Lecture #17

EE 313 Linear Systems and Signals
Preview of today’s lecture

- Fourier transform properties
  - Use frequency shifting to find the corresponding time domain signal
  - Use Parseval’s theorem to compute energy in time or frequency
  - Explain the application of duality

- Application of Fourier properties to communications

- Relevant sections of Oppenheim and Willsky: 4.3
Brief review – Fourier transform properties

Key points

- Use Fourier series properties to simplify calculation & build intuition
- Analyze problems that include FS properties
Fourier transform properties

- Linearity
- Time shifting
- Differentiation and integration
- Time and frequency scaling
- Frequency shifting
- Parseval’s theorem
- Duality (defined more formally)
Fourier transform properties thus far

◆ Linearity
  ✦ If \( x(t) \leftrightarrow X(j\omega) \), \( y(t) \leftrightarrow Y(j\omega) \)
  ✦ Then \( ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega) \)

◆ Time shifting
  ✦ If \( x(t) \leftrightarrow X(j\omega) \)
  ✦ Then \( x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega) \)
Fourier transform properties thus far

- Differentiation
  - If
    \[ x(t) \leftrightarrow X(j\omega) \]
  - Then
    \[ \frac{dx}{dt} \leftrightarrow j\omega X(j\omega) \]

- Integration
  - If
    \[ x(t) \leftrightarrow X(j\omega) \]
  - Then
    \[
    \int_{-\infty}^{t} x(\tau) \, d\tau \leftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)
    \]

DC component
Fourier transform properties thus far

- **Time and frequency scaling**
  - If
    \[ x(t) \leftrightarrow X(j\omega) \]
  - Then
    \[ x(at) \leftrightarrow \frac{1}{|a|} X \left( \frac{j\omega}{a} \right) \]

- **Time and frequency scaling special cases**

  **Inversion**
  \[ x(-t) \leftrightarrow X(-j\omega) \]

  **Rescaling**
  \[ x \left( \frac{t}{b} \right) \leftrightarrow |b| X(jb\omega) \]
Example of time / frequency scaling

- What is the inverse Fourier transform of $\text{sinc}(\omega)$
- We know that
  - From the rect-sinc Fourier pair $\text{rect}(t) \leftrightarrow \text{sinc} \left( \frac{\omega}{2\pi} \right)$
  - From the scaling law $x(at) \leftrightarrow \frac{1}{|a|} X \left( \frac{j\omega}{a} \right)$
- Using the scaling property $\text{rect}(t/2\pi) \leftrightarrow 2\pi \text{sinc} (\omega)$
- Therefore using linearity $\frac{1}{2\pi} \text{rect}(t/2\pi) \leftrightarrow \text{sinc} (\omega)$
Example with scaling and time shift

- Determine the Fourier transform of \( \text{sinc}(1 - 2t) \)

\[
\begin{align*}
  x(t) &= \text{sinc}(1 - 2t) \\
        &= y(2t) \\
y(t) &= \text{sinc}(1 - t) \\
      &= \text{sinc}(-(t - 1)) \\
      &= z(t - 1) \\
z(t) &= \text{sinc}(-t) \\
      &= \text{sinc}(t) \\
\text{rect}(t) &\leftrightarrow \text{sinc}(\omega/2\pi) \\
\text{sinc}(t/2\pi) &\leftrightarrow 2\pi \text{rect}(-\omega) = 2\pi \text{rect}(\omega) \\
\text{sinc}(t) &\leftrightarrow \text{rect}(\omega/(2\pi)) \\
Z(j\omega) &= \text{rect}(\omega/(2\pi)) \\
Y(j\omega) &= e^{-j\omega} Z(j\omega) \\
X(j\omega) &= \frac{1}{2} Y \left( j \frac{\omega}{2} \right) \\
      &= \frac{1}{2} e^{-j\omega/2} \text{rect}(\omega/(4\pi))
\end{align*}
\]
Frequency shifting

- If
  \[ x(t) \leftrightarrow X(j\omega) \]

- Then
  \[ x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega - \omega_0)) \]

  this is called modulation

- Corollary
  \[ x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0)) \]

Modulate in time leads to shift in frequency
Example combining shift and scaling

- Determine the inverse Fourier transform of

\[ X(j\omega) = \frac{2\sin(3(\omega - 2\pi))}{(\omega - 2\pi)} \]
Example (continued)

- **Given**

  \[ X(j\omega) = \frac{2 \sin(3(\omega - 2\pi))}{(\omega - 2\pi)} \]

- **To solve use the following fact**

  \[ \text{rect}(t) \leftrightarrow \text{sinc} \left( \frac{\omega}{2\pi} \right) = \frac{\sin(\omega/2)}{\omega/2} \]

- **For convenience let**

  \[ R(j\omega) = \frac{\sin(\omega/2)}{\omega/2} \]
Example (continued)

◆ Given

\[ X(j\omega) = \frac{2 \sin (3(\omega - 2\pi))}{(\omega - 2\pi)} \]

