Lecture #1

EE 313 Linear Systems and Signals
Preview of today’s lecture

◆ Going through the practicalities (via course syllabus)
  ✦ Syllabus review
  ✦ Lecture outline
  ✦ How the class will work
  ✦ How to succeed in class

◆ What is the class about?

◆ Brief review of some important mathematical background
  ✦ Sinusoids, Euler’s identity, complex numbers, logs, decibels, integration
How this class will work

◆ Lectures
  ✦ Slides supplemented by handwritten examples
  ✦ Examples will be scanned
  ✦ You should write your own notes and be an active thinker

◆ Homework
  ✦ Problems related to material covered the previous week
  ✦ No way around solving lots of problems
  ✦ Start early!

◆ Quizzes
  ✦ Test that you are following the material, typically occur on Thursdays at the beginning of class
Tips for success

◆ Review your calculus book
  ✦ You will need to perform integrals and derivatives
  ✦ You will need trigonometric identities
  ✦ You will need to master complex numbers and especially complex exponentials (more on this in a second)

◆ Don’t fall behind!
  ✦ Keep up with the material every week, read the book each week
  ✦ Attend class
  ✦ Get help if you are struggling: TA and my office hours, your classmates, tutoring

◆ Attempt the homework problems on your own and get as far as you can using the book and course materials. This process is critical to developing the problem-solving and self-teaching skills you need for this class and beyond.
What is this class about?
Signals

Value or information relative to an independent variable

Building a signal from other signals

\[ x(t) = \frac{1}{2} + \frac{1}{4} \]

coefficient

sinusoids
Systems

mathematical description of how the input signal is transformed into the output signal

Convolution
Transfer function
Frequency response
Brief review of key fundamentals
**Sinusoids**

\[ x(t) = A \cos(\omega_0 t + \phi) = A \cos(2\pi f_0 t + \phi) \]

- **\( A \)** amplitude
- **\( \phi \)** phase in radians
- **\( \omega_0 \)** frequency (radians/sec)
- **\( f_0 = \frac{\omega_0}{2\pi} \)** frequency (in Hertz)
- **\( T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0} \)** is the period

Frequency is \( 100\pi \) radians = 50 Hertz

Period is \( T_0 = 1/50 = 0.02 \) secs
Complex numbers

\[ z = x + jy \]

\[ z = re^{j\theta} \]

- Complex number is essentially a pair of independent real numbers \( z = (x, y) \)
  - Widely used in engineering and science
  - \( j = \sqrt{-1} \) is the imaginary number (EE’s use j because i is historically used for current)

\[ r = \sqrt{x^2 + y^2} \]

\[ \theta = \begin{cases} \tan^{-1} \left( \frac{y}{x} \right) & x > 0 \\ \tan^{-1} \left( \frac{y}{x} \right) + \pi & x < 0, y \geq 0 \\ \tan^{-1} \left( \frac{y}{x} \right) - \pi & x < 0, y < 0 \end{cases} \]
**Euler’s formula**

\[ e^{j\theta} = \cos(\theta) + j \sin(\theta) \]

- \( e^{j\pi/2} = j \)
- \( e^{j\pi/4} = \cos(\pi/4) + j \sin(\pi/4) \)
- \( e^{j0} = 1 \)
- \( e^{j\pi} = \cos(\pi) + j \sin(\pi) = -1 \)

**Useful facts**

- \( e^{jx} = \cos x + j \sin x \)
- \( e^{-jx} = \cos x - j \sin x \)
- \( e^{jx} + e^{-jx} = 2 \cos x \)
- \( e^{jx} - e^{-jx} = 2j \sin x \)

\[
\cos(\theta) = \frac{1}{2} \left( e^{j\theta} + e^{-j\theta} \right) \\
\sin(\theta) = \frac{1}{2j} \left( e^{j\theta} - e^{-j\theta} \right)
\]
Working with complex numbers

consider these two complex numbers
\[ z_1 = a + jb = r_1e^{j\theta_1} \]
\[ z_2 = c + jd = r_2e^{j\theta_2} \]

