find its FT using the FS
  \[
  x(t) = \cos \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})
  \]

- Leveraging the frequency impulse results
  \[
  X(j\omega) = \frac{1}{2}(2\pi \delta(\omega - \omega_0) + 2\pi \delta(\omega + \omega_0))
  \]
  \[
  = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)
  \]
Fourier transform of a sine

- Consider the signal
  \[ x(t) = \sin \omega_0 t \]

- To get its FT, we note that
  \[ x(t) = \sin \omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) \]

- Leveraging the frequency impulse results, we have
  \[ X(j\omega) = \frac{1}{2j}(2\pi \delta(\omega - \omega_0) - 2\pi \delta(\omega + \omega_0)) \]
  \[ = \pi j(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)) \]
Fourier transform of a rectangle function

- Consider the signal
  
  \[ x(t) = \text{rect}(t) = \Pi(t) = \begin{cases} 
  0, & |t| > \frac{1}{2} \\
  1, & |t| \leq \frac{1}{2} 
\end{cases} \]

- Note that
  
  \[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = 1 \]

- Can create other related shapes
  
  \[ A \text{ rect}\left(\frac{t}{B}\right) \]
Fourier transform of a rectangle function

- Its FT is given by

\[ X(j\omega) = \int_{-\infty}^{\infty} \text{rect}(t)e^{-j\omega t} dt \]

\[ = \int_{-1/2}^{1/2} e^{-j\omega t} dt \]

\[ = -\frac{1}{j\omega}e^{-j\omega t}|_{-1/2}^{1/2} \]

\[ = -\frac{1}{j\omega}(e^{-j\omega/2} - e^{j\omega/2}) \]

\[ = \frac{2}{\omega} \cdot \frac{1}{2j}(e^{j\omega/2} - e^{-j\omega/2}) \]

\[ = \frac{2}{\omega} \sin \frac{\omega}{2} \]

\[ = \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \]

\[ = \text{sinc} \left( \frac{\omega}{2\pi} \right) \]
About the sinc function

Maximum value of 1

Zero crossings at +/-1, +/- 2, ....

\[ \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \]

Be aware, sometimes sinc is defined like this

\[ \text{sinc}(t) = \frac{\sin(t)}{t} \]
Fourier transform of rectangle function

Maximum value of 1

Zero crossings at +/- 2π, +/- 4π...

\[ X(j\omega) = \text{sinc} \left( \frac{\omega}{2\pi} \right) \]
Fourier transform of a scaled rectangle function

- Its FT is given by

\[ X(j\omega) = \int_{-\infty}^{\infty} \text{rect}(2t/T_1) e^{-j\omega t} \, dt \]

\[ = \int_{-T_1/2}^{T_1/2} e^{-j\omega t} \, dt \]

\[ = -\frac{1}{j\omega} e^{-j\omega T_1/2} \bigg|_{-T_1/2}^{T_1/2} \]

\[ = -\frac{1}{j\omega} (e^{-j\omega T_1/2} - e^{-j\omega T_1/2}) \]

\[ = \frac{2}{\omega} \cdot \frac{1}{2j} (e^{j\omega T_1/2} - e^{-j\omega T_1/2}) \]

\[ = \frac{2}{\omega} \sin \frac{\omega T_1}{2} \]

\[ = \frac{T_1}{2\pi} \frac{\sin \frac{\omega T_1}{2}}{\frac{\omega}{2}} \]

\[ = T_1 \text{sinc} \left( \frac{\omega T_1}{2\pi} \right) \]
Fourier transform of rectangle

- Crossings at $\pm \pi/T_1, \pm 2\pi/T, \ldots$
- For $T_1 \to \infty$, FT is $\delta(\omega)$
- So
  - $T_1 \uparrow$, pulse narrow, sinc wide
  - $T_1 \downarrow$, sinc becomes narrow, pulse wide

Narrow pulse in time is broad in frequency
Inverse Fourier transform of the rectangle function

- Can compute the FT directly

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\frac{B}{2}}^{\frac{B}{2}} e^{j\omega t} d\omega \]

\[ = \frac{1}{j2\pi t} e^{j\omega t} \bigg|_{-\frac{B}{2}}^{\frac{B}{2}} \]

\[ = \frac{1}{j2\pi t} (e^{\frac{jB}{2}t} - e^{-\frac{jB}{2}t}) \]

\[ = \frac{B}{2\pi} \text{sinc} \left( \frac{Bt}{2\pi} \right) \]

\[ \text{rect}(\omega/B) \]

Zeros at

\[ \pm 2\pi/B, \pm 4\pi/B, \ldots \]
Preview of duality

\[ x(t) \]

\[ \begin{array}{c}
- T_1 \\
0 \\
T_1 \\
\end{array} \]

\[ \begin{array}{c}
1 \\
\text{Zeros at} \\
\pm 2\pi/B, \pm 4\pi/B, \ldots \\
\end{array} \]

\[ F \]

\[ \begin{array}{c}
T_1 \\
T_1 \text{sinc} \left( \frac{\omega T_1}{2\pi} \right) \\
\end{array} \]

\[ \begin{array}{c}
\frac{B}{2\pi} \text{sinc} \left( \frac{Bt}{2\pi} \right) \\
\end{array} \]

\[ \begin{array}{c}
- \frac{2\pi}{T_1} \\
- \frac{\pi}{T_1} \\
\frac{\pi}{T_1} \\
\frac{2\pi}{T_1} \\
\end{array} \]

\[ \begin{array}{c}
\text{rect}(\omega/B) \\
1 \\
0 \\
B/2 \\
- B/2 \\
\end{array} \]

It is possible to figure out one set of transforms from the other.
Example

- Determine the FT of the following signal

\[ \sin \left( 2\pi t + \frac{\pi}{8} \right) \]

- Solution
Example

◆ Determine the inverse Fourier transform of

\[ X(\omega) = 2\pi \delta(\omega) + \pi \delta(\omega - 4\pi) + \pi \delta(\omega + 4\pi) \]

◆ Solution
Example

Compute the Fourier transform of

\[
x(t) = \begin{cases} 
1 - t^2 & t \in [0, 1] \\
0 & \text{else}
\end{cases}
\]
Summary

◆ The Fourier transform is a generalization of the Fourier series
   ✦ Applies to aperiodic signals

◆ Existence of the Fourier transform
   ✦ Similar conditions as in the case of the Fourier series
   ✦ Use the Fourier series and delta function to define the Fourier transform for periodic signals

◆ Calculating the Fourier transform and its inverse
   ✦ Apply the analysis and synthesis equations
   ✦ Derive and use the transforms of special functions
   ✦ Next lecture properties will be exploited to simplify calculations