Lecture #14

EE 313 Linear Systems and Signals
Preview of today’s lecture

- Brief review

- Frequency response of a system
  - Explain how an LTI system acts as a filter

- Filters
  - First-order low-pass filters
  - Low-pass, highpass, and bandpass filters

- Introduction to Fourier transform
  - Explain the connection between Fourier series and Fourier transform

- Relevant sections of Oppenheim and Willsky: 3.8, 3.9, and 4.1.1
Building intuition on the frequency response

Key points
- Explain how some systems represent frequency filters
- Distinguish between different kinds of frequency filters
Frequency response from eigenfunctions with $s = j \omega_0$

$e^{j\omega_0 t} \xrightarrow{H(j\omega)} H(j\omega_0)e^{j\omega_0 t}$

$\cos(\omega_0 t) \xrightarrow{H(j\omega)} |H(j\omega_0)| \cos(\omega_0 t + \angle H(j\omega_0))$

$H(j\omega_0)$ is the gain of the system at frequency $\omega_0$
How to find the frequency response?

- If you already know the system response then

\[ H(j\omega) = H(s)|_{s=j\omega} \]

- If you have the impulse response, compute the Fourier transform

\[ H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt \]

- If the system is described by a differential equation

\[ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \]

\[ H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \]
Example: Low-pass filtering a periodic signal

- Consider an **ideal low-pass filter** whose frequency response is

\[
H(j\omega) = \begin{cases} 
1, & |\omega| \leq 100 \\
0, & |\omega| > 100 
\end{cases}
\]

We call this **ideal** due to the sharp transition.
Example: Low-pass filtering a periodic signal 2

- Find the output if the input signal is a square wave with period $T = \frac{1}{5}$.

$$T = \frac{1}{5}$$
$$T_1 = \frac{1}{20}$$
$$\omega_0 = \frac{2\pi}{1/5} = 10\pi$$
Example: Low-pass filtering a periodic signal 3

- From past lecture, can deduce the Fourier series coefficients of \( x(t) \)

\[
a_k = \frac{1}{\pi k} \sin \left( 2\pi k \frac{T_1}{T} \right)
\]

\[
a_0 = \frac{2T_1}{T}
\]

\[
k \neq 0 \quad a_k = \frac{4}{\pi k} \sin \left( \pi k \frac{1}{2} \right)
\]

\[
k = 0 \quad a_0 = 2
\]
Example: Low-pass filtering a periodic signal

\[ \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \]

\[ \rightarrow \]

LTI System

\[ \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt} \]

Cutoff is 100 rad/s

Harmonics are at frequencies

\[ k\omega_0 \]

\[ \omega_0 = 10\pi \approx 31.4 \]

\[ 2\omega_0 = 2 \cdot 10\pi \approx 62.8 \]

\[ 3\omega_0 = 3 \cdot 10\pi \approx 94.2 \]

\[ 4\omega_0 = 4 \cdot 10\pi \approx 125.6 \]

Fourier series coefficients are modified by the frequency response of the system.
Example: Low-pass filtering a periodic signal

Fourier coefficients before the LTI system

Fourier coefficients after the LTI system
Example: Low-pass filtering a periodic signal

Time domain signal before the LTI system

Time domain signal after the LTI system
Types of filters

Key points
- Distinguish between different kinds of frequency filters
- Explain low-pass, high-pass, and bandpass filter concepts
Low-pass filter

- Systems that pass low frequencies, attenuate high frequencies

![Diagram showing the frequency response of a low-pass filter with a pass region and an attenuate region, along with the ideal and practical low-pass filter characteristics.](image-url)
1st order low-pass filters

Consider the 1st order differential equation

\[
\frac{dy}{dt} + Ay(t) = x(t)
\]

\[
\underbrace{(s + A)}_{Q(s)} Y(s) = \underbrace{1}_{P(s)} X(s)
\]

\[
H(j\omega_0) = \frac{P(j\omega_0)}{Q(j\omega_0)} = \frac{1}{j\omega_0 + A}
\]
**1st order low-pass filters**

- What is the system gain $|H(j\omega)|$?

$$|H(j\omega)| = \frac{1}{\sqrt{\omega^2 + A^2}}$$

- In dB

$$|H(j\omega)| (dB) = 20 \log_{10} |H(j\omega)|$$

$$= 20 \log_{10} \left( (\omega^2 + A^2)^{-\frac{1}{2}} \right)$$

$$= -10 \log_{10} (\omega^2 + A^2)$$
**1st order low-pass filters**

- What does this gain mean in terms of the frequency response?

- Three regimes

\[ |H(j\omega)| (dB) = -10 \log_{10} (\omega^2 + A^2) \]

- \( \omega \ll A, \quad |H(j\omega)| = -20 \log_{10} |A| \)

- \( \omega \gg A, \quad |H(j\omega)| = -20 \log_{10} \omega, \quad \omega > 0 \)

- \( \omega \approx A, \quad \text{Transition region, usually can be ignored} \)
1st order low-pass filters

- What does this gain mean in terms of the frequency response?

\[ |H(j\omega)| (dB) = -10 \log_{10} (\omega^2 + A^2) \]
Example– Designing a simple audio filter

- Design a LPF
  - 1st order filter
  - Low pass cutoff of 1.6 KHz or 10,000 radians/sec
  - Passband amplification of 40 dB (power increase of 10,000x)

- General 1st order filter has the form

\[ H(j\omega) = \frac{G}{j\omega + A} \]

- We want to determine the values of \( A \) and \( G \)
Example – Designing a simple audio filter (cont.)

