Lecture #23

EE 313 Linear Systems and Signals

Professor Robert W. Heath Jr.
Announcements

◆ Turn in midterm regrade requests if you have not already
  ✤ Final requests are due by April 30

◆ Please fill in survey about topics for final review in class
Preview of today’s lecture

◆ Quiz review

◆ Z-transform
  ✦ Review
  ✦ Z-transform properties
  ✦ Using Z-transform to characterize differential equations

◆ DT Fourier transform (DTFT)
  ✦ Definition
  ✦ Examples
Quiz review
Quiz review 1/4

1. (100 points) Let \( x(t) \) be the sampled signal specified as

\[
x(t) = \sum_{n=0}^{\infty} e^{-nT} \delta(t - nT),
\]

where \( T > 0 \).

(a) Determine \( X(s) \), and its region of convergence.

(b) Sketch the pole-zero plot of \( X(s) \)
Quiz review 2/4

basic Laplace transform identity

Note that

\[ \delta(t - nT) \xrightarrow{\mathcal{L}} e^{-snT}, \text{ all } s. \]

Therefore, the Laplace transform of \( x(t) \) is given by

\[
X(s) = \sum_{n=0}^{\infty} e^{-nT} e^{-snT} = \frac{1}{1 - e^{-(s+1)T}}.
\]
To determine the ROC, we first find the poles of $X(s)$. The poles of $X(s)$ occur when $e^{-(s+1)T} = 1$. This means that the poles of $X(s)$ satisfies the following equation

$$e^{-(s_k+1)T} = e^{j2\pi k}, \quad k = 0, \pm 1, \pm 2, \ldots$$

(5)

Taking the logarithm of both sides of the equation and simplifying, we get

$$s_k = -1 + \frac{j2\pi k}{T}, \quad k = 0, \pm 1, \pm 2, \ldots$$

(6)

Hence, all the poles of $X(s)$ lie on a vertical line (parallel to the $j\omega$ axis) and passing through $s = -1$. Since the signal is right-sided, the ROC of $X(s)$ is $\text{Re}\{s\} > -1$. 
Quiz review 4/4

Figure 1: The pole-zero plot of $X_s$. 

The pole-zero plot is as shown in the Figure 1.
Z-transform review

Key points
- The Z-transform is the discrete-time equivalent of the Laplace transform
- Is associated with a Region of Convergence
Definition of Z-transform

For any DT signal $x[n]$, the Z-transform of this signal is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where $z$ is a complex number, more naturally written in polar form

$$z = Re^{j\theta}$$
**Region of convergence (ROC) of Z-transform**

- ROC of a Z-transform is the region of the complex number $z$ where the Z-transform converges (exists)

$$z = R e^{j\theta}$$

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Z-transform pairs (more common)

<table>
<thead>
<tr>
<th>TABLE 10.2 SOME COMMON z-TRANSFORM PAIRS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Signal</strong></td>
</tr>
<tr>
<td>1. ( \delta[n] )</td>
</tr>
<tr>
<td>2. ( u[n] )</td>
</tr>
<tr>
<td>3. ( -u[-n-1] )</td>
</tr>
<tr>
<td>4. ( \delta[n-m] )</td>
</tr>
<tr>
<td>5. ( \alpha^n u[n] )</td>
</tr>
<tr>
<td>6. ( -\alpha^n u[-n-1] )</td>
</tr>
</tbody>
</table>
# Z-transform pairs (less common)

<table>
<thead>
<tr>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. $n\alpha^n u[n]$</td>
<td>$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>8. $-n\alpha^n u[-n-1]$</td>
<td>$\frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>9. $[\cos \omega_0 n]u[n]$</td>
<td>$\frac{1 - [\cos \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>10. $[\sin \omega_0 n]u[n]$</td>
<td>$\frac{[\sin \omega_0]z^{-1}}{1 - [2 \cos \omega_0]z^{-1} + z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>11. $[r^n \cos \omega_0 n]u[n]$</td>
<td>$\frac{1 - [r \cos \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>12. $[r^n \sin \omega_0 n]u[n]$</td>
<td>$\frac{[r \sin \omega_0]z^{-1}}{1 - [2r \cos \omega_0]z^{-1} + r^2 z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>
Z-transform properties

Key points
- Enumerate key properties of the Z-transform
- Use Z-transform properties to solve signals & systems problems
Z-transform properties

