Lecture #19

EE 313 Linear Systems and Signals

Professor Robert W. Heath Jr.
Announcements

- Enoch will take Prof. Heath’s office hours the next two weeks due to travel

- Midterm regrade requests due on April 30
Midterm #2 results 😞

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Major confusion about eigenfunction property and filtering periodic signals
Problem 3. In-N-Out. 25 points

Consider the LTI system in Figure 1, where the frequency response of the system with the impulse response $h_1(t)$ is plotted in Figure 2. Determine the output of the cascaded system in Figure 1 if the input of this system is the periodic signal $x(t)$, with period $T = 4$ defined on $t \in [-2, 2]$ as

$$x(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & \frac{1}{2} < |t| < 2. \end{cases}$$

The Fourier series coefficients of the signal $x(t)$ is given by

$$a_k = \sin \frac{\pi k}{4} \pi k.\$$

This was derived in class and used in several homeworks. If you did not have this formula, you could still continue the problem using a general $a_k$. Given the filter frequency response $H_1(j\omega)$, the output of this filter, $w(t)$, is

$$w(t) = \sum_{k=-\infty}^{\infty} a_k H_1(jk\omega) e^{jk\omega t} \approx \sum_{k=-2}^{2} a_k H_1(jk\omega) e^{jk\omega t}.$$ 

From the Fourier series of the signal $x(t)$, we have $a_2 = a_{-2} = \frac{1}{2} \pi$, and $a_6 = a_{-6} = \frac{1}{6} \pi$.

Therefore, the Fourier series coefficients of $w(t)$, denoted $d_k$, are $d_2 = d_{-2} = \frac{1}{2} \pi$, and $d_6 = d_{-6} = \frac{1}{6} \pi$.

Finally, the output of the system is the integral of $w(t)$ in the time domain. Using

$$y(t) = \int_{-\infty}^{t} w(\tau) d\tau.$$ 

Figure 1: System Block Diagram.

Figure 2: The frequency response of the system $h_1(t)$, with $\omega_0 = \frac{\pi}{2}$. 
Periodic inputs to LTI systems

From lecture #12

\[ e^{st} \rightarrow \text{LTI System} \rightarrow H(s)e^{st} \]

\[ \alpha e^{s_1 t} + \beta e^{s_2 t} \rightarrow \text{LTI System} \rightarrow \alpha H(s_1)e^{s_1 t} + \beta H(s_2)e^{s_2 t} \]

\[ \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \rightarrow \text{LTI System} \rightarrow \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k)e^{j\omega_0 kt} \]

Fourier series coefficients are modified by frequency response of the system
Solution 1/2

The Fourier series coefficients of the signal $x(t)$ is given by $a_k = \frac{\sin(\pi k/4)}{\pi k}$. This was derived in class and used in several homeworks. If you did not have this formula, you could still continue the problem using a general $a_k$. Given the filter frequency response $H_1(j\omega)$, the output of this filter, $w(t)$, is

$$w(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0}$$

$$= \pi a_2 e^{j2\omega_0} + \pi a_{-2} e^{-j2\omega_0} - \pi a_6 e^{j6\omega_0} - \pi a_{-6} e^{-j6\omega_0}$$

From the Fourier series of the signal $x(t)$, we have $a_2 = a_{-2} = \frac{1}{2\pi}$, and $a_6 = a_{-6} = \frac{-1}{6\pi}$. Therefore, the Fourier series coefficients of $w(t)$, denoted $d_k$, are $d_2 = d_{-2} = \frac{1}{2}$ and $d_6 = d_{-6} = \frac{1}{6}$. 
Solution 2/2

Finally, the output of the system is the integral of \( w(t) \) in the time domain. Using Fourier series properties, the Fourier series of the output signal \( y(t) \), denote them \( b_k \) will be \( b_k = \left( \frac{1}{jk\omega_0} \right) d_k \). Therefore, we have \( b_2 = -b_{-2} = \frac{1}{j2\pi} \) and \( b_6 = -b_{-6} = \frac{1}{j18\pi} \), and the output is then \( y(t) = \frac{1}{\pi} \sin(2\omega_0 t) + \frac{1}{9\pi} \sin(6\omega_0 t) \).
Preview of today’s lecture

◆ Brief review
  ✦ FT Multiplication property
  ✦ FT Convolution property
  ✦ FT Symmetry, even and odd properties

◆ Laplace transform
  ✦ Definition of Laplace transform
  ✦ Region of convergence of Laplace transform
  ✦ Some Laplace transform pairs
Brief review: FT convolution, multiplication, and symmetry properties

Key points
- Convolution in time is multiplication in frequency
- Multiplication in time is convolution in frequency
- Use these facts to compute convolutions
- Connect signal symmetry properties in time & frequency domains
**Convolution property**

- If \( h(t) \xrightarrow{\mathcal{F}} H(j\omega) \quad x(t) \xrightarrow{\mathcal{F}} X(j\omega) \quad y(t) \xrightarrow{\mathcal{F}} Y(j\omega) \)

