Lecture #13

EE 313 Linear Systems and Signals
Preview of today’s lecture

◆ Key properties of Fourier series
  ✪ Connect time-domain & frequency domain structure
  ✪ Incorporate FS properties into your problem solving

◆ Application of Fourier series properties
  ✪ Use the FS properties to discover the signal, given clues
  ✪ Use the structure of a signal to determine its FS coefficients

◆ Relevant sections of Oppenheim and Willsky: 3.5
Properties of Fourier series:

Learning objectives

- Use Fourier series properties to simplify calculation & build intuition
- Analyze problems that include FS properties
Properties of the Fourier series

- The following notation is used to denote a signal and its FS coefficients

\[ x(t) \overset{FS}{\leftrightarrow} a_k \]

- Properties are used to figure out how changes in one domain correspond to changes in the other domain

  - Example: what are the FS coefficients of \( x(t-2) \)?

A main objective is to avoid having to re-compute the new coefficients from scratch.
Fourier series properties

- Let $x(t)$ and $y(t)$ both have period $T = \frac{2\pi}{\omega_0}$, and

$$x(t) = \mathcal{FS} a_k$$
$$y(t) = \mathcal{FS} b_k$$

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Fourier series properties (continued)

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Property #7: Parseval’s theorem

- Consider a periodic signals with FS representation

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

- The power in the signal is

\[ \frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \]

Power is the same whether in the time or frequency domain
Proof of Parseval’s theorem

\[
\frac{1}{T} \int |x(t)|^2 dt = \frac{1}{T} \int_T \left| \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right|^2 dt
\]

\[
= \frac{1}{T} \int_T \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k e^{jk\omega_0 t} a_\ell^* e^{-j\ell\omega_0 t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \frac{1}{T} \int_T a_k e^{jk\omega_0 t} a_\ell^* e^{-j\ell\omega_0 t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \frac{1}{T} \int_T e^{j(k-\ell)\omega_0 t} dt
\]

Use orthogonality property
Proof of Parseval’s theorem (cont.)

\[
\frac{1}{T} \int |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \delta[k - \ell]
\]

\[
= \sum_{k=-\infty}^{\infty} |a_k|^2
\]

Orthogonality is key to the proof
Parseval’s theorem – Example

- Consider the signal  $x(t) = \cos(\omega_0 t)$

  $$= \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

- The FS coefficients:  $a_0 = 0, \quad a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0$  else

  - Find the power using Parseval’s theorem

    $$\frac{1}{T} \int_T |\cos(\omega_0 t)|^2 dt = \sum |a_k|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

  - Find the power directly in the time domain

    $$\cos^2 \omega_0 t = \frac{1}{2}(1 + \cos 2\omega_0 t) \quad \frac{1}{T} \int_0^T \frac{1}{2} dt + \frac{1}{T} \int_T \cos 2\omega_0 t dt = \frac{1}{2}$$
Property #8: Derivative

- Consider a periodic signal \(x(t)\) with \(T = \frac{2\pi}{\omega_0}\) and

\[
x(t) \xrightarrow{FS} a_k
\]

- Then

\[
\frac{dx(t)}{dt} \leftrightarrow a_k(jk\omega_0)
\]

Each FS coefficient scaled as a function of the frequency
Proof of the derivative property

\[
x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}
\]

\[
\frac{d}{dt} x(t) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} e^{j k \omega_0 t}
\]

\[
= \sum_{k=-\infty}^{\infty} a_k (j k \omega_0) e^{j k \omega_0 t}
\]
Application of Fourier series properties

Learning objectives

- Use the Fourier series properties to infer information about signals

Note: Several extra examples provided here for review after class
Application Example 1

- Consider the following three CT signals with a fundamental period of $T=1/2$

$$x(t) = \cos(4\pi t)$$

$$y(t) = \sin(4\pi t)$$

$$z(t) = x(t)y(t)$$

- Determine the FS coefficients of the three signals

(from O&W 3.25)
Application Example 1

\[
\sin x \cdot \cos y = \frac{1}{2} \left[ \sin (x - y) + \sin (x + y) \right]
\]

(a) The nonzero FS coefficients of \( x(t) \) are \( a_1 = a_{-1} = 1/2 \).

(b) The nonzero FS coefficients of \( x(t) \) are \( b_1 = b_{-1}^* = 1/2j \).

