Announcements
Preview of today’s lecture

◆ Brief review
  ✦ Laplace and inverse Laplace transforms
  ✦ Laplace transform properties

◆ LTI system characterization

◆ Revising filtering
  ✦ Low-pass filters
  ✦ High-pass filters
Brief review – Laplace transform

Key points
- Determine LT and inverse LT of signals & systems
- Use LT and its properties to solve signals and systems problems
Laplace and transform definition

- The Laplace transform for a general signal $x(t)$ is defined as $\mathcal{L}\{x(t)\}$

\[
X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt
\]

- For $s$ imaginary, i.e., $s = j\omega$, this integral correspond to the FT
- For general values of the complex variable $s$, it is referred to as the LT

- LT is used to analyze systems described by LCCDEs
  - Design systems or check for stability of systems
Inverse Laplace transform

- Given a LT $X(s)$ of the signal $x(t)$, this signal can be written as

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$$

- Notes
  - The contour of the integration is the straight line in the s-plane corresponding to all points satisfying $\text{Re}\{s\} = \sigma$
  - We can choose any such line only for $\sigma$ in the ROC
Poles and zeros of

Consider

\[ H(s) = \frac{F(s)}{G(s)} = \frac{F(s)}{(s + a)(s + b)(s + c)} \]

\[ H(s) = \frac{A}{s + a} + \frac{B}{s + b} + \frac{C}{s + c} \]

✦ Poles of \( H(s) \): \( s = -a, -b, -c \)

Where in the s-plane \( H(s) \rightarrow \infty \)

Determine the output format

✦ Zeros of \( H(s) \): Roots of \( F(s) \)

Where in the s-plane \( H(s) = 0 \)
LT properties - Linearity

- If
  \[ x_1(t) \xrightarrow{\mathcal{L}} X_1(s) \]  
  \[ x_2(t) \xrightarrow{\mathcal{L}} X_2(s) \]
  with ROC \( R_1 \) and \( R_2 \), respectively.

- Then
  \[ a x_1(t) + b x_2(t) \xrightarrow{\mathcal{L}} a X_1(s) + b X_2(s) \]
  with ROC containing \( R_1 \cap R_2 \).
LT properties – Shifting in the time and s-domain

* Shifting in the time domain
  
  ✦ If
  \[ x(t) \xrightarrow{\mathcal{L}} X(s) \quad \text{with ROC R} \]

  ✦ Then
  \[ x(t - t_0) \xrightarrow{\mathcal{L}} e^{-st_0}X(s) \quad \text{with ROC R} \]

* Shifting in the s-domain
  
  ✦ If
  \[ x(t) \xrightarrow{\mathcal{L}} X(s) \quad \text{with ROC R} \]

  ✦ Then
  \[ e^{s_0t}x(t) \xrightarrow{\mathcal{L}} X(s - s_0) \quad \text{with ROC R+ Re\{s_0\}} \]
LT properties – Time scaling

- If
  \[ x(t) \xrightarrow{\mathcal{L}} X(s) \]
  with ROC R

- Then
  \[ x(at) \xrightarrow{\mathcal{L}} \frac{1}{|a|} X \left( \frac{s}{a} \right) \]
  with ROC \( R_1 = aR \)

With 0 < a < 1

With 0 > a > -1
LT properties – Conjugation and differentiation

- If
  \[ x(t) \overset{L}{\rightarrow} X(s) \] 
  with ROC R

- Then
  \[ x^*(t) \overset{L}{\rightarrow} X^*(s^*) \] 
  with ROC R

- If
  \[ x(t) \overset{L}{\rightarrow} X(s) \] 
  with ROC R

- Then
  \[ \frac{dx(t)}{dt} \overset{L}{\rightarrow} sX(s) \] 
  with ROC R
LT properties – Convolution property

- If
  \[ x_1(t) \overset{\mathcal{L}}{\rightarrow} X_1(s) \quad \text{with ROC } R_1 \]
  \[ x_2(t) \overset{\mathcal{L}}{\rightarrow} X_2(s) \quad \text{with ROC } R_2 \]

- Then
  \[ x_1(t) * x_2(t) \overset{\mathcal{L}}{\rightarrow} X_1(s)X_2(s) \quad \text{with ROC containing } R_1 \cap R_2 \]
LTI system characterization

Key points
- Connect causality and stability to the Laplace transform
- Determine the Laplace transform of a LCCDE
Laplace transform is used for system analysis

\[ x(t) \rightarrow h(t) \rightarrow y(t) \rightarrow Y(s) = H(s)X(s) \]

- Many properties that were studied about systems can be interpreted in terms of the LT
  - Causality – related to the type of ROC
  - Stability – whether the ROC includes the unit circle

- Linear constant coefficient differential equations have a nice interpretation in terms of the Laplace transform
Causal systems

Example

\[ h(t) = e^{-t}u(t) \]

\[ H(s) = \frac{1}{s + 1}, \quad \text{Re}(s) > -1 \]

ROC associated with the system function for a causal system is a right-half plane.
An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$ axis.