addition
\[ z_1 + z_2 = (a + jb) + (c + jd) \]
\[ = (a + c) + j(b + d) \]

multiplication
\[ z_1z_2 = (a + jb)(c + jd) \]
\[ = (ac - bd) + j(bc + ad) \]
\[ = r_1r_2e^{j(\theta_1+\theta_2)} \]

conjugate
\[ z_1^* = a - jb \]
\[ z_1 + z_1^* = a - jb + a + jb \]
\[ = 2a = 2\text{Re}(z_1) \]
\[ z_1z_1^* = r_1r_1e^{j(\theta_1-\theta_1)} \]
\[ = r_1^2 = x^2 + y^2 \text{ magnitude squared} \]

division
\[ z_1/z_2 = (r_1/r_2)e^{j(\theta_1-\theta_2)} \]
Example

Consider \( z = -1 - 2j \)

- Plot this complex number
- Find its polar form

\[
\begin{align*}
  r &= \sqrt{(-1)^2 + (-2)^2} \\
  &= \sqrt{5} \\
  \\
  \theta &= \tan^{-1}(-1/-2) - \pi \\
  &= -2.0344 \\
  &= -0.6476\pi
\end{align*}
\]
Example

- Let
  \[ z_1 = 1 - j2 \]
  \[ z_2 = 2 + j3 \]

- Compute
  \[ z_1 + z_2 \]
  \[ z_1 z_2 \]
  \[ z_1/z_2 \]
  \[ z_1/z_1^* \]
  \[ z_1 - z_1^* \]
Logs

\[ c = \log_b a \iff b^c = a \]

- Most common in signals and systems
  - Natural log \( \ln = \log_e \) and log base 10 \( \log_{10} \)

- What is neat about logs?
  - Huge numbers \( \rightarrow \) small positive numbers
    \[ \log_{10} 10,345,034,896 = 10 \approx 10^{10} \]
  - Tiny numbers \( \rightarrow \) small negative numbers
  - Multiplication \( \rightarrow \) addition
  - Division \( \rightarrow \) subtraction

- Used to compute amplifier gain, antenna gain, losses, etc.
Relationship between logarithms and exponentials

\[ \log a^x = x \log a \]

\[ y = e^{-at} \]

\[ \log \text{exponential} \rightarrow \text{linear} \]

\[ \ln(y) = -at \]

exponential

linear
**Application of logs in information theory/communications**

*transmitted signal* → *received signal*

**Capacity** of this communication channel is:

\[ C = B \log_2 \left( 1 + \frac{P_{\text{signal}}}{P_{\text{noise}}} \right) \]

- *units are bits per second*
- *bandwidth of the communication channel*
- *signal-to-noise ratio (SNR)*

Claude Shannon
(Bell Labs)
Decibel: describing system gain

- The Decibel (Deci = ten, bel = “Bell Labs”)

- If $H > 1 \rightarrow$ “gain”
- If $H < 1 \rightarrow$ “attenuation” (also called gain)

$$H = \frac{Y}{X} \quad H(\text{dB}) = 20 \log_{10} \left| \frac{Y}{X} \right|$$

$$= 10 \log_{10} \left| \frac{Y}{X} \right|^2$$

- Decibels are a unitless ratio of powers

10 log$_{10}$ 10 = 10 dB
10 log$_{10}$ 1 = 0 dB
10 log$_{10}$ 2 = 3 dB
10 log$_{10}$ 0.5 = -3 dB
Example dB calculation

- Determine the gain (in dB) of the amplifier with linear gain

- **H = 20**
  \[20 \log_{10} 20 = 20 \log_{10} 2(10) = 2 \cdot 10 \log_{10} 2 + 2 \cdot 10 \log_{10} 10 = 26\text{dB}\]

- **H = 0.1**
  \[20 \log_{10} 0.1 = 20(-1) = -20\text{dB}\]

- **H = 50**
  \[20 \log_{10} 50 = 20 \log_{10} 5(10) = 20 \log_{10} 5 + 20 \log_{10} 10 = 14\text{dB} + 20\text{dB} = 34\text{dB}\]
Integration

- Integrals are an essential part of calculus
  - Computes the area under a curve
  - “Opposite” of the derivative operation