- Why study properties?
  - Use the Z-transform properties to simplify calculation of the Z-transform and its inverse

- Properties are similar to other transforms we have studied
  - Be sure to make note of what happens to the ROC

- Full properties are listed in Table 10.1 of the book
**Left or right shift (advance / delay)**

- Recall that
  \[
  \delta[n - m] \leftrightarrow z^{-m} \cdot \frac{1}{z\{\delta[n]\}}
  \]
- For general signals
  \[
  x[n - m] \leftrightarrow z^{-m} X(z)
  \]
  \[
  x[n + m] \leftrightarrow z^m X(z)
  \]

Delay equivalent to \(z^{-1}\)

\[
\begin{align*}
  x[n] & \rightarrow z^{-1} \rightarrow x[n - 1]
\end{align*}
\]
Conjugate symmetry

\[ x_1[n] \ast x_2[n] \leftrightarrow X_1(z)X_2(z) \]

ROC is at least the intersection of each separate RCO

- Note the application to LTI system output

\[ x[n] \rightarrow h[n] \rightarrow y[n] = h[n] \ast x[n] \]

- Z-transform

\[ Y(z) = H(z)X(z) \]

- Transfer function

\[ H(z) = \frac{Y(z)}{X(z)} = \mathcal{Z}\{h[n]\} \]
Conjugate symmetry

\[ x^*[n] \leftrightarrow X^*(z^*) \quad \text{Same ROC} \]

- Application: if \( x[n] \) is real, then

\[ X(z) = X^*(z^*) \]

- For real signals, poles of \( X(z) \) have to be complex conjugates

\[ X(z) = \frac{1}{(z - a_1)(z - a_2)} \quad \rightarrow \quad X^*(z^*) = \frac{1}{(z^* - a_1)(z^* - a_2)} \]
Example

- Determine the Z-transform of the following sequence, sketch the poles and zeros, and determine the ROC

\[ x[n] = \left(\frac{1}{2}\right)^n \{u[n + 4] - u[n - 5]\} \]

- Solution
Inverse Z-transform

Key points

- Determine the inverse Z-transform of a DT signal using partial fraction expansion
Inverse Z-transform

- Actual Z-transform computation requires complex integration

\[ x[n] = \frac{1}{2\pi j} \oint X(z)z^{n-1} \, dz \]

- Involves counter clockwise contour integration
- Calculation is beyond the scope of this course

- Use partial fraction expansions with factors of the form

\[ \frac{1}{1 - az^{-1}} \]

- \( a^n u[n] \) for \( |z| > |a| \)
- \( -a^n u[-n - 1] \) for \( |z| < |a| \)
Determining the impulse response using the Z-transform

Key points
- Use Z-transform to determine the impulse response for a difference equation
- Use partial fraction expansions and transforms from the table to find the impulse response in the time domain
Using Z-transform to characterize system response

- Consider the difference equation
  \[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]

- Determine \( H(z) \)
  - Take Z-transform of both sides, use linearity & time-shift properties
    \[ \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \]
  - Then
    \[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum b_k z^{-k}}{\sum a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots} \]
Using Z-transform to characterize system response

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum b_k z^{-k}}{\sum a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots} \]

◆ Observations
  ✤ Numerator of \( H(z) \) can be expressed by an \((M+1)\times1\) vector \( \mathbf{a} \)
  ✤ Denominator of \( H(z) \) can be expressed by an \((N+1)\times1\) vector \( \mathbf{b} \)
  ✤ Using MATLAB: \texttt{poles} = \texttt{root(a)}, \texttt{zeros} = \texttt{root(b)}
  ✤ Other useful MATLAB commands

\texttt{fft(.)} \quad \texttt{freqz(.)} \quad \texttt{zplane(.)} \quad \texttt{filter(.)} \quad \texttt{impz(.)}
Example

- Consider the causal system described by the DE

\[ y[n] - 2y[n - 1] = x[n - 1] \]

- Determine the system transfer function \( H(z) \)
- What is the system impulse response \( h[n] \)?
- Is this system stable?