(c) Using the multiplication property, we know that

\[
z(t) = x(t)y(t) \quad \xrightarrow{FS} \quad c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}.
\]

Therefore,

\[
c_k = a_k * b_k = \frac{1}{4j} \delta[k - 2] - \frac{1}{4j} \delta[k + 2].
\]

This implies that the nonzero Fourier series coefficients of \( z(t) \) are \( c_2 = c_{-2}^* = (1/4j) \).
Application Example 2

- Consider the impulse train signal

\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \text{period } T \]

- Calculate the FS coefficients

\[ a_k = \frac{1}{T} \int_{T} x(t) e^{-jk\frac{2\pi}{T}t} \, dt \]

\[ = \frac{1}{T} \int_{0}^{T} \delta(t) e^{-jk\frac{2\pi}{T}t} \, dt \]

\[ = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{0} \, dt \]

\[ = \frac{1}{T} \quad \forall k \]
Application Example 3

- Consider the pulse train
- Calculate the FS coefficients
- Because $y(t) = x\left(t - \frac{T}{2}\right)$

$$b_k = a_k e^{-jk\omega_0 t_0}$$
$$= a_k e^{-jk\frac{2\pi}{T} \frac{T}{2}}$$
$$= a_k e^{-jk\pi}$$
$$= \frac{1}{T} \cos k\pi$$
$$= \frac{(-1)^k}{T}$$
Application Example 4

Let \( x(t) \) be a periodic signal with a fundamental period \( T \), and FS coefficients \( a_k \). Derive the FS coefficients of the following signal

\[
\frac{d^2 x(t)}{dt^2}
\]
Application Example 4

The Fourier series synthesis equation gives

\[ x(t) = \sum_{k=\pm \infty} a_k e^{j(2\pi/T)kt}. \]

Differentiating both sides wrt \( t \) twice, we get

\[ \frac{d^2 x(t)}{dt^2} = \sum_{k=\pm \infty} -k^2 \frac{4\pi^2}{T^2} a_k e^{j(2\pi/T)kt}. \]

By inspection, we know that the Fourier series coefficients of \( d^2 x(t)/dt^2 \) are \( -k^2 \frac{4\pi^2}{T^2} a_k \).
Application Example 5

- Consider the FS coefficients of a CT signal that is periodic with period 4. Determine the signal \( x(t) \)

\[
    a_k = \begin{cases} 
        0, & k = 0 \\
        (j)^k \frac{\sin(k\pi/4)}{k\pi}, & \text{otherwise}
    \end{cases}
\]

- Approach
  - Start with a known FS
  - Make transformations to reach the required signal
Application Example 5 (continued)

- Use the known FS and the FS properties to recover signals from their FS coefficients

- Consider this function from an earlier lecture

\[
a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin \left(k \frac{2\pi}{T} T_1 \right)}{k\pi}
\]
Application Example 5 (continued)

Solution:

Consider the signal $y(t)$ with FS coefficients $b_k$ with

$$b_k = \frac{\sin \frac{k\pi}{4}}{k\pi}$$

As $T=4$

$$b_k = \frac{\sin \frac{k\pi}{4}}{k\pi} = \frac{\sin \left( k \cdot \frac{2\pi}{T} \cdot T_1 \right)}{k\pi}$$

$$= \frac{\sin \left( k \frac{\pi}{2} \cdot T_1 \right)}{k\pi} \rightarrow T_1 = \frac{1}{2}$$
Application Example 5 (continued)

- The dc component of the signal $y(t)$ is

$$b_0 = \frac{1}{T} \int_T y(t) dt$$

$$= \frac{1}{4} \cdot 1 = \frac{1}{4}$$

- But the DC component of $y(t)$ is 0, so subtract it

  - Define the signal $w(t) \leftrightarrow c_k$ as $\omega(t) = y(t) - \frac{1}{4}$.