- Most integrals in this course are definite integrals
  \[ c = \int_{a}^{b} x(t) \, dt \]

- Sometimes the limits will be a variable, not a fixed number
  \[ y(t) = \int_{0}^{t} x(\tau) \, d\tau \]
Integration example 1

\[
\int_0^\infty e^{-2t} \, dt = \left. \frac{1}{-2} e^{-2t} \right|_0^\infty \\
= \frac{1}{-2} (0 - e^{-2 \cdot 0}) \\
= \frac{1}{2}
\]
Integration example 2

\[
\int_0^3 \cos(\pi t) \sin(2\pi t) \, dt = \frac{1}{2} \int_0^3 \sin((2\pi - \pi)t) + \sin((2\pi + \pi)t) \, dt
\]

since \( \cos A \sin B = \frac{1}{2} [\sin(B - A) + \sin(B + A)] \)

\[
= \frac{1}{2} \int_0^3 \sin(\pi t) + \sin(3\pi t) \, dt
\]

\[
= -\frac{1}{2\pi} \cos(\pi t) \bigg|_0^3 + \frac{-1}{2 \cdot 3\pi} \cos(3\pi t) \bigg|_0^3
\]

\[
= \frac{1}{2\pi} (\cos(0) - \cos(3\pi)) + \frac{1}{3} (\cos(0) - \cos(3\pi))
\]

\[
= \frac{1}{2\pi} (1 + 1 + \frac{1}{3} (1 + 1)) = \frac{4}{3\pi}
\]
Integration example 3

\[ \int_0^\infty te^{-t}dt = t(-1)e^{-t}\big|_0^\infty - \int_0^\infty (-1)e^{-t}dt \]

\[ = - \lim_{t \to \infty} te^{-t} - e^{-t}\big|_0^\infty \]

\[ = - \lim_{t \to \infty} \frac{t}{e^t} + 1 \]

\[ = - \lim_{t \to \infty} 0 + 1 \]

\[ = 1 \]
Geometric sum

- For discrete-time, will deal with sums instead of integrals

- Infinite geometric sum

  \[ \sum_{n=0}^{\infty} a^n = \begin{cases} \frac{1}{1-a} & |a| < 1 \\ \infty & \text{otherwise} \end{cases} \]

- Finite geometric sum

  \[ \sum_{n=0}^{N} a^n = \begin{cases} \frac{1-a^{N+1}}{1-a} & a \neq 1 \\ \frac{1}{N+1} & a = 1 \end{cases} \]
Useful reference material
Trigonometric identities

Euler’s theorem: $e^{\pm j\theta} = \cos \theta \pm j \sin \theta$

\[
\begin{align*}
\cos \theta &= \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \\
\sin \theta &= (e^{j\theta} - e^{-j\theta})/2j \\
\sin^2 \theta + \cos^2 \theta &= 1 \\
\cos^2 \theta - \sin^2 \theta &= \cos 2\theta \\
2 \sin \theta \cos \theta &= \sin 2\theta \\
\cos^2 \theta &= \frac{1}{2}(1 + \cos 2\theta) \\
\sin^2 \theta &= \frac{1}{2}(1 - \cos 2\theta) \\
\sin(\theta \pm \phi) &= \sin \theta \cos \phi \pm \cos \theta \sin \phi \\
\cos(\theta \pm \phi) &= \cos \theta \cos \phi \mp \sin \theta \sin \phi \\
\sin \theta \sin \phi &= \frac{1}{2} \left[ \cos(\theta - \phi) - \cos(\theta + \phi) \right] \\
\cos \theta \cos \phi &= \frac{1}{2} \left[ \cos(\theta - \phi) + \cos(\theta + \phi) \right] \\
\sin \theta \cos \phi &= \frac{1}{2} \left[ \sin(\theta - \phi) + \sin(\theta + \phi) \right]
\end{align*}
\]
Indefinite integrals

