Lecture #12

EE 313 Linear Systems and Signals
Preview of today’s lecture

- Output of an LTI system for a periodic input

- When does the Fourier series work?
  - Determine whether a given signal satisfies Dirichlet conditions

- Key properties of Fourier series, Part I
  - Connect time-domain & frequency domain structure
  - Incorporate FS properties into your problem solving

- Relevant sections of Oppenheim and Willsky: 3.2 and 3.3
Brief review

Key points

- Eigenfunctions of LTI systems are complex exponentials
- Analysis and synthesis of a Fourier series
- Importance of orthogonality
Eigenfunctions of a LTI systems

\[ x(t) = e^{st} \]

LTI System

\[ y(t) = H(s)e^{st} \]

CT complex exponential with complex \( s \)

DT complex exponential with complex \( z \)

\[ x[n] = z^n \]

LTI System

\[ y[n] = H(z)z^n \]

Eigenvalue (a complex scalar)

Eigenfunction
Analysis and synthesis equations

- Finding the coefficients: Use the **analysis** equation
  
  \[ a_k = \frac{1}{T} \int_{T} x(t) e^{-jkw_0t} dt = \frac{1}{T} \int_{T} x(t) e^{-jk\frac{2\pi}{T}t} dt \]

- Finding the signal: Use the **synthesis** equations
  
  \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0t} = \sum_{k=0}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \]

You can integrate over any time interval of duration T.
Orthogonality of complex exponentials

- Inner products of vectors

\[ x^*y = \|x\| \|y\| \cos(\theta) \]

zero if \( \theta = \frac{\pi}{2} \)

- “Orthogonal” means that the inner product is 0, intuitively:
  - “x and y have nothing in common”
  - The projection of y onto x (or vice versa) is zero

Orthogonality is a crucial concept in signal processing
Orthogonality of functions

- Inner product of two functions (or signals) $x(t)$ and $y(t)$

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t)y^*(t)dt$$

- Orthogonality of harmonics of complex sinusoids
  - Consider two different harmonics $k$ and $n$
    $$x(t) = e^{jkw_0t}, \quad y(t) = e^{jn\omega_0t}$$
    $$\langle x(t), y(t) \rangle = \int_{0}^{T} e^{j(k-n)\omega_0t}dt = \int_{0}^{T} \cos((k-n)\omega_0t) + j\sin((k-n)\omega_0t)dt$$
    $$= \begin{cases} 
      T & k = n \\
      0 & k \neq n 
    \end{cases}$$
Example: Analyzing a cosine

- Consider \( \cos(2\pi 50t) \), it follows that \( \omega_0 = 2\pi 50 \) and \( T = \frac{1}{50} \)

\[
a_k = 50 \int_0^{\frac{1}{50}} \cos(2\pi 50t) e^{-j2\pi 50kt} dt
\]

\[
= 50 \int_0^{\frac{1}{50}} \frac{1}{2} \left( e^{j2\pi 50t} + e^{-j2\pi 50t} \right) e^{-j2\pi 50kt} dt
\]

\[
= 50 \int_0^{\frac{1}{50}} \frac{1}{2} \left( e^{j2\pi 50t} + e^{-j2\pi 50t} \right) e^{-j2\pi 50kt} dt
\]

\[
= \frac{50}{2} \int_0^{\frac{1}{50}} e^{j2\pi 50t} e^{-j2\pi 50kt} dt + \frac{50}{2} \int_0^{\frac{1}{50}} e^{-j2\pi 50t} e^{-j2\pi 50kt} dt
\]
Example: Analyzing a cosine (continued)

\[ a_1 = \frac{1}{2} \]  
\[ a_{-1} = \frac{1}{2} \]  
\[ \cos(100\pi t) = \sum_k a_k e^{j2\pi k50t} = \frac{1}{2} e^{j2\pi 50t} + \frac{1}{2} e^{-j2\pi 50t} \]
What happens to periodic signals when input to LTI systems?

Learning objectives

- Compute the output of an LTI system to a periodic input
Periodic inputs to LTI systems

\[ e^{st} \rightarrow \text{LTI System} \rightarrow H(s)e^{st} \]

\[ \alpha e^{s_1 t} + \beta e^{s_2 t} \rightarrow \text{LTI System} \rightarrow \alpha H(s_1)e^{s_1 t} + \beta H(s_2)e^{s_2 t} \]

\[ \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \rightarrow \text{LTI System} \rightarrow \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k)e^{j\omega_0 kt} \]

Fourier series coefficients are modified by the frequency response of the system.
When does the Fourier series work?

Learning objectives

- Understand what the Gibbs phenomenon is
- Determine whether a periodic signal satisfy Dirichlet conditions
Synthesizing a square wave

$$x(t) = \frac{2T_1}{T} + 2 \sum_{n=0}^{\infty} \frac{\sin(k\omega_0 T_1)}{k\pi} \cos(k\omega_0 t)$$

DC

k=-1,0,1

k=-2,…,2
Adding more terms

\[ k = -3, \ldots, 3 \]

\[ k = -4, \ldots, 4 \]

\[ k = -1000, \ldots, 1000 \]
Complications with the Fourier series

- Some technical issues with using Fourier series on certain signals
  - Challenge arises from the discontinuous nature of the signal
  - Ripple is known as the Gibbs phenomena

Where does the ripple go?

