Announcements

◆ Change in late homework policy
  ➤ Due at the beginning of class when the bell rings
  ➤ Turn in at the back of the room with Enoch
  ➤ 4 points late per minute
Preview of today’s lecture

◆ Quiz #3 solution

◆ Brief review
  ✦ Impulse response of a CT LTI system
  ✦ Characterizing the output of a CT LTI system using the convolution

◆ CT convolution
  ✦ Define key properties of the convolution
  ✦ Compute the convolutional of two CT signals

◆ LTI systems properties in terms of the impulse response
  ✦ Relate system properties to impulse response characteristics

◆ CT LTI systems as differential equations
  ✦ Explain how a differential equation can represent and input-output relationship
Quiz #3 Solution
Quiz #3

1. (100 points) For an input of $x[n] = 2\delta[n]$ into an LTI system we observe an output

$$y[n] = 2\delta[n - 1] + 2\delta[n - 2] + \delta[n - 3].$$

What is the output of this system for $x[n] = u[n] - u[n - 3]$?

Answer

The output to $x[n] = 2\delta[n]$ is

$$y[n] = 2\delta[n - 1] + 2\delta[n - 2] + \delta[n - 3]$$

therefore the output to $\frac{1}{2}x[n] = \delta[n]$ is the impulse response

$$h[n] = \delta[n - 1] + \delta[n - 2] + \frac{1}{2}\delta[n - 3].$$
Quiz #3 Solution Continued

- Write the input signal as a sum of deltas

\[ x[n] = u[n] - u[n - 3] = \delta[n] + \delta[n - 1] + \delta[n - 2] \]

- Simplify the output

\[ y[n] = h[n] + h[n - 1] + h[n - 2] = \delta[n - 1] + \delta[n - 2] + \frac{1}{2} \delta[n - 3] \delta[n - 2] + \delta[n - 3] + \frac{1}{2} \delta[n - 4] \delta[n - 3] + \delta[n - 4] + \frac{1}{2} \delta[n - 5] \]
\[ = \delta[n - 1] + 2\delta[n - 2] + \frac{5}{2} \delta[n - 3] + \frac{3}{2} \delta[n - 4] + \frac{1}{2} \delta[n - 5] \]

Can also be done using convolution, but possibly more work
Brief review

Learning objectives

- Determine the output of an LTI system in terms of the input
Impulse response of an LTI system

- If the system has an impulse response \( h(t) \), then

\[
\delta(t) \rightarrow h(t) \rightarrow h(t)
\]

- Time shift (follows from time invariance)

\[
\delta(t - T) \rightarrow h(t) \rightarrow h(t - T)
\]

- Sum of shifted deltas (follows from linearity and time invariance)

\[
\alpha \delta(t) + \beta \delta(t - T_0) \rightarrow h(t) \rightarrow \alpha h(t) + \beta h(t - T_0)
\]
Writing the input in terms of delta functions

- Write the stair case approximation of a function as

\[ \hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta \]

- For small step size

\[ \lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau = x(t) \]
Output of an LTI system to a simple input

- The output to an infinite sum of shifted delta functions

\[ \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \rightarrow h(t) \rightarrow \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

- Output of an LTI system can be computed by convolution of input with the impulse response
Continuous-time convolution

Learning objectives
- Define key properties of the convolution
- Compute the convolutional of two CT signals
Basic convolution properties

◆ Commutative

\[ x(t) \ast h(t) = h(t) \ast x(t) \]

◆ Associative

\[ f(t) \ast [g(t) \ast h(t)] = [f(t) \ast g(t)] \ast h(t) \]

◆ Distributive

\[ f(t) \ast (h(t) + g(t)) = f(t) \ast h(t) + f(t) \ast g(t) \]

Same properties hold in DT case as well
Convolution with an impulse function

- Convolution with $\delta(t)$

\[
\begin{align*}
f(t) \star \delta(t) &= \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau) d\tau \\
&= \int_{-\infty}^{\infty} f(t)\delta(t - \tau) d\tau \\
&= f(t) \int_{-\infty}^{\infty} \delta(t - \tau) d\tau \\
&= f(t)
\end{align*}
\]
Length of the output of a convolution

- **Length of the output signal**
  - If \( f_1(t) \) has a length \( T_1 \) seconds, and \( f_2(t) \) has a length \( T_2 \) seconds, then the signal \( f_1(t) \ast f_2(t) \) has a length \( T_1 + T_2 \)

  ![Convolution Diagram](image)

  - **Note:** Checking the length is a good sanity check
  - **For discrete-time signals,** if \( f_1[n] \) has \( N_1 \) samples and \( f_2[n] \) has \( N_2 \) samples, then \( f_1[n] \ast f_2[n] \) has \( N_1+N_2-1 \) samples

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Lecture 8 EE 313 Heath
Example

- Determine the convolution output of the following two signals

Solution
LTI systems properties in terms of the impulse response

Learning objectives
- Relate system properties to impulse response characteristics
Memoryless

- A memoryless system has impulse response that is a scaled impulse
  - For discrete-time systems
    \[ h[n] = k\delta[n] \]
    \[ y[n] = h[n] \ast x[n] = kx[n] \]
  - For continuous-time systems
    \[ h(t) = k\delta(t) \]
    \[ y(t) = kx(t) \]
Invertibility

- An LTI system is invertible if there exists $g[n]$ such that

$$h[n] * g[n] = \delta[n]$$

Inverting System
Invertibility Example 1

Is this system defined by the impulse response below invertible?

\[ h(t) = \delta(t - \epsilon), \quad \epsilon > 0 \]