- Gain in terms of dB

\[ |H(j\omega)|(dB) = 20 \log_{10} G - 10 \log_{10}(\omega^2 + A^2) \]

Shift up by \(20 \log_{10} G\).
Example – Designing a simple audio filter (cont.)

- Cutoff frequency is still determined by $A=10\text{Krad/s}$
- For $G$, we need amplification of 40 dB

$$|H(j\omega)| (dB) = 20 \log_{10} G - 10 \log_{10}(\omega^2 + A^2)$$

$$40 = 20 \log_{10} G - 10 \log_{10}(10^8)$$
$$40 = 20 \log_{10} G - 80$$
$$120 = 20 \log_{10} G$$
$$10^6 = G$$
Example – Designing a simple audio filter (cont.)

- Resulting frequency response

\[ H(j\omega) = \frac{10^6}{j\omega + 10^4} \quad \rightarrow \quad \begin{cases} 
P(j\omega) = 10^6 \\
Q(j\omega) = j\omega + 10^4 \\
Q(D) = D + 10^4 
\end{cases} \]

- Resulting differential equation

\[ \frac{dy}{dt} + 10^4 y(t) = 10^6 x(t) \]
High-pass filters (HPFs)

- Systems that pass high frequencies, attenuate low frequencies

![Diagram of high-pass filter with cut-off frequency](image)
High-pass filters (HPFs)

- Example: a differentiator

\[
y(t) = \frac{dx}{dt}.
\]

- For \( x(t) = e^{j\omega_0 t} \) \( \Rightarrow \) \( y(t) = j\omega_0 e^{j\omega_0 t} \)

\[
y(t) = H(j\omega_0)e^{j\omega_0 t}
\]

\[
H(j\omega_0) = j\omega_0
\]

\[
|H(\omega)| = \omega_0
\]

High gain with high frequencies
Band-pass filters

- A specific band is passed, and outside this band is attenuated
Band-stop (notch) filters

- **Stop** (attenuates) a **certain band**, and **passes the other frequencies**

![Diagram showing the behavior of a band-stop (notch) filter. The system gain is plotted against frequency. The filter passes frequencies outside the notch band and attenuates frequencies within the notch band.](image-url)
Introduction to the Fourier transform

Key points
- Explain the connection between Fourier series and Fourier transform
Fourier series review

- Periodic signals (that are well behaved i.e. satisfy Dirichlet conditions) can be represented as a sum of complex sinusoids

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

- All of the information in \( x(t) \) is contained in \( \omega_0 \) and \( \{a_k\}_{k=-\infty}^{\infty} \)
What about aperiodic signals?

- Most interesting real-world signals are not periodic
  - The Fourier series for a non-periodic signal does not have any meaning

- Is it possible to represent an aperiodic signal as a sum of sinusoids?

Yes using the Fourier transform
Consider the periodic unit pulse signal

For this signal, the FS coefficients are
(special case of what was derived in an earlier in lecture)

\[ a_k = \frac{\sin \frac{k\pi}{T}}{k\pi} \]
What happens if the period increases?
Example for T=2

\[ a_k = \frac{\sin \frac{\pi k}{2}}{\pi k} \]

\[ \omega_0 = \pi \]

\[ = \frac{1}{2} = \frac{1}{T} \]

\[ = a_{-1} = \frac{\sin \frac{\pi}{2}}{\pi} = \frac{1}{\pi} \]

\[ = a_{-2} = \frac{\sin \pi}{2\pi} = 0 \]

\[ = a_{-3} = \frac{\sin \frac{3\pi}{2}}{3\pi} = -\frac{1}{3\pi} \]
Example for $T=4$

$$a_k = \sin \frac{\pi k}{4} \quad \pi k$$

$$\omega_0 = \frac{\pi}{2}$$

$$a_0 = \frac{1}{4} = \frac{1}{T}$$

$$a_1 = a_{-1} = \frac{\sin \frac{\pi}{4}}{\pi} = \frac{1}{\sqrt{2\pi}}$$

$$a_2 = a_{-2} = \frac{\sin \frac{\pi}{2}}{2\pi} = \frac{1}{2\pi}$$

$$a_3 = a_{-3} = \frac{\sin \frac{3\pi}{4}}{3\pi} = \frac{1}{3\sqrt{2\pi}}$$

Twice the number of samples but with the same sinc envelope
As $T$ goes towards $T = \infty$

- In the limit the signal becomes aperiodic
- We get the full sinc wave (continuous frequency response)
Fourier transform

◆ As $T$ grows large

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$\omega_0 = \frac{2\pi}{T}$$

Fourier transform (analysis)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier transform (synthesis)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k x(t) e^{jk\omega_0 t}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Note: $T$ grows big while $\omega_0$ grows small (see O&W 4.1.1)
Summary

◆ Frequency response of an LTI system
  ✦ Describes how the system behaves for different sinusoidal inputs
  ✦ Is a simplification of results on eigenfunctions
  ✦ Frequency response is a special case of the system transfer function

◆ Filters are ways to describe how an LTI system acts on sinusoids
  ✦ Certain systems have special structure
  ✦ Low-pass, high-pass, band-pass, notch are common filters
  ✦ Filter design deals with finding systems that meet performance specifications

◆ Introduction to Fourier transform
  ✦ An aperiodic signal can be thought of as the limit of a periodic signal
  ✦ Fourier transform falls out from this limit
  ✦ More in the next lecture!