Lecture #3

EE 313 Linear Systems and Signals

Professor Robert W. Heath Jr.
Announcements

◆ Lecture 4 is also posted

◆ Office hours are finalized
  ✦ Enoch Yeh – MW 3-6pm in ACA 112
  ✦ Ahmed Alkhateeb – Thursday 1:30-4:30 in ACA 114
  ✦ Prof. Heath – Tuesday 2:30-5:00pm in ACA 151

◆ Homework #1 is due this Thursday
Preview of today’s lecture

- **Calculations using the dB**
  - The dB is way of making relative power measurements
  - Perform calculations using the dB

- **Complex exponential review**
  - Determine the period and fundamental frequency
  - Explain the differences between continuous and discrete-time

- **Unit-impulse and unit-step functions**
  - Define and use the unit-pulse and unit-step functions
Calculations using the dB

Learning objectives

- Use the dB as a relative power measurement
- Perform calculations using the dB
Decibel

Input

\[ x(t) \]

System

Output

\[ y(t) = Ax(t) \]

\[ P_x = \frac{1}{T} \int_T^T |x(t)|^2 dt \]

\[ P_y = \frac{1}{T} \int |y(t)|^2 dt \]

\[ = \frac{1}{T} \int |Ax(t)|^2 dt \]

\[ = \frac{1}{T} A^2 \int |x(t)|^2 dt \]

\[ P_y = A^2 P_x \]
Decibel

- Gain in decibels (dB) is a relative power measurement

\[
\text{Gain (dB)} = 10 \log_{10} \frac{P_y}{P_x} = 10 \log_{10} A^2 = 20 \log_{10} A
\]

- It has to be relative so the units (e.g. Watts) disappear
- The dB is also used as a measure of power relative to 1W or 1mW

\[
P_x(dBW) = 10 \log_{10} \frac{P_x}{1W}
\]

\[
P_x(dBm) = 10 \log_{10} \frac{P_x}{1mW}
\]

To add further confusion, dBW is often just written dB
Example

\[ P_x \rightarrow -104 \text{ dB} \rightarrow P_y = A^2 P_x \]

\[ P_x = 1W = 1000mW \]

Because \[ \text{Gain (dB)} = 10 \log_{10} \frac{P_y}{P_x} = 10 \log_{10} A^2 = 20 \log_{10} A \]

And \[ G(dB) = -104 \text{ dB} \text{ it follows that } A = 10^{-104/20} \]

The received power is

\[ P_y = P_x + G(dB) = 30dBm - 104dB = -74dBm \]

Where we use the fact that

\[ 30dBm \rightarrow 1 \text{ Watt} \rightarrow 1000mW \]
Complex exponential review

Learning objectives

- Determine the period and fundamental frequency
- Explain the differences between continuous and discrete-time
CT complex exponential

General form

\[ x(t) = C e^{at} = |C| e^{j\theta} e^{at}, \quad C, a \text{ are complex} \]

\[ C = c_1 + j c_2 \]

Cartesian to polar

\[ |C| = \sqrt{c_1^2 c_2^2} \quad \theta = \tan^{-1} \frac{c_2}{c_1} \]

Polar to cartesian

\[ c_1 = \cos \theta \cdot |C| \]
\[ c_2 = \sin \theta \cdot |C| \]
CT complex exponential

- **Rewriting in polar form**

\[ x(t) = Ce^{at} = |C|e^{j\theta}e^{at}, \quad C, a \text{ are complex} \]

\[ a = r + j\omega \]

Increasing or decaying exponential

Complex sinusoid

\[ \omega \quad \text{is the frequency} \]

**Figure 1:**

- Increasing or decaying exponential
- Complex sinusoid
- \( x(t) = |C|e^{rt}e^{j(\theta + \omega t)} \)
- \( r < 0 \)
- \[ \text{Re}\{x(t)\} = |C|e^{rt} \cdot \cos(\omega t + \theta), \quad T = \frac{2\pi}{\omega} \]
DT complex exponential

- General form is \( x[n] = C \alpha^n \)

- Rewriting using the polar form

\[
x[n] = |C| e^{j\theta} (Re^{j\omega_0})^n
\]

\[
= |C| R^n e^{j\theta} e^{j\omega_0 n}
\]

\[
= |C| R^n e^{j(\omega_0 n + \theta)}
\]

- Note: \(|C|R^n\) is complex envelope, \(\omega_0\) is frequency and \(\theta\) is the phase
DT complex exponential

Example

\[ x[n] = \alpha^n, \quad \alpha = -2 \]

\[ = (-2)^n \]

\[ = (-1)^n 2^n, \quad (ab)^n = a^n b^n \]
CT complex exponentials

- Consider the complex sinusoid

\[ x(t) = e^{j\omega t + \theta} \]

\[ = \cos(\omega t + \theta) + j \sin(\omega t + \theta) \]

- \( \omega \) is the frequency of the sinusoid
- \( \theta \) is the phase
- \( T = \frac{2\pi}{\omega} \) is the period

- Larger \( \omega \) leads to higher frequency and shorter period
DT complex exponentials

- Consider the complex sinusoid

\[ x[n] = e^{j\omega_0 n + \theta} \]

\[ = \cos(\omega_0 n + \theta) + j \sin(\omega_0 n + \theta) \]

- \( \omega_0 \) is the frequency of the sinusoid
- \( \theta \) is the phase
- Periodic only if \( \frac{2\pi}{\omega_0} \) is a rational number
- If periodic, then the period is \( N = m \frac{2\pi}{\omega_0} \)
  - \( m \) is the smallest integer such that \( N \) is an integer
DT complex exponentials – high & low frequency