Assume system is at rest if no other information is given
Solution

To get $H(z)$

\[ Y(z) - 2z^{-1}Y(z) = z^{-1}X(z) \]
\[ H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 2z^{-1}} = \frac{1}{z - 2} \]

To get $h[n]$

\[ h[n] = \mathcal{Z}^{-1}\{H(z)\} \]
\[ \mathcal{Z}^{-1}\left\{z^{-1} \cdot \frac{1}{1 - 2z^{-1}} \right\} = z^n u[n] \]
\[ = 2^{n-1} u[n - 1] \]

The system is unstable (ROC does not include the unit circle)

this is the notation we use in our book

this advance notation is used in some books
Example

◆ Consider the causal system

\[ y[n] + \frac{3}{4}y[n - 1] + \frac{1}{8}y[n - 2] = x[n] \]

◆ If the input is

\[ x[n] = \left(\frac{3}{4}\right)^n u[n] \]

◆ What is \( y[n] \)?
Solution

- Use the Z-transform to find the output

\[ y[n] + \frac{3}{4}y[n - 1] + \frac{1}{8}y[n - 2] = x[n] \]

\[ Y(z) \left( 1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \right) = \frac{1}{1 - \frac{3}{4}z^{-1}} \]

\[ Y(z) = \frac{1}{(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{4}z^{-1})(1 + \frac{1}{2}z^{-1})} \]
Solution

- Via PFE

\[
Y(z) = \frac{1}{(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{4}z^{-1})(1 + \frac{1}{2}z^{-1})} \\
= \frac{A}{1 + \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{4}z^{-1}} + \frac{C}{1 - \frac{3}{4}z^{-1}}
\]

- Use the “Cover up” method to determine A, B, C

\[
A = Y(z) \left(1 + \frac{1}{2}z^{-1}\right) \bigg|_{z=\frac{1}{2}} = \frac{1}{(1 + \frac{3}{4})\left(1 - \frac{1}{4}\right)} = \frac{4}{5}
\]

\[
B = \frac{1}{(1 + \frac{3}{4})(1 - \frac{1}{2})} = -\frac{1}{4}
\]

\[
C = \frac{1}{(1 + \frac{1}{4})\left(1 + \frac{1}{2}\right)}
\]
Solution

◆ Using the table

\[ Y(z) = \frac{\frac{4}{5}}{1 + \frac{1}{2}z^{-1}} - \frac{\frac{1}{4}}{1 + \frac{1}{4}z^{-1}} + \frac{\frac{9}{20}}{1 - \frac{3}{4}z^{-1}} \]

\[ y[n] = \frac{4}{5} \left(-\frac{1}{2}\right)^n u[n] - \frac{1}{4} \left(\frac{1}{4}\right)^n u[n] + \frac{9}{20} \left(\frac{3}{4}\right)^n u[n] \]

◆ Sanity check: \( y[0] = x[0] = 1 \)

\[ \frac{4}{5} \cdot \frac{1}{4} + \frac{9}{20} = \frac{16 - 5 + 9}{20} = \frac{20}{20} = 1 \]
Discrete-time Fourier transform

Key points

- Define DTFT and explain its connection to Z-transform
- Use DTFT properties to solve problems
Recall the CT Fourier transform

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

- We saw that this is a special case of LT with \( \sigma = 0 \) (\( s = j\omega \))

\[ X(s) = \int x(t) e^{-st} dt, \quad s = \sigma + j\omega, \text{(a complex \#)} \]
Eigenfunctions for discrete-time signals

\[ x[n] \xrightarrow{\gamma^n} H(z) \xrightarrow{H(\gamma)\gamma^n} \]

\[ \gamma = Re^{j\omega} \quad \text{complex #} \]

\[ x[n] \xrightarrow{\gamma^n} H(z) \xrightarrow{H(e^{j\omega})e^{j\omega n}} \]

\[ e^{j\omega n} \xrightarrow{H(z)} H(e^{j\omega})e^{j\omega n} \]

purely oscillatory input \( \gamma = e^{j\omega} \)

Discrete-time Fourier transform of \( h[n] \), Z-transform at \( z = e^{j\omega} \)
What is DTFT?

