Lecture #22

EE 313 Linear Systems and Signals
Preview of today’s lecture

◆ Sampling and aliasing
  ✦ Determine when there is aliasing
  ✦ Illustrate the effects of aliasing

◆ Reconstruction of a signal from its samples
  ✦ Reconstruction theorem
  ✦ Role of the sinc function in reconstruction

◆ Practical sampling and reconstruction
  ✦ Analog-to-digital converters
  ✦ Digital-to-analog converters

◆ Relevant sections of Oppenheim and Willsky: 7.1-7.3
Sampling and aliasing

Key points
- Determine when there is aliasing
- Illustrate the effects of aliasing
Sampling theorem

◆ Let \( x(t) \) be a continuous-time signal

◆ Then, \( x(t) \) is uniquely determined by its samples \( x(nT), n = 0, \pm 1, \pm 2, \ldots \)

(1) If the signal is bandlimited, i.e. \( X(j\omega) = 0 \) for \( |\omega| > \omega_M \)

(2) And the sampling frequency is chosen such that

◆ The sampling frequency is \( \omega_s = \frac{2\pi}{T} \)

◆ The sampling period is \( T = \frac{2\pi}{\omega_s} < \frac{2\omega_M}{2\pi} = \frac{\omega_M}{\pi} \)
## Sampling in the time and frequency domains

<table>
<thead>
<tr>
<th></th>
<th>Time domain</th>
<th>Frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CT signal</strong></td>
<td>$x(t)$</td>
<td>$X(j\omega)$</td>
</tr>
<tr>
<td><strong>impulse train of samples</strong></td>
<td>$x_p(t) = \sum_n x(nT) \delta(t - nT)$</td>
<td>$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$</td>
</tr>
<tr>
<td><strong>DT signal</strong></td>
<td>$x_d[n] = x(nT)$</td>
<td>$X_d(e^{j\Omega}) = X_p(j\Omega/T)$</td>
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</table>
Sampling in the frequency domain – Nyquist OK

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<td>$X(j\omega)$</td>
<td>$X_p(j\omega)$</td>
<td>$X_d(e^{j\Omega})$</td>
</tr>
<tr>
<td>$\omega_s &gt; 2\omega_M$</td>
<td></td>
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</table>

- $\omega_M$ is the maximum frequency of the continuous-time (CT) signal.
- $\omega_s$ is the sampling frequency.
- $\omega_M$ and $\omega_s$ are related by the Nyquist-Shannon sampling theorem.
- The diagram shows the frequency spectrum of the CT signal, the impulse train of samples, and the DT signal.
Sampling in the frequency domain – Nyquist Not OK

CT signal

impulse train of samples

DT signal

E.g. $\omega_M < \omega_s < 2\omega_M$

Aliasing! The shape of the original signal in frequency domain is distorted.
Reconstruction of a signal from its samples

Key points

- Define the reconstruction formula
- Explain the role of the sinc function in reconstruction
- Illustrate reconstruction in time and frequency domains
How to extract the original signal?

\( \Omega_M \) this is the bandwidth of the DT system

Ideally the original signal comes out if Nyquist was satisfied
Step 1: convert to impulse train of samples

$x_d[n] \xrightarrow{\text{Convert to pulse train of samples}} y_p(t)$

time domain
Step 1: convert to impulse train of samples

\[ X_d(e^{j\Omega}) \]

\[ Y_p(j\omega) \]

frequency domain
Step 2: Filtering to reconstruct the signal

Ideal low pass filter
Cutoff $\frac{\pi}{T}$
Gain $T$

Frequency domain

$T \text{rect} \left( \frac{\omega}{2\pi/T} \right) \leftrightarrow \text{sinc} \left( \frac{t}{T} \right)$
Step 2: Filtering to reconstruct the signal

\[ x_d[n] \rightarrow \text{Convert to pulse train of samples} \rightarrow \text{Lowpass filter cutoff } \pi/T, \text{ gain } T \rightarrow y(t) \]

\[
y(t) = y_p(t) \ast h(t) \\
= h(t) \ast \sum_{n=-\infty}^{\infty} x_d[n] \delta(t - nT) \\
= \sum_{n=-\infty}^{\infty} x_d[n] h(t - nT) \\
= \sum_{n=-\infty}^{\infty} x_d[n] \text{sinc} \left(\frac{t - nT}{T}\right)
\]

sinc interpolation

Reconstruction formula!
Step 2: Filtering to reconstruct the signal

Bunch of scaled sinc functions
Ideal discrete-to-continuous converter

\[ x_d[n] = x(nT) \]