Convergence of the Fourier Series
The ripple

3 terms

Gap shrinks

5 terms

Gap goes to zero for very large k

100 terms
Signals that satisfy the “Dirichlet” conditions

- Dirichlet (“Diri-klay”) conditions tell us when the Fourier Series exists
- A periodic signal $x(t)$ has a Fourier series representation when it satisfies the following conditions
  - (1) Absolute integrability
  - (2) Finite number of minima and maxima for a given time period
  - (3) Finite number of discontinuities for a period $T$

http://www.dictionary.com/browse/dirichlet
#1: Absolute integrability

\[ \int_{T} |x(t)| \, dt < \infty \]

- Example of violation

\[ x(t) = \begin{cases} \frac{1}{|t|}, & t \in (-2, 2) \\ \text{repeat for all } T \end{cases} \]

---

**Bad signal**
#2: Finite number of min and max for a given period

- Example of violation (just one period shown)

\[ x(t) = \begin{cases} 
\sin \left( \frac{1}{t} \right), & t \in (0, \frac{1}{\pi}) \\
\text{repeat every } \frac{1}{\pi} \text{ secs} 
\end{cases} \]
#3: Finite number of discontinuities for a period $T$

- Example of violating signal (shown and defined over one period)

$$x(t) = \sum_{k=0}^{\infty} \left[ u(t - \frac{1}{2^k}) - u(t) \right] (-1)^k$$

Bad signal

Infinite # of discontinuities
Historical Subnote
Laplace (French, 1749-1827)

Lagrange (Italian, 1736-1812)

Euler (Swiss, 1707-83)

J. Fourier (French, 1768-1830)

S. Poisson (French, 1781-1840)

D. Bernoulli (Swiss, 1700-82)
A small world of true geniuses

These six men all made unbelievable contributions to mathematics, science, and engineering, working and writing by hand! (Often the math came about from trying to explain a natural phenomenon, or design something).

Lagrange had an unbelievably prolific career in algebra, analysis, number theory, e.g. “Lagrange multipliers” for finding maxima of functions.

Fourier was Lagrange’s PhD student.
  - But Lagrange, who everyone deferred to on important matters, would not let him publish his work on Fourier Series because he thought it was wrong
  - Laplace, however, one of the greatest scientists of all time, was in favor of publishing the Fourier Series

Poisson, a giant in physics, mathematics, and especially probability, was co-supervised by Lagrange and Laplace for his PhD.

Euler, who developed the notations f(x), e, and π, as well as the equivalence between complex exponentials and sinusoids (and a great many other things) was succeeded as Director of Mathematics at Prussian Academy (Berlin) by Lagrange.

Bernoulli, who pioneered amongst other things the principle of energy conservation, was a close childhood friend of Euler and had a notable family:
  - His nearly equally brilliant father (Johann), an early developer of calculus, plagiarized his work and refused to speak to him, he was so ashamed of being outshone by his son
  - His Uncle Jacob Bernoulli discovered the Theory of Probability
Properties of Fourier series: Part I

Learning objectives

- Use Fourier series properties to simplify calculation & build intuition
- Analyze problems that include FS properties
Properties of the Fourier series

- The following notation is used to denote a signal and its FS coefficients:

\[ x(t) \xrightarrow{FS} a_k \]

- Properties are used to figure out how transformations of the input signal lead to transformations of the FS coefficients.

- Properties considered in this lecture (more in next lecture):
  1. Linearity
  2. Time shifting
  3. Time reversal
  4. Time scaling
  5. Multiplication
  6. Conjugate Symmetry
Property #1: Linearity

- If \( x(t) \) and \( y(t) \) both have period \( T = \frac{2\pi}{\omega_0} \), and

\[
x(t) \overset{FS}{\leftrightarrow} a_k
\]

\[
y(t) \overset{FS}{\leftrightarrow} b_k
\]

\[
z(t) = Ax(t) + By(t)
\]

- Then

\[
z(t) \overset{FS}{\leftrightarrow} Aa_k + Bb_k
\]

FS of a sum of signals is the sum of their FS coefficients
Property #2: Time Shifting

- Let \( x(t) \) have period \( T = \frac{2\pi}{\omega_0} \), and \( x(t) \) \( \xrightarrow{FS} \) \( a_k \)

- If \( y(t) = x(t - t_0) \), \( y(t) \) is periodic with the same period
  \[ y(t) \xrightarrow{FS} b_k \]

- Then \( b_k = a_k e^{-jk\omega_0 t_0} \)

Note \( |b_k| = |a_k| \) since \( |e^{jk}| = 1 \)