  - Then $c_0 = 0$

$$c_k = \frac{\sin \frac{\pi k}{4}}{\pi k}$$
Application Example 5 (continued)

- Now, what is remaining is to add $j^k$

- We know that $j^k = (e^{j\frac{\pi}{2}})^k = e^{j\frac{\pi}{2} k}$

- So, now consider $x(t) = w(t - t_0)$

- Using the FS properties
  
  If $w(t) \leftrightarrow c_k$ then $w(t - t_0) \leftrightarrow c_k \cdot e^{j\frac{\pi}{2} k} = a_k$

  $$e^{j\frac{\pi}{2} k} = e^{-j k\omega_0 t_0}, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$

  $$= e^{-j k \frac{\pi}{2} t_0} \quad \longrightarrow \quad t_0 = -1$$
Application Example 5 (concluded)

Hence, \( x(t) = w(t - t_0) = w(t + 1) \)

\[ a_k = \begin{cases} 
0, & \text{if } k = 0 \\
(j)^k \frac{\sin(k\pi/4)}{k\pi}, & \text{otherwise}
\end{cases} \]
Application Example 6

Consider the FS coefficients of a CT signal that is periodic with period 4. Determine the signal \( x(t) \)

\[
a_k = \begin{cases} 
1, & k \text{ even} \\
2, & k \text{ odd}
\end{cases}
\]

Solution: Use the fact that

\[
\sum_{k=\infty}^{\infty} \delta(t - k) \leftrightarrow \{1\}
\]
Application Example 6 (continued)

- Note from the scaling property

\[ \sum_{k=-\infty}^{\infty} \delta(t/4 - k) \leftrightarrow \{1\} \]

- Simplifying and using properties of the delta function

\[ \sum_{k=-\infty}^{\infty} \delta(t/4 - k) = \sum_{k=-\infty}^{\infty} \delta \left( \frac{t - 4k}{4} \right) \]

\[ = \sum_{k=-\infty}^{\infty} 4\delta(t - 4k) \]
Application Example 6 (continued)

Therefore for a signal with period 4 note that

\[
\sum_{k=-\infty}^{\infty} \delta(t - 4k) \leftrightarrow \left\{ \frac{1}{4} \right\}
\]

\[
4 \sum_{k=-\infty}^{\infty} \delta(t - 4k) \leftrightarrow \{1\}
\]
Application Example 6 (continued)

What about a signal with

\[ c_k = \begin{cases} 
1 & k \text{ even} \\
0 & k \text{ odd} 
\end{cases} \]

Inserting into the synthesis equation

\[ x_2(t) = \sum_{k} c_k e^{j\frac{2\pi}{4}kt} \]

\[ = \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{4}2kt} \]

\[ = \sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{2}kt} \]

Signal with period 2 and FS coefficients \{1\}
Application Example 6 (concluded)

- Write signal with period 4 and FS coefficients
  
  \[ a_k = \begin{cases} 
  1, & k \text{ even} \\
  2, & k \text{ odd} 
  \end{cases} \]

- As the sum of signals with FS coefficients
  
  \[ b_k = 1 \quad \text{and} \quad c_k = \begin{cases} 
  1, & k \text{ even} \\
  0, & k \text{ odd} 
  \end{cases} \]

- Time domain signal is then
  
  \[ x(t) = 4 \sum_{k=-\infty}^{\infty} \delta(t - 4k) + 2 \sum_{k=-\infty}^{\infty} \delta(t - 2k) \]
Application Example 7

In the following, we specify the FS coefficients of a CT signal that is periodic with period 4. Determine the signal $x(t)$

$$a_k = \begin{cases} jk, & k < 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution:

$$x(t) = a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t}$$

$$= je^{j\omega_0 t} - je^{-j\omega_0 t} + 2je^{j2\omega_0 t} - 2je^{-j2\omega_0 t}$$

$$= -2\sin(\omega_0 t) - 4\sin(2\omega_0 t)$$
Application Example 8

Let \( x(t) \) be a periodic signal whose FS coefficients are
\[
a_k = \begin{cases} 
2 & k = 0 \\
\pm j (1/2)^{|k|} & \text{otherwise}
\end{cases}
\]

Is \( x(t) \) real?
- Real signals must satisfy \( x(t) = x^*(t) \) or \( a_k = a_{-k}^* \) not satisfied here

Is \( x(t) \) even?
- Even signals satisfy \( x(t) = x(-t) \) or \( a_k = a_{-k} \) yes is satisfied

Is \( \frac{dx(t)}{dt} \) even?
- The FS coefficients of \( \frac{dx(t)}{dt} \) are \( (j\omega_0 k) a_k \) for which \( (j\omega_0 k) a_k \neq -(j\omega_0 k) a_{-k} \)
Summary of lecture

- FS properties connect manipulations in time and frequency domains
  - Use the properties to simplify calculations
  - Make inferences about signals from their characteristics

- Application of Fourier series properties
  - Use known FS expansions to find new signals with similar coefficients
  - Use FS properties to answer questions about signals