◆ Rewrite as another shifted function

\[ X(j\omega) = Y(j(\omega - 2\pi)) \]
\[ Y(j\omega) = \frac{2 \sin (3\omega)}{\omega} \]

◆ Rewrite again as

\[ Y(j\omega) = 3 \frac{2 \sin (6\omega/2)}{6\omega/2} \]
\[ = 6R(j6\omega) \]
Example (continued)

- Using the scaling property
  \[ Y(j\omega) = 6R(j6\omega) \quad \Rightarrow \quad y(t) = \text{rect}(t/6) \]

- Using the shift property
  \[ X(j\omega) = Y(j(\omega - 2\pi)) \quad \Rightarrow \quad x(t) = e^{j2\pi t} \text{rect}(t/6) \]
Parseval’s theorem

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \]

Energy of the signal in the time domain
Energy of the signal in the frequency domain

- This is a result of conservation of energy
- Scaling factor is because of radians
Example with Parseval’s theorem

- If the signal $x(t)$ has the FT below $X(j\omega)$

$$x(t) = t \left( \frac{\sin t}{\pi t} \right)^2$$

$$X(j\omega) = \begin{cases} 
\frac{j}{2\pi}, & -2 \leq \omega < 0 \\
-\frac{j}{2\pi}, & 0 \leq \omega \leq 2 \\
0, & \text{otherwise}
\end{cases}$$

- Calculate

$$A = \int_{-\infty}^{\infty} t^2 \left( \frac{\sin t}{\pi t} \right)^4 dt$$
Example with Parseval’s theorem (cont.)

Using Parseval’s relation,

\[
\int_{-\infty}^{\infty} t^2 \left( \frac{\sin t}{\pi t} \right)^4 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi^3}
\]
Another example with Parseval’s theorem

- Calculate

\[ \int_{-\infty}^{\infty} \text{sinc}^4(t) t^2 \, dt \]

- Solution:

\[ \int_{-\infty}^{\infty} \text{sinc}^4(t) t^2 \, dt = \int t^2 \frac{\sin^2(\pi t)}{(\pi t)^2} \frac{\sin^2(\pi t)}{(\pi t)^2} \, dt \]

\[ = \frac{1}{\pi^2} \int \text{sinc}^2(t) \sin^2(\pi t) \, dt \]
Another example with Parseval’s theorem (cont.)

- Define $f(t) = \frac{1}{\pi} \text{sinc}(t) \sin \pi t$ and remember $\mathcal{F}\{\text{sinc}(t)\} = \text{rect}\left(\frac{\omega}{2\pi}\right)$

- Then $F(j\omega) = \frac{1}{\pi} \frac{\pi}{2j\pi} \left[ \text{rect}\left(\frac{\omega - \pi}{2\pi}\right) - \text{rect}\left(\frac{\omega + \pi}{2\pi}\right) \right]$
Another example with Parseval’s theorem (cont.)

- By Parseval’s theorem,

\[ \int |f(t)|^2 dt = \frac{1}{2\pi} \int |F(j\omega)|^2 d\omega \]

- The integral is then equal to

\[ \frac{1}{2\pi} \int |F(j\omega)|^2 d\omega = \frac{1}{2\pi} \cdot 2\pi \cdot width \cdot \left( \frac{1}{2\pi} \right)^2 \]
Duality in the Fourier transform

- If

\[ x(t) \leftrightarrow X(j\omega) \]

- Then

\[ X(jt) \leftrightarrow 2\pi x(-\omega) \]

If you know one Fourier pair then you know the other Fourier pair
Applications of duality

- Reproving frequency shift
  - Consider
    \[ x(t - t_0) \leftrightarrow e^{-j\omega_0 t} X(j\omega) \]
  - Then
    \[ x(t)e^{j\omega_0 t} \leftrightarrow X(j(\omega - \omega_0)) \]

- Impulse in time and frequency
  - Consider
    \[ \delta(t) \leftrightarrow 1 \]
  - Then
    \[ 1 \leftrightarrow 2\pi\delta(\omega) \]
Duality example

◆ Consider

\[ \text{rect}(t) \leftrightarrow \text{sinc} \left( \frac{\omega}{2\pi} \right) \]

◆ Then

\[ \text{sinc} \left( \frac{t}{2\pi} \right) \leftrightarrow 2\pi \cdot \text{rect}(-\omega) \]

\[ = 2\pi \cdot \text{rect}(\omega) \]

since rect is an even function.
Application to communication systems

Key points

- Explain how frequency domain concepts are used in communication systems
Spectrum allocation

Numbers are the frequency given in MHz, multiply by 2 \( \pi \) to get Mrad/s

UHF band

Cellular is Land mobile

ISM band here is used for cordless phones, sensors
Practical application – Modulation

- Modulation is used in communication systems

- Generally refers to modifying a sinusoid to carry information

- Many types of modulation
  - Amplitude modulation (AM)
  - Frequency modulation (FM)
  - Phase modulation (PM)

- Focus on an example with amplitude modulation
Practical application – Amplitude modulation

- One type of modulation is AM

- What happens in the frequency domain?