Shift in time results in a phase shift in frequency
Example 1

Let \( x(t) \) be a periodic signal with a fundamental period \( T \), and FS coefficients \( a_k \). Derive the FS coefficients of the following signal

\[
x(t - t_0) + x(t + t_0)
\]

**Solution**

\[
x(t) \overset{FS}{\longleftrightarrow} a_k
\]

\[
x(t - t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0}
\]

\[
x(t + t_0) \overset{FS}{\longleftrightarrow} a_k e^{jk\omega_0 t_0}
\]

\[
x(t - t_0) + x(t + t_0) \overset{FS}{\longleftrightarrow} a_k e^{-jk\omega_0 t_0} + a_k e^{jk\omega_0 t_0}
\]

\[
= 2 \cos(k\omega_0 t_0) a_k
\]
Property #3: Time reversal

- Let \( x(t) \) have period \( T = \frac{2\pi}{\omega_0} \), and \( x(t) \xleftarrow{FS} a_k \)

- Then \( y(t) = x(-t) \), \( y(t) \) is periodic with the same period

- and

\[
y(t) \xleftarrow{FS} a_{-k}
\]

Reverse in time results in reverse in frequency
**Time reversal proof**

- Suppose that
  
  \[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

- Then
  
  \[ y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t} \]

- Changing variables
  
  \[ y(t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t} \]
Implications of time reversal on even and odd

- Let \( x(t) \) have period \( T = \frac{2\pi}{\omega_0} \), and \( x(t) \xrightarrow{FS} a_k \)

- If \( x(t) \) is even then \( x(t) = x(-t) \) and it follows that

  \[ a_k = a_{-k} \]

- If \( x(t) \) is odd then \( x(-t) = -x(t) \) and it follows that

  \[ a_k = -a_{-k} \]

Symmetry in the signal leads to structure in FS coefficients
Property #4: Time scaling

- Let \( x(t) \) have period \( T = \frac{2\pi}{\omega_0} \), and \( x(t) \xrightarrow{FS} a_k \).

- If \( y(t) = x(\alpha t) \), \( \alpha > 0 \)
  
  - \( \alpha < 1 \) → stretching
  - \( \alpha > 1 \) → compression

- Then \( y(t) = x(\alpha t) \) is periodic with period \( T/\alpha \) and

\[
x(\alpha t) \xrightarrow{FS} a_k
\]

Scale in time does not change the FS coefficients
Visualizing time scaling

- Example

\[ y(t) = x(\alpha t) \]

\[ \alpha = \frac{1}{4} \]

Stretched signal has same structure
Time scaling proof

- Since \( x(t) \xrightarrow{FS} a_k \) it follows that \( x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \)

- Then

\[
x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t}
\]

\[
= \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha \omega_0) t}
\]

\( x(\alpha t) \xrightarrow{FS} a_k \)
Property #5: Multiplication

- If \( x(t) \) and \( y(t) \) both have period \( T = \frac{2\pi}{\omega_0} \), and

\[
x(t) \xrightleftharpoons{FS} a_k \\
y(t) \xrightleftharpoons{FS} b_k
\]

- Then for \( z(t) = x(t)y(t) \)

\[
z(t) = x(t)y(t) \xrightleftharpoons{FS} h_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}
\]

Product in time leads to convolution in frequency.
Property #6: Conjugate symmetry

- If \( x(t) \) is periodic with period \( T = \frac{2\pi}{\omega_0} \) and \( x(t) \xrightarrow{FS} a_k \)

- Then \( x^*(t) \xrightarrow{FS} a_{-k}^* \)

- Implications
  - If \( x(t) \) is real, then the FS coefficients are conjugate symmetric
    \[ a_{-k}^* = a_k \]
  - If \( x(t) \) is real and even, then the FS coefficients are real and even
    \[ a_k = a_k^* \]
  - If \( x(t) \) is real and odd, then the FS coefficients are imaginary and odd
Example

Let $x(t)$ be a periodic signal with a fundamental period $T$, and FS coefficients $a_k$. Derive the FS coefficients of the following signal:

$\star Even\{x(t)\}$

Note that $Ev\{x(t)\} = [x(t) + x(-t)]/2$. The FS coefficients of $x(-t)$ are

$$b_k = \frac{1}{T} \int_T x(-t)e^{-jk(2\pi/T)t} \, dt$$

$$= \frac{1}{T} \int_T x(\tau)e^{jk(2\pi/T)\tau} \, d\tau$$

$$= a_{-k}$$

Therefore, the FS coefficients of $Ev\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}}{2}. $$
Summary of lecture

◆ Outputs of LTI systems are periodic
  ✦ Output can be found using Eigenfunction principle
  ✦ Fourier series coefficients depend on the system frequency response

◆ Existence of the Fourier Series for a given period signal
  ✦ Provides a zero-energy representation of finite energy signals
  ✦ Provides an exact characterization of signals that satisfy the Dirichlet conditions (except at a finite number of points)

◆ Fourier series properties
  ✦ Time-domain structure leads to frequency domain structure
  ✦ Incorporate FS properties into your problem solving