- Causal system example

Corollary: for a causal system the poles must be in the left hand plane.
Laplace transforms of LCCDE

- Consider the LCCDE

\[
\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}
\]

- The corresponding Laplace transform is

\[
H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}
\]

- Roots are called zeros

- Roots are called poles

- Need further information to specify the ROC

If the system is causal, the ROC is to the right of the rightmost pole
Example

A causal LTI system with impulse response \( h(t) \) has input & output \( x(t), y(t) \) related through the LCCDE

\[
\frac{d^3 y}{dt^3} + (1 + \alpha) \frac{d^2 y}{dt^2} + \alpha(1 + \alpha) \frac{dy}{dt} + \alpha^2 y(t) = x(t)
\]

- If \( g(t) = \frac{dh(t)}{dt} + h(t) \). How many poles does \( G(s) \) have?

- For what real values of the parameter \( \alpha \) is \( S \) guaranteed to be stable?
Revising filtering

Key points
- Explain low-pass filters and the relation to the integrators
- Explain high-pass filters and the relation to the integrators
Low-pass filters (LPFs)

- Systems that pass low frequencies, attenuate high frequencies

System gain

\[ |H(j\omega)| \]

Pass

Attenuate

Cut-off frequency

Frequency

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**1st order low-pass filters**

- Consider the 1st order differential equation

\[
\frac{dy}{dt} + Ay(t) = x(t)
\]

\[
\left( s + A \right) Y(s) = \frac{1}{P(s)} X(s)
\]

\[
H(j\omega) = \frac{P(j\omega_0)}{Q(j\omega_0)} = \frac{1}{j\omega_0 + A}
\]

\[
|H(j\omega)|(dB) = -10 \log_{10} (\omega^2 + A^2)
\]

-20 dB/decade
The integrator is a LPF

\[ x(t) \xrightarrow{\int_{-\infty}^{t}} y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]

- Why is it a LPF?
  - Consider  
    \[ y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]
  - Its frequency response  
    \[ \mathcal{F} \left\{ \int_{-\infty}^{t} x(\tau) d\tau \right\} = \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega) \]

\[ X(0) = \int_{-\infty}^{\infty} x(t) dt \]

Attenuation at high frequency
The integrator is a LPF

- Integrator kills the high frequency components [smoother]

\[ x(t) \rightarrow \int_{-\infty}^{t} . \, dt \rightarrow y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau \]

\[ \mathcal{F} \left\{ \int_{-\infty}^{t} x(\tau) \, d\tau \right\} = \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega) \]
The integrator is a LPF

- Difference between \( H(j\omega) = \frac{1}{j\omega} \) \& \( H(j\omega) = \frac{1}{j\omega + a} \)

- In the time domain, this corresponds to

\[
H(j\omega) = \frac{1}{j\omega} \quad \Rightarrow \quad x(t) = u(t)
\]
\[
H(j\omega) = \frac{1}{j\omega + a} \quad \Rightarrow \quad x(t) = e^{-at}u(t)
\]
The integrator is a LPF

\[ |H(j\omega)| \]

- **Example: CAT-V cable (ethernet)**
  - High frequency get attenuated at larger distances
  - This is due to the accumulated capacitance (integrator effect)
  - Can not send above 50 MHz over a few hundred feet
The integrator is a LPF

Example (cont.): Impact on the time domain

One solution for this problem → shaping the transmitted signal

high frequencies

interference between successive symbols

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High-pass filters (HPFs)

- Systems that pass high frequencies, attenuate low frequencies
Differentiator is a HPF

\[ y(t) = \frac{dx(t)}{dt} \]

- Why is it a HPF?
  - Consider
  - Its frequency response is \( Y(j\omega) = j\omega X(j\omega) \)

Attenuation at low frequency
Differentiator is a HPF

- Filter frequency response

\[ H(j\omega) = j\omega \]

\[ |H(j\omega)| = \omega \]

- Gain in dB

\[ |H(j\omega)| = 20 \log |\omega| \]
Differentiator is a HPF

- In discrete time, HPF can be easily implemented

\[
\frac{d}{dt} y(t) = \lim_{\Delta t \to 0} \frac{y(t) - y(t - \Delta t)}{\Delta t}
\]

\[
y[n] = x[n] - x[n - 1]
\]
HPF’s

- What do HPF’s do?
  - Amplify high frequencies
  - Find or identify rapid changes
  - In DT, they find the “edges”

- HPF’s are somewhat uncommon in nature
- HPF’s are useful for many applications
  - Compression
  - Computer vision
  - Synchronization

*source: http://cs.brown.edu/courses/cs143/proj5*
Summary

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◆ LTI system characterization

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  ✦ Low-pass filters
  ✦ High-pass filters