\[
\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax)
\]
\[
\int \cos(ax) \, dx = \frac{1}{a} \sin(ax)
\]
\[
\int \sin^2(ax) \, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}
\]
\[
\int \cos^2(ax) \, dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}
\]
\[
\int x \sin(ax) \, dx = \frac{\sin(ax) - ax \cos(ax)}{a^2}
\]
\[
\int x \cos(ax) \, dx = \frac{\cos(ax) + ax \sin(ax)}{a^2}
\]
\[
\int x^m \sin(x) \, dx = -x^m \cos(x) + m \int x^{m-1} \cos(x) \, dx
\]
\[
\int x^m \cos(x) \, dx = x^m \sin(x) - m \int x^{m-1} \sin(x) \, dx
\]
\[
\int \sin(ax) \sin(bx) \, dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2
\]
\[
\int \sin(ax) \cos(bx) \, dx = -\left[ \frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)} \right], \quad a^2 \neq b^2
\]
\[
\int \cos(ax) \cos(bx) \, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2
\]
\[
\int e^{ax} \, dx = \frac{e^{ax}}{a}
\]
\[
\int x^m e^{ax} \, dx = \frac{x^m e^{ax}}{a} - \frac{m}{a} \int x^{m-1} e^{ax} \, dx
\]
\[
\int e^{ax} \sin(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx) - b \cos(bx)]
\]
\[
\int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx) + b \sin(bx)]
\]
Definite integrals

\[
\int_0^\infty \frac{ax}{a^2 + x^2} \, dx = \frac{\pi}{2}, \quad a > 0
\]

\[
\int_0^{\pi/2} \sin^n(x) \, dx = \int_0^{\pi/2} \cos^n(x) \, dx = \begin{cases} 
\frac{1 \cdot 3 \cdot 5 \cdots (n-1) \pi}{2 \cdot 4 \cdot 6 \cdots (n)} & n \text{ even, } n \text{ an integer} \\
\frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 3 \cdot 5 \cdots (n)} & n \text{ odd}
\end{cases}
\]

\[
\int_0^\pi \sin^2(nx) \, dx = \int_0^\pi \cos^2(mx) \, dx = \frac{\pi}{2}, \quad n \text{ an integer}
\]

\[
\int_0^\pi \sin(mx) \sin(nx) \, dx = \int_0^\pi \cos(mx) \cos(nx) \, dx = 0, \quad m \neq n, \ m \text{ and } n \text{ integer}
\]

\[
\int_0^\pi \sin(mx) \cos(nx) \, dx = \begin{cases} 
\frac{2m(m^2 - n^2)}{m + n} & m + n \text{ odd} \\
0 & m + n \text{ even}
\end{cases}
\]

\[
\int_0^\infty \frac{\sin(ax)}{x} \, dx = \frac{\pi}{2}, \quad a > 0
\]

\[
\int_0^\infty \frac{\sin^2x}{x^2} \, dx = \frac{\pi}{2}
\]

\[
\int_0^\infty e^{-a^2x^2} \, dx = \sqrt{\frac{\pi}{2a}}, \quad a > 0
\]

\[
\int_0^\infty x^ne^{-ax} \, dx = n!/a^{n+1}, \quad n \text{ an integer and } a > 0
\]

\[
\int_0^\infty x^2n e^{-ax^2} \, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1}a^n} \sqrt{\frac{\pi}{a}}
\]

\[
\int_0^\infty e^{-ax} \cos(bx) \, dx = \frac{a}{a^2 + b^2}, \quad a > 0
\]

\[
\int_0^\infty e^{-ax} \sin(bx) \, dx = \frac{b}{a^2 + b^2}, \quad a > 0
\]

\[
\int_0^\infty e^{-a^2x^2} \cos(bx) \, dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/4a^2}
\]
Summary of Lecture

- **EE 313** leverages the power of mathematics to abstract the key traits of signals and systems
  - Allows sophisticated and precise design and prediction
  - Built on top of calculus in particular

- **Important mathematical concepts**
  - Sinusoids and exponentials – key signal class of interest
  - Complex numbers – indispensable tool when dealing with sinusoids
  - Euler’s identity – connects sinusoids and exponentials via complex numbers
  - Logs – makes large numbers manageable, and system response intuitive
  - Integrals and finite sums – system responses and transforms