Preview of today’s lecture

◆ Brief review
  ✦ Revisiting filtering

◆ Z-transform
  ✦ Definition of Z-transform
  ✦ Region of convergence
  ✦ Some Z-transform pairs
  ✦ Z-transform properties
Brief review: Revising filtering

Key points
- Explain low-pass filters and the relation to the integrators
- Explain high-pass filters and the relation to the integrators
Low-pass filters (LPFs)

- Systems that pass low frequencies, attenuate high frequencies

System gain

$$|H(j\omega)|$$

Pass

Attenuate

cut-off frequency

$$\omega$$ frequency
**1st order low-pass filters**

- Consider the 1st order differential equation

\[
\frac{dy}{dt} + Ay(t) = x(t)
\]

\[
(s + A)Y(s) = \frac{1}{Q(s)} X(s) = \frac{1}{P(s)}
\]

\[
H(j\omega) = \frac{P(j\omega)}{Q(j\omega)} = \frac{1}{j\omega_0 + A}
\]

\[
|H(j\omega)|(dB) = -10\log_{10}(\omega^2 + A^2)
\]

-20 dB/decade

(log scale)
The integrator is a LPF

\[ x(t) \rightarrow \int_{-\infty}^{t} (\cdot) \, dt \rightarrow y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau \]

- Why is it a LPF?
  - Consider \( y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau \)
  - Its frequency response
    \[ \mathcal{F} \left\{ \int_{-\infty}^{t} x(\tau) \, d\tau \right\} = \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega) \]
    \[ X(0) = \int_{-\infty}^{\infty} x(t) \, dt \]
    Attenuation at high frequency
The integrator is a LPF

- Integrator kills the high frequency components [smoother]

\[
x(t) \rightarrow \int_{-\infty}^{t} (.) dt \rightarrow y(t) = \int_{-\infty}^{t} x(\tau) d\tau
\]

\[
\mathcal{F} \left\{ \int_{-\infty}^{t} x(\tau) d\tau \right\} = \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)
\]
The integrator is a LPF

- Difference between \( H(j\omega) = \frac{1}{j\omega} \) & \( H(j\omega) = \frac{1}{j\omega + a} \)

- In the time domain, this corresponds to

\[
H(j\omega) = \frac{1}{j\omega} \quad \Rightarrow \quad x(t) = u(t)
\]

\[
H(j\omega) = \frac{1}{j\omega + a} \quad \Rightarrow \quad x(t) = e^{-at}u(t)
\]
The integrator is a LPF

\[ |H(j\omega)| \]

- Example: CAT-V cable (ethernet)
  - High frequency get attenuated at larger distances
  - This is due to the accumulated capacitance (integrator effect)
  - Can not send above 50 MHz over a few hundred feet
The integrator is a LPF

◆ Example (cont.): Impact on the time domain

One solution for this problem → shaping the transmitted signal

high frequencies

interference between successive symbols

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High-pass filters (HPFs)

- Systems that pass high frequencies, attenuate low frequencies

![Diagram showing high-pass filter behavior with cut-off frequency]

- **Attenuate** above cut-off frequency
- **Pass** below cut-off frequency

System gain vs. frequency
Differentiator is a HPF

\[ x(t) \xrightarrow{\frac{d(.)}{dt}} y(t) = \frac{dx}{dt}, \]

Why is it a HPF?
- Consider
  \[ y(t) = \frac{dx(t)}{dt} \]
- Its frequency response is
  \[ Y(j\omega) = j\omega X(j\omega) \]

Attenuation at low frequency
Differentiator is a HPF

- Filter frequency response

\[ H(j\omega) = j\omega \]

\[ |H(j\omega)| = \omega \]

- Gain in dB

\[ |H(j\omega)| = 20 \log |\omega| \]
Differentiator is a HPF

- In discrete time, HPF can be easily implemented

\[
\frac{d}{dt} y(t) = \lim_{\Delta t \to 0} \frac{y(t) - y(t - \Delta t)}{\Delta t}
\]

\[
y[n] = x[n] - x[n - 1]
\]
What do HPF's do?
- Amplify high frequencies
- Find or identify rapid changes
- In DT, they find the “edges”

HPF’s are somewhat uncommon in nature
HPF’s are useful for many applications
- Compression
- Computer vision
- Synchronization

Edge detection example*

*source: http://cs.brown.edu/courses/cs143/proj5
Z-transform

Key points
○ Calculate Z-transform of DT functions
○ Determine the region of convergence of Z-transforms
○ Use Z-transform properties to solve signals & systems problems
Z-transform