- Generally

\[ X(z) = X(e^{j\omega}) \rightarrow H(z) \rightarrow H(e^{j\omega})X(e^{j\omega}) \]

Where

\[ X(e^{j\omega}) \triangleq DTFT\{x[n]\} \]

\[ H(e^{j\omega}) \triangleq DTFT\{h[n]\} \]

- So, DTFT is the Z-transform \( z = e^{j\omega} \)
Defining the DTFT

- For any DT signal $x[n]$, the DTFT is defined as

$$\text{DTFT}\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- For signals where the unit circle is in the ROC then

$$\text{DTFT}\{x[n]\} = \mathcal{Z}\{x[n]\}|_{z=e^{j\omega}}$$

- Certain special periodic signals have “defined” transforms
Example – Simple DTFT calculation

- Calculate the DTFT of the signal
  
  \[ x[n] = \alpha^n u[n], \quad |\alpha| < 1 \]

- Solution
  
  - Applying the DTFT definition
    
    \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} \]
    
    \[ = \sum_{n=0}^{\infty} \alpha^n (e^{-j\omega})^n \]
    
    \[ = \frac{1}{1 - \alpha e^{-j\omega}} \]
Example – Plotting simple DTFT

- Plot $|X(e^{j\omega})|$ for

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$= \frac{1}{1 - \alpha (\cos \omega - j \sin \omega)}$$

$$= \frac{1}{(1 - \alpha \cos \omega) + j \alpha \sin \omega}$$

- Note that

$$|X(e^{j\omega})|^2 = \frac{1}{(1 - \alpha \cos \omega)^2 + (\alpha \sin \omega)^2}$$

$$= \frac{1}{1 - 2\alpha \cos \omega + \alpha^2 \cos^2 \omega + \alpha^2 \sin^2 \omega}$$

$$= \frac{1}{1 + \alpha^2 - 2\alpha \cos \omega}$$
Example (cont.)

Let us plot $|X(e^{j\omega})|$

Very important note: DTFT is periodic with period $2\pi$

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)}) \text{ as } e^{j(\omega+2\pi)} = e^{j\omega}$$
Example – System that adds a delayed replica

Consider a digital communication system

\[ x[n] \xrightarrow{H(z)} y[n] = x[n] + x[n - 1] \]

- What is \( H(z) \) and \( H(e^{j\omega}) \)?
- Plot \(|H(e^{j\omega})|\)

Example of an intersymbol interference channel
Example

\[ x[n] \xrightarrow{H(z)} y[n] = x[n] + x[n - 1] \]

◆ To get \( H(z) \)

\[ Y(z) = X(z) + z^{-1}X(z) \]

\[ \rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 + z^{-1} \]

◆ Then

\[ h[n] = \delta[n] + \delta[n - 1] \]

\[ H(e^{j\omega}) = 1 + e^{-j\omega} = 1 + \cos \omega - j \sin \omega \]
Example

To plot $|H(e^{j\omega})|$:

$$|H(e^{j\omega})|^2 = (1 + \cos \omega)^2 + (-\sin \omega)^2$$

$$= 1 + 2 \cos \omega + \cos^2 \omega + \sin^2 \omega$$

$$= 2 + 2 \cos \omega = 2(1 + \cos \omega)$$

$$|H(e^{j\omega})| = \sqrt{2(1 + \cos \omega)}$$

This is a LPF

Figure 1:

- $H(e^{j\omega})$ with DC gain
- DT frequency response is periodic! (with period $2\pi$)

Example 2: The “Delay Channel”

$$H(z) = x[k] + x[k-1]$$

data stream has perfect delay reflection output

(a) What's $H(z)$ and $H(e^{j\omega})$?

(b) Plot $H(e^{j\omega})$ and interpret it
Inverse DTFT

- Given the DTFT $X(e^{j\omega})$ of a DT signal $x[n]$, this signal can be recovered as

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

- The integration is over any $2\pi$ period
- Generally it is possible to compute the integral
Example

- Determine the DT signal that corresponds to the following FT

\[ X(e^{j\omega}) = \begin{cases} 
1, & \frac{\pi}{4} \leq |\omega| \leq \frac{3\pi}{4} \\
0, & \frac{3\pi}{4} \leq |\omega| \leq \pi, 0 \leq |\omega| < \frac{\pi}{4} 
\end{cases} \]

- Solution

\[ x[n] = \text{IDFT}\{X(e^{j\omega})\} \]

\[ = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{\pi/4}^{3\pi/4} e^{j\omega n} d\omega \]

\[ = \frac{1}{\pi n} [\sin(3\pi n/4) - \sin(\pi n/4)] \]
Summary

♦ Z-transform
  ✦ Z-transform properties
  ✦ Using Z-transform to characterize differential equations

♦ DT Fourier transform (DTFT)
  ✦ Definition
  ✦ Examples