- Then

\[
y(t) = h(t) \ast x(t) \xrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega)X(j\omega)
\]

**Convolution in time is multiplication in frequency**
Block diagrams

\[ x(t) \rightarrow h(t) \rightarrow y(t) \]

\[ x(t) \rightarrow H(j\omega) \rightarrow y(t) \]
Multiplication property

- If \( h(t) \overset{\mathcal{F}}{\rightarrow} H(j\omega) \) \( x(t) \overset{\mathcal{F}}{\rightarrow} X(j\omega) \) \( y(t) \overset{\mathcal{F}}{\rightarrow} Y(j\omega) \)

- Then

\[
y(t) = h(t)x(t) \overset{\mathcal{F}}{\rightarrow} Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\theta)X(j(\omega - \theta))d\theta
\]

Product in time is convolution in frequency
Example

- AM example revisited
  - Output of the AM modulator
    \[
    y(t) = x(t) \cos(\omega_0 t) = x(t) \cdot \frac{1}{2} (e^{j\omega_0 t} + e^{-j\omega_0 t})
    \]
  - In the frequency domain
    \[
    Y(\omega) = \frac{X(\omega) \ast \mathcal{F}\{\cos \omega_0 t\}}{2\pi}
    \]
    \[
    \mathcal{F}\{\cos \omega_0 t\} = \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]
    \]
    \[
    Y(\omega) = \frac{X(\omega)}{2} \ast [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]
    \]
Symmetry of the Fourier transform

Note that

\[ x(t) \xrightarrow{\mathcal{F}} X(j\omega) \]

\[ x^*(t) \xrightarrow{\mathcal{F}} X(-j\omega) \]

Implications

- Conjugate symmetry \( x(t) \) is real \( \rightarrow X(j\omega) = X^*(-j\omega) \)

- Real signals are even in amplitude since \( |X(j\omega)| = |X(-j\omega)| \)
**Even and odd**

- Can decompose a signal into even and odd components

\[ x(t) = e(t) + o(t) \]

- Even part is

\[ e(t) = \frac{1}{2}(x(t) + x(-t)) \]

- Odd part is

\[ o(t) = \frac{1}{2}(x(t) - x(-t)) \]

- Can similarly decompose

\[ X(j\omega) = E(j\omega) + O(j\omega) \]
Connecting the properties

\[ x(t) = \text{Re}\{e(t)\} + j\text{Im}\{e(t)\} + \text{Re}\{o(t)\} + j\text{Im}\{o(t)\} \]

\[ X(j\omega) = \text{Re}\{E(j\omega)\} + j\text{Im}\{E(j\omega)\} + \text{Re}\{O(j\omega)\} + j\text{Im}\{O(j\omega)\} \]
Laplace transform - Definition

Key points
- Define the Laplace transform
- Explain the difference with the Fourier transform
Laplace transform

✦ What will be covered
  ✪ What is the Laplace transform (we will use “LT”)?
  ✪ The region of convergence (ROC) of Laplace transform
  ✪ Some Laplace transform pairs
  ✪ Laplace transform properties
  ✪ Interconnections of LTI systems
  ✪ Partial fraction expansion (PFE)

Today
Laplace transform definition

- Recall the output of LTI system to the complex exponential $e^{st}$

$$y(t) = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

- For $s$ imaginary, i.e., $s = j\omega$, this integral correspond to the FT of $h(t)$.
- For general values of the complex variable $s$, it is referred to as the Laplace transform.
Laplace and transform definition

- The Laplace transform for a general signal $x(t)$ is defined as $\mathcal{L}\{x(t)\}$

\[
X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt
\]

- LT is usually applied to understand systems (FT more for signals)
- The LT is associated with a region of convergence (ROC) that will help check system stability
Laplace transform – Region of Convergence

Key points
- Determine the region of convergence of Laplace transform
Example of the Region of Convergence of the LT

- Find the LT of the signal  \( x(t) = e^{-at}u(t), \ a \) is real

- Solution:

\[
X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
= \int_{0}^{\infty} e^{-at}e^{-st}dt \\
= \int_{0}^{\infty} e^{-(a+s)t}dt \\
= \left. \frac{-1}{a+s} \right|_{t=0}^{t=\infty} \lim_{t\to\infty} e^{-(a+s)t} - 1
\]

This term will not go to \( \infty \) (will converge) when

\[ \Re \{ s + a \} > 0 \]

So, we say the ROC of LT is

\[ \Re \{ s \} > -a \]
Example of the Region of Convergence of the LT

Example (to illustrate what is meant by ROC of LT)

- Find the LT of the signal $x(t) = e^{-at}u(t)$, $a$ is real

Solution (continued)

- In the ROC, $\mathcal{L}\{x(t)\}$ exists
- To emphasize this point, we write

$$X(s) = \frac{1}{s + a}, \quad \text{Re}\{s\} > -a$$

ROC

- Im

ROC

- Re

$-a$
Why is only the $\text{Re}\{s\}$ important?