- Definition of Z-transform
- Region of convergence of Z-transform
- Some Z-transform pairs
- Z-transform properties
Review of the basics

- DT signals and LTI DT systems

\[ \delta[n] \rightarrow \text{DT system} \rightarrow h[n] \]
\[ x[n] \rightarrow h[n] \rightarrow y[n] = h[n] \ast x[n] \]

- \( x[n] \) input sequence \( x[n] \triangleq x(nT), \ T = \text{sampling period} \)
- \( h[n] \) impulse response
- \( y[n] \) output sequence

\[ y[n] = h[n] \ast x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n - k] \]
Eigenfunction and eigenvalue

- Consider the input
  \[ x[n] = \alpha^n \]

  any complex number

- Then, the output will be

  \[
  y[n] = \sum h[k] \alpha^{n-k}
  \]

  \[
  = \alpha^n \sum_{k=-\infty}^{\infty} h[k] \alpha^{-k}
  \]

  eigenfunction

  \[
  H(\alpha) = \text{eigenvalue}
  \]
Definition of Z-transform

- For any DT signal $x[n]$, the Z-transform of this signal is defined as

$$X(z) = Z \{x[n]\} \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- where $z$ is a complex number, $z = Re^{j\theta}$
Analogy with FT & LT

- In LT, we have
  \[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt, \quad (s \text{ is complex} = \sigma + j\omega) \]

  \[ e^{st} \quad \rightarrow \quad h(t) \quad \rightarrow \quad e^{st}H(s) \]

- Similarly, in Z-transform
  \[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

  \[ z^n \quad \rightarrow \quad h[n] \quad \rightarrow \quad H(z)z^n \]
Region of convergence (ROC) of Z-transform

- ROC of a Z-transform is the region of the complex number \( z \) where the Z-transform converges (exists).

\[ z = R e^{j\theta} \]
Example

- Consider the signal

\[ x[n] = \left( \frac{1}{5} \right)^n u[n - 3] \]

- Evaluate its Z-transform and specify its region of convergence

- Solution:
Example

Let

\[ x[n] = (-1)^n u[n] + \alpha^n u[-n - n_0] \]

Determine the constraints on the complex number \( \alpha \) and the integer \( n_0 \), given that the ROC of \( X(z) \) is

\[ 1 < |z| < 2 \]
Some Z-transform pairs

- Consider the signal

\[ x[n] = a^n u[n] \]

- Determine its Z-transform and ROC

**Solution**

\[ X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \]

\[ = \sum_{n=0}^{\infty} \left( \frac{a}{z} \right)^n \]

\[ = \frac{1}{1 - \frac{a}{z}}, \quad \text{if } \left| \frac{a}{z} \right| < 1, \text{ else } = \infty \]

This decides the ROC
Some Z-transform pairs

- Z-transform of \( x[n] = a^n u[n] \) (cont.)
  - Solution
  \[
  X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\
  = \frac{1}{1 - a z^{-1}} \\
  \]
  - ROC: \(|z| > |a|\)

\( a \) is a unit circle in the \( z \)-plane.
Some Z-transform pairs

- Notes on ROC of
  - $z = a$ is a pole of $X(z)$
  - If the ROC includes the unit circle $\rightarrow$ stable

- Example, for the shown system
  $$a = \frac{1}{2} + \frac{1}{2}j$$

and the ROC includes the unit circle
Some Z-transform pairs

- Consider the signal
  \[ x[n] = u[n] \]

- Determine its Z-transform and ROC

- Solution:
  - This is a special case of the signal \( x[n] = a^n u[n] \) with \( a = 1 \)
  - Z-transform \( X(z) = \frac{z}{z - 1}, \quad |z| > 1 \)
  - Note: ROC does not include the unit circle \( \Rightarrow \) Not stable
Some Z-transform pairs

- Consider the signal

\[ x[n] = \delta[n] \]

- Determine its Z-transform and ROC

- Solution:

\[ X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1, \quad \text{(same as FT of } \delta(t)) \]

- Another solution

\[ \delta[n] = u[n] - u[n - 1] \]

\[ \mathcal{Z}\{\delta[n]\} = \mathcal{Z}\{u[n]\} - \mathcal{Z}\{u[n - 1]\} = 1 \]
Some Z-transform pairs

- Consider the signal

\[ x[n] = e^{j\omega_0 n} u[n] \]