Convert sequence to pulse train

Lowpass filter with gain \( T \) and cutoff \( \pi/T \)

\[ y(t) \]

\[ x_d[n] = x(nT) \]

Discrete-to-continuous converter

\[ y(t) \]

\[ T \]

\[ x_d[n] = x(nT) \]

D/C

\[ y(t) \]

\[ T \]
Reconstruction in the time and frequency domains

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<tr>
<td>impulse train of</td>
<td>$y_p(t) = \sum_{n=-\infty}^{\infty} x_d[n] \delta(t - nT)$</td>
<td>$Y_p(j\omega) = X_d(e^{j\omega T})$</td>
</tr>
<tr>
<td>samples</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>CT signal</strong></td>
<td>$y(t) = \sum_{n=-\infty}^{\infty} x_d[n] \text{sinc}\left(\frac{t - nT}{T}\right)$</td>
<td>$Y(j\omega) = T\text{rect}\left(\frac{\omega}{2\pi/T}\right) Y_p(j\omega)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= T\text{rect}\left(\frac{\omega}{2\pi/T}\right) X_d(j\omega T)$</td>
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If Nyquist is satisfied then

- In the frequency domain
  \[ Y(j\omega) = X(j\omega) \]

- In the time domain
  \[ x(t) = \sum_{n=-\infty}^{\infty} x(nT) \text{sinc} \left( \frac{t - nT}{T} \right) \]
Sampling in practice

Key points
- Define the reconstruction formula
- Explain the role of the sinc function in reconstruction
Impulse sampling is not realistic

- Impulse Sampling (Theoretical – not implemented in practice)
- Natural Sampling (Theoretical - multiplier is a switch)
- Zero-order hold Sampling (Ideal Sample/Hold - instantaneous acquisition time is impractical)
- Track/Hold (Real Sample/Hold – Result is sampled and stored in a memory element)

Real device is an analog-to-digital converter ADC

From notes by N. Prelcic, University of Vigo
Oversampling is often used to avoid aliasing.

A signal is not actually bandlimited and has a tail, there is also noise.

Anti-aliasing filter.
Practical D/C model

$x_c(nT_s) \rightarrow \text{Convert to impulses} \rightarrow x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t-nT_s) \rightarrow \text{Zero-order hold} \rightarrow x_0(t) \rightarrow \text{Reconstruction Filter} \rightarrow x_c(t)$

-2w_s -w_s 0 w_s 2w_s

introduces distortion in reconstruction, which is removed by reconstruction

Real device is an digital-to-analog converter DAC
ADCs and DACs use quantized signals

$N = \text{total number of bits (including sign bit)}$

Full Scale level = FS

Quantization step $\Delta = \text{weight of the LSB} = (2 \cdot \text{FS}) / 2^N$

$2^N$ quantized levels, from $-\text{FS}$ to $(\text{FS} - \text{LSB})$

Weight of the MSB = FS/2
Summary

- Aliasing occurs when Nyquist is not satisfied

- Signals are reconstructed from their samples using sinc interpolation
  - Reconstruction is perfect if Nyquist is satisfied

- Practical sampling and reconstruction issues
  - Zero-order hold and track/hold better modeling sampling
  - Oversampling is used to avoid aliasing due to non-ideal bandlimited signals
  - Anti-aliasing filters help to avoid aliasing before sampling
  - Reconstruction uses steps not a sinc function, which means an additional layer of reconstruction is needed