◆ Consider

\[ X(s) = \lim_{t \to \infty} e^{-(s+a)t} \]

◆ It can be written as

\[
X(s) = \lim_{t \to \infty} e^{-(\text{Re}\{s\} + j\text{Im}\{s\} + a)t} \\
= \lim_{t \to \infty} e^{-(\text{Re}\{s\} + a)t} e^{-j\text{Im}\{s\}t}
\]

◆ Note that the amplitude does not depend on the imaginary part

\[
\left| e^{-(s+a)t} \right| = \left| e^{-(\text{Re}\{s\} + a)t} \right|
\]
Why is only the Re\{s\} important? (cont.)

- As we have \[ |e^{-(s+a)t}| = |e^{-(\text{Re}\{s\}+a)t}| \]
- If \(- (\text{Re}\{s\} + a) < 0\) \(\text{the limit converges}\)
- If \(- (\text{Re}\{s\} + a) > 0\) \(\text{the limit does not exist}\)
- If \((\text{Re}\{s\} + a) = 0\) \(\text{the limit does not exist (rotation)}\)
Example

Find the Laplace transform and ROC of the signal

\[ x(t) = -e^{-at}u(-t) \]

Laplace Transform (LT) is given by

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \]

\[ = \int_{-\infty}^{0} -e^{-at}e^{-st}dt \]

\[ = \int_{-\infty}^{0} -e^{-(a+s)t}dt \]

\[ = \frac{1}{s + a} \lim_{t \to \infty} \frac{1}{s + a} e^{-(s+a)t} \]
Example (cont.)

- So,

\[ X(s) = \frac{1}{s + a} - \lim_{t \to -\infty} \frac{1}{s + a} e^{-(s+a)t} \]

This term converges when \( \text{Re}\{s + a\} < 0 \)

- Then, we have

\[ -e^{-at} u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s + a} \quad \text{ROC} \quad \text{Re}\{s\} < -a \]
Example

◆ For each of the following integrals, specify the values of the real parameter $\sigma$ which ensure that the integral converges

A. $\int_{-\infty}^{\infty} e^{-5|t|} e^{-(\sigma+j\omega)t}$

Solution:
Example

For the following integrals, specify the values of the real parameter which ensure that the integral converges

B. \[ \int_{-5}^{5} e^{-5t} e^{-(\sigma + j\omega) t} \]

Solution:
Laplace transform – Pole-zero plots

Key points
- Plot the poles and zeros given a rational transfer function and the ROC
ROC is often illustrated using a pole-zero plot.

- If \( x(t) \) is a linear combination of exponentials then
- If \( \text{order of } D(s) > N(s) \) then there are “zeros at infinity”
  similarly if \( \text{order of } N(s) < D(s) \) then “poles at infinity”

Example with 1 real pole and 2 complex zeros

\[
X(s) = \frac{N(s)}{D(s)}
\]

Can never have poles inside the ROC
Example

- You would be given a transform and a ROC
- ROC right half plane

\[ X(s) = \frac{1}{(s - 1)(s - 2)} \]

Poles at 1 and 2

2 zeros at infinity
Laplace transform – Some transform pairs

Key points
- Derive common Laplace transform pairs
Some Laplace transform pairs – Unit Delta

- Find the Laplace transform and ROC of
  \[ x(t) = \delta(t) \]

- Solution:
  ✴ LT is
  \[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \]
  \[ X(s) = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt \]
  \[ X(s) = e^{s\cdot 0} = 1 \]

- ROC is all \( s \) in the complex plane
Some Laplace transform pairs – Unit-step

- Find the Laplace transform and ROC of

\[ x(t) = u(t) \]

- Solution:
  - LT is
  
  \[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \]
  
  \[ = \int_{-\infty}^{\infty} u(t)e^{-st}dt \]
  
  \[ = \frac{-1}{s}e^{-st}\bigg|_{t=0}^{\infty} \]
  
  \[ = \lim_{t\to\infty} \frac{-1}{s}e^{-st} + \frac{1}{s} = \frac{1}{s} \]

\[ \text{ROC} \quad \text{Re}\{s\} > 0 \]
Example

Consider the signal

\[ x(t) = e^{-5t}u(t) + e^{-\beta t}u(t) \]

and denote its Laplace transform by \( X(s) \). What are the constraints placed on the real and imaginary parts of \( \beta \) if the ROC of \( X(s) \) is given by \( \Re\{s\} > -3 \)

Solution:
Example

◆ Determine the Laplace transform and ROC of the signal

\[ x(t) = e^{-4t}u(t) + e^{-5t}(\sin 5t)u(t) \]

◆ Solution
Summary

- Laplace transform is used for the analysis of systems
  - Similar in operation to the Fourier transform but is associated with a region of convergence

- Region of convergence
  - Determines where the Laplace transform works
  - Often plot the region of convergence associated with the poles and zeros of the transfer function