- Determine its Z-transform and ROC

- Solution:
  - This is a special case of the signal \( x[n] = a^n u[n] \) with \( a = 1 \cdot e^{j\omega_0} \)
  - Z-transform

\[ X(z) = \frac{z}{z - e^{j\omega_0}} \quad \text{ROC} \quad |z| > 1 \]
Some Z-transform pairs

- Consider the signal \( x[n] = \left( \frac{1}{2} \right)^n \cos(10n)u[n] \)

- Determine its Z-transform and ROC

Solution:

- The signal can be written as

\[
x[n] = \left( \frac{1}{2} \right)^n \cos(10n)u[n] \\
= \left( \frac{1}{2} \right)^n \frac{1}{2} (e^{j10n} + e^{-j10n})u[n] \\
= \frac{1}{2} \left[ \left( \frac{1}{2} e^{10j} \right)^n + \left( \frac{1}{2} e^{-10j} \right)^n \right] u[n]
\]
Some Z-transform pairs

- Z-transform of $x[n] = \left(\frac{1}{2}\right)^n \cos(10n)u[n]$ (cont.)

- It can be written as

$$x[n] = \frac{1}{2} \left[ \left(\frac{1}{2}e^{10j}\right)^n + \left(\frac{1}{2}e^{-10j}\right)^n \right] u[n]$$

- Z-transform is

$$X(z) = \frac{1}{2} \cdot \frac{z}{z - a_1} + \frac{1}{z} \cdot \frac{z}{z - a_2}$$

$$= \frac{z}{2z - e^{10j}} + \frac{z}{2z - e^{-10j}}$$

$$= \frac{4z^2 - z(e^{-10j} + e^{10j})}{4z^2 - 2z(e^{10j} - e^{-10j}) + 1}$$

$$X(z) = \frac{4z^2 - 2\cos(10) \cdot z}{4z^2 - 4z \cos(10) + 1}$$
Some Z-transform pairs

- **Z-transform of** \( x[n] = \left( \frac{1}{2} \right)^n \cos(10n) u[n] \) **is** (cont.)

\[
X(z) = \frac{4z^2 - 2 \cos(10) \cdot z}{4z^2 - 4z \cos(10) + 1}
\]

- What are the zeros and poles?
  - **Zeros** at \( X(z)=0 \)
    \[
z = 0 \quad | \quad 4z - 2 \cos(10) = 0
    \]
    \[
z = \frac{1}{2} \cos(10) = -0.42
    \]
  - **Poles** at \( X(z)=\infty \)
    \[
2z - e^{10j} = 0 \rightarrow z_1 = \frac{1}{2} e^{10j}
    \]
    \[
2z + e^{10j} = 0 \rightarrow z_2 = \frac{1}{2} e^{-10j}
    \]

\[z_1 = z_2^*, \text{ (complex conjugate)}\]
Some Z-transform pairs

- Z-transform of $x[n] = \left(\frac{1}{2}\right)^n \cos (10n)u[n]$ is (cont.)

$$X(z) = \frac{4z^2 - 2 \cos (10) \cdot z}{4z^2 - 4z \cos (10) + 1}$$

- ROC
- Poles at $2z - e^{10j} = 0 \rightarrow z_1 = \frac{1}{2}e^{10j}$
  $2z + e^{10j} = 0 \rightarrow z_2 = \frac{1}{2}e^{-10j}$

- Notes $z_1 = z_2^*$, (complex conjugate)
  $10 = 3.18\pi$

- ROC $|z| > \frac{1}{2} \rightarrow$ stable
Some Z-transform pairs

- Consider the signal

\[ x[n] = \delta[n - m], \quad \text{Delay by } m \]

- Determine its Z-transform and ROC

- **Solution:**

\[
X(z) = \sum_{n=-\infty}^{\infty} \delta[n - m]z^{-n}
\]

\[
= \sum_{n} \delta[n - m]z^{-m}
\]

\[
= z^{-m} \sum_{n} \delta[n - m]
\]

\[
= z^{-m}
\]
Some Z-transform pairs

- Consider the signal

- Determine its Z-transform and ROC

- Solution: \( X(z) = 1 + z^{-2} - 0.5z^{-3} + 2z^{-4} \)

- Note: For finite DT signals, you can get the ZT immediately

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Example

- Determine the Z-transform of the following sequence, sketch the poles and zeros, and determine the ROC

\[ x[n] = \left( \frac{1}{2} \right)^n \{ u[n + 4] - u[n - 5] \} \]

- Solution
Summary

- Brief review
  - Revisiting filtering

- Z-transform
  - Definition of Z-transform
  - Region of convergence
  - Some Z-transform pairs
  - Z-transform properties