Yes, it is invertible because

\[ g(t) = \delta(t + \epsilon), \quad \text{(shift back)} \]

satisfies

\[ h(t) \ast g(t) = \delta(t) \]
Invertibility Example 2

- Consider the system defined by the impulse response \( h[n] = u[n] \).
- The output of the system is given by

\[
y[n] = h[n] \ast x[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k], \quad \text{(definition of conv)}
\]

\[
= \sum_{k=-\infty}^{n} x[k] u[n-k] \quad \text{stops at time } k=n
\]

\[
= \sum_{k=-\infty}^{n} x[k]
\]
Invertibility Example 2 (continued)

\[ y[n] = \sum_{k=-\infty}^{n} x[k] \]

◆ This function is called an accumulator

◆ Note that the output can also be written as

\[ y[n] = x[n] + x[n - 1] + x[n - 2] \cdots \]

\[ = y[n - 1] \]

◆ Therefore, \( x[n] \) can be recovered from \( y[n] \) as

\[ x[n] = y[n] - y[n - 1] \]
Invertibility Example 2 (continued)

- The inverting system impulse response is

\[ g[n] = \delta[n] - \delta[n - 1] \]

- Check:

\[ h[n] \ast g[n] = \delta[n] \]
\[ u[n] \ast (\delta[n] - \delta[n - 1]) = u[n] - u[n - 1] = \delta[n] \]
Causality

- An LTI system is causal if its impulse satisfies
  - For discrete-time systems  \( h[n] = 0, \quad \forall n < 0 \)
  - For continuous-time systems  \( h(t) = 0, \quad \forall t < 0 \)

- Causal example

- Noncausal example
Stability

- An LTI system is BIBO stable if its impulse response is absolutely summable
  - For discrete-time systems
    \[ \sum_{k=-\infty}^{\infty} |h[k]| < \infty \]
  - For continuous-time systems
    \[ \int_{-\infty}^{\infty} |h(t)| \, dt < \infty \]
Why absolute summability for stability?

- Condition ensures that the output of a bounded input is bounded
  ✤ Consider $|x[n]| < B$, $\forall n$, where $B$ is a constant
  ✤ The output will be
    $$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n - k] \right|$$
    $$\leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n - k]|$$
    $$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

If this term is bounded then, the output is bounded
Step response – DT systems

◆ The output of the system when the input is a unit step function

\[ u[n] \rightarrow \text{System} \rightarrow s[n] \triangleq h[n] * u[n] \]

step response

\[ \sum_{k=-\infty}^{\infty} h[k] u[n-k] = \sum_{k=-\infty}^{n} h[k] \]

◆ Note: \[ h[n] = s[n] - s[n-1] \]
Example: DT step response calculation

- Determine the step response of the system with impulse response

\[ h[n] = \alpha^n u[n] \]

- Convolve the impulse response with the step function

\[
\begin{align*}
s[n] &= \sum_{k=-\infty}^{n} \alpha^k u[k] = \sum_{k=0}^{n} \alpha^k \\
&= \frac{1 - \alpha^{n+1}}{1 - \alpha}
\end{align*}
\]
Step response – CT systems

◆ The output of the system when the input is a unit-step function

\[ s(t) \triangleq h(t) \ast u(t) = \int_{-\infty}^{\infty} h(\tau)u(t - \tau) d\tau \]

\[ = \int_{-\infty}^{t} h(\tau) d\tau \]

◆ Note: \[ \frac{d}{dt}s(t) = \frac{d}{dt} \int_{-\infty}^{t} h(\tau) d\tau = h(t) \]
Example: CT step response calculation

- What is the impulse response of the system with the step response

\[ s(t) = t^2 u(t) \]

- Impulse response

\[ h(t) = \frac{d}{dt} s(t) \]

\[ = 2tu(t) = 2r(t) \]

**Unit-ramp function**
Example

- Determine if the following systems is (a) causal and/or (b) stable or not?

\[ h[n] = n \left( \frac{1}{3} \right)^n u[n - 1] \]

- Solution:
  - Causal?
  - Stable?
Continuous-time systems as differential equations

Learning objectives

- Describe how some systems are described by differential equations
- Analyze problems with systems modeled by differential equations
Systems described with differential equations

- Many systems are modeled by differential equations
  - RLC circuits
  - Mechanical systems
  - Heat transfer systems
  - Chemical systems
  - Communications channels

Linear constant coefficient differential equations are an important special case
A simple differential equation example

Constant coefficients

\[ a_0 y(t) + a_1 \frac{d}{dt} y(t) = b_0 x(t) + b_1 \frac{d}{dt} x(t) \]

output

input

derivative of output

derivative of input
General format relating input and output

\[ a_N \frac{d^N}{dt^N} y + a_{N-1} \frac{d^{N-1}}{dt^{N-1}} y + \cdots + a_1 \frac{dy}{dt} + a_0 y(t) \]

\[ = b_M \frac{d^M}{dt^M} x + b_{M+1} \frac{d^{M+1}}{dt^{M+1}} x + \cdots + b_1 \frac{dx}{dt} + b_0 x(t) \]

- In general can have several orders of derivatives
  - Coefficients on the left-hand side are related to the state polynomial
  - Coefficients on the right-hand side are related to the input polynomial
  - In EE313, usually \( N > M \), \( N = 1 \) or \( 2 \)

You are assumed to remember differential equations ☺️
Summary of Lecture

◆ CT convolution
  ✦ Convolution obeys associative, distributive, commutative properties
  ✦ Use these properties to simplify systems

◆ LTI systems properties in terms of the impulse response
  ✦ System properties relate to impulse response characteristics

◆ Linear constant coefficient differential equations
  ✦ Can represent and input-output relationship for an LTI system