\[ x_1(t) \cos(\omega_1 t) = x_1(t) \frac{1}{2} e^{j\omega_1 t} + x_1(t) \frac{1}{2} e^{-j\omega_1 t} \]

\[ \mathcal{F} \{x_1(t) \cos(\omega_1 t)\} = \frac{1}{2} X_1(j(\omega - \omega_1)) + \frac{1}{2} X_1(j(\omega + \omega_1)) \]
**Example 1**

**Parseval's Theorem**

- **Amplitude Modulation** (AM)

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**Fourier Transform Properties**

- Parseval's Theorem
- 

\[ x(t) = x_1(t) \cos(\omega_1 t) \]

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- **Carrier frequency**
- 

\[ X(j\omega) \]

---

- **Frequency**
- 

\[ \omega \]

---

- **Bandwidth**
- 

\[ \omega_B \]

---

- **Carrier frequency**
- 

\[ \omega_1 \]

---

- **Power**
- 

\[ P \]
Practical application – Demodulation

- The reverse of modulation is called demodulation
  ✦ Exploit the fact that

  \[ y(t) = x(t) \cos^2(\omega_1 t) = x(t) \frac{1}{2} (1 + \cos(2\omega_1 t)) \]

  Can eliminate this with a lowpass filter!

\[ y(t) \rightarrow \cos(2\omega_1 t) \rightarrow \text{LPF} \rightarrow x(t) \]
Practical application – Frequency Division Multiplexing

- Consider a communication system with two users
- Suppose that the signal for each user has frequency response

- Objective: Create a communication signal that carries the information contained in each users’ signal
Practical application – Frequency Division Multiplexing (cont.)

- One possible solution is AM with different carrier frequencies
Practical application – Frequency Division Multiplexing (cont.)

◆ How to prevent possible overlap in the signals?
  ✦ At the transmitter

\[ x_1(t) \xrightarrow{\text{LPF}} \cos(\omega_1 t) \xrightarrow{\text{Gain} = 2} x_1(t) \]

✦ At the receiver

\[ y(t) \xrightarrow{\text{BPF}} \cos(\omega_1 t) \xrightarrow{\text{LPF}} x_1(t) \]
Practical application – Inphase and quadrature

What if two information signals are sent as follows?

![Diagram](image)

- Inphase signal: $x_I(t)$
- Quadrature signal: $x_Q(t)$

Note the sign and sine here, which will make sense shortly.
Practical application – Inphase and quadrature (cont.)

- What happens in the frequency domain?
  - Inphase term
    \[
    \mathcal{F}\{x_I(t) \cos(\omega_1 t)\} = \frac{1}{2} X_I(j(\omega - \omega_1)) + \frac{1}{2} X_I(j(\omega + \omega_1))
    \]
  - Quadrature term
    \[
    \mathcal{F}\{ -x_Q(t) \sin(\omega_1 t)\} = \frac{i}{2} X_Q(j(\omega - \omega_1)) - \frac{i}{2} X_Q(j(\omega + \omega_1))
    \]

mixture of inphase and quadrature terms but not the same mixture at positive and negative frequencies
Practical application – Inphase and quadrature (cont.)

What about demodulation?

Trig identities

\[
\sin u \sin v = \frac{1}{2} [\cos(u - v) - \cos(u + v)]
\]

\[
\cos u \cos v = \frac{1}{2} [\cos(u - v) + \cos(u + v)]
\]

\[
\sin u \cos v = \frac{1}{2} [\sin(u - v) + \sin(u + v)].
\]

Applying the identities

\[
x(t) \cos(\omega_1 t) = \frac{1}{2} x_I(t) + \frac{1}{2} x_I(t) \cos(2\omega_1 t) - \frac{1}{2} x_Q(t) \sin(2\omega_1 t)
\]

\[
x(t) \sin(\omega_1 t) = -\frac{1}{2} x_Q(t) + \frac{1}{2} x_Q(t) \cos(2\omega_1 t) + \frac{1}{2} x_I(t) \sin(2\omega_1 t).
\]
Practical application – Inphase and quadrature (cont.)

- IQ demodulator

\[ x(t) \]
\[ \cos(\omega_1 t) \]
\[ -\sin(\omega_1 t) \]

\[ x_I(t) \]
Gain = 2

\[ x_Q(t) \]
Gain = 2
Practical application – Inphase and quadrature (cont.)

Why do we use complex signals?

\[ x_{bb}(t) = x_I(t) + jx_Q(t) \]

\[ \text{Re}\{x_{bb}(t)\} \quad \text{Im}\{x_{bb}(t)\} \]

Complex signals become a convenient way to work with inphase and quadrature together, avoiding the need for matrix notation.
Summary

- Fourier transform properties
  - Use frequency shifting to find the corresponding time domain signal
  - Use Parseval’s theorem to compute energy in time or frequency
  - Explain the application of duality
  - Justify the need to define bandwidth based on the uncertainty principle

- Explain connections between Fourier and communications
  - Amplitude modulation shifts the spectrum of a signal to a carrier
  - Frequency division multiplexing is a way to share spectrum
  - Bandwidth occupation of a signal is defined