- Observe the following fact for integers k and n
  \[ e^{j((\omega_0 + 2\pi k)n + \theta)} = e^{j(\omega_0 n + \theta + 2\pi kn)} = e^{j(\omega_0 n + \theta)}e^{j2\pi kn} = e^{j\omega_0 n + \theta} \]

- This means that frequencies \( \omega_0 + 2\pi k \) are equivalent!!
  - We normally report the smallest value as the frequency
  - Use range \( \omega_0 \in [0, 2\pi] \) or \( \omega_0 \in [-\pi, \pi] \)

- \( |\omega_0| \) near 0 is low while \( |\omega_0| \) near \( \pi \) is high
Illustrations of DT frequency

- Cosine with frequency 0.1
- Cosine with frequency 0.2
- Cosine with frequency 1.1
- Cosine with frequency 1.8
- Cosine with frequency 2
Example

- Consider the periodic discrete-time exponential time signal

\[ x[n] = e^{jm(2\pi/N)n} \]

Show that the fundamental period of this signal is \( N_0 = N / \text{gcd}(m, N) \)

- Solution
Unit–step and unit-impulse functions

Learning objectives

- Explain the properties of unit-impulse and unit-step functions
- Analyze problems that include unit-impulse and unit-step functions
DT unit-impulse function

$$\delta[n] = \begin{cases} 
1, & n = 0 \\
0, & n \neq 0 
\end{cases}$$

Also known as the Kronecker delta function
DT unit-impulse function

Example

\[ 5\delta[n - 3] \]

\[ 3\delta[n] - 2\delta[n + 1] + 2\delta[n - 1] \]
DT unit-impulse function

- Can build any DT sequence using the unit-impulse
Sifting property of the impulse function

Consider

\[ x[n] \delta[n] = x[0] \delta[n] \]
Sifting property of the impulse function

What about?

In general the sifting property

\[ x[n] \delta[n - n_0] = x[n_0] \delta[n - n_0] \]
DT unit-step function

\[ u[n] = \begin{cases} 
1, & n \geq 0 \\
0, & n < 0 
\end{cases} \]
DT unit-step function

- Example

Note: $u[n] - u[n - 3] = \delta[n] + \delta[n - 1] + \delta[n - 2]$
Using the sifting property with unit step functions

Example

\[ x[n] = \alpha^n u[n], \quad \alpha = 2 \]
\[ = 2^n u[n] \]

\[ y[n] = x[n] \delta[n - 10] \]
\[ = x[10] \delta[n - 10] \]
\[ = 2^{10} \delta[n - 10] \]
Connections between impulse and step functions

Important relations

\[
\delta[n] = u[n] - u[n - 1]
\]

\[
\delta[n - n_0] = u[n - n_0] - u[n - (n_0 + 1)]
\]

\[
u[n] = \sum_{m=-\infty}^{n} \delta[m]
\]

\[
u[n] = \sum_{m=0}^{n} \delta[n - m]
\]

Also

\[
\delta[n]\delta[n] = \delta[n]
\]

\[
u[n]u[n] = u[n]
\]
Example using connections

Consider the signal

\[ x[n] = 1 - \sum_{k=3}^{\infty} \delta[n - 1 - k] \]

Determine the values of \( M \) and \( n_0 \) so that \( x[n] \) can be expressed as

\[ x[n] = u[Mn - n_0] \]

Solution
CT unit step function

\[ u(t) \triangleq \begin{cases} 
0, & t < 0 \\
1, & t > 0 
\end{cases} \]

- At \( t=0 \), may be either 0, 1, 0.5 depending on the book
  - The specific choice is only important in a mathematical analysis class
CT unit step function examples

- Examples

\[ u(t - 3) \]

\[ u(t + \frac{1}{2}) - u(t - \frac{1}{2}) = \text{rect}(t) \]

This is another common function
CT unit-impulse function

- Impulse function is a very short pulse, with unit area $\int_{-\infty}^{\infty} \delta(t)dt = 1$

- Think about a rectangle function

  - Unit area $\frac{1}{2} \cdot 2\Delta \cdot \frac{1}{\Delta} = 1$.

  - As $\Delta \to 0$ this is $\delta(t)$

Also known as the Dirac delta function
CT unit-impulse function
CT unit-impulse function

Important relations

\[
\frac{du(t)}{dt} = \delta(t)
\]

\[
u(t) = \int_{-\infty}^{t} \delta(t) \, dt
\]
Key use of impulse function is sifting

\[
x(t) \delta(t) = x(0) \delta(t)
\]
\[
x(t) \delta(t - t_0) = x(t_0) \delta(t - t_0)
\]

Example

\[
x(t) = 2t
\]
\[
y(t) = x(t)(\delta(t - 2) + \delta(t - 4))
\]
\[
y(t) = 4\delta(t - 1) + 8\delta(t - 2)
\]
Cautionary note

◆ The unit-impulse function, also called the Direct delta function is really a generalized function
  ✦ It does not really behave like a normal function
  ✦ It is either zero or infinity 😞

◆ We should only be using it under the integral sign
  ✦ It is well defined in the integral sign
  ✦ Some Professors will complain if not in the integral sign

◆ Take real analysis in the math department for further enlightenment
Example

- Consider a periodic signal with a period of 2

\[ x(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
-2 & 1 \leq t < 2 
\end{cases} \]

- Relate the derivative of this signal to the impulse train

\[ \sum_{n} \delta(t - nT) \]
Summary of today’s lecture

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