Preview of today’s lecture

- Introduction to the Laplace transform
  - Compute the Laplace transform from the integral

- Region of convergence of a Laplace transform
  - Determine the values of $s$ where the Laplace transform converges

- Pole-zero plots
  - Illustrate the region of convergence for a Laplace transform
  - Plot the poles and zeros along with the region of convergence

- Relevant sections of Oppenheim and Willsky: 9.0, 9.1, 9.2, and 9.6
Defining the Laplace Transform

Key points
- Define the Laplace transform
- Explain the difference with the Fourier transform
- Determine the region of convergence
The Laplace transform from eigenfunctions

- Recall the output of an LTI system to the complex exponential $e^{st}$

$$y(t) = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

- For $s$ imaginary, i.e., $s = j\omega$, this integral correspond to the FT of $h(t)$.
- For general values of the complex variable $s$, it is referred to as the Laplace transform.
Laplace transform as an integral

- The Laplace transform for a general signal \( x(t) \) is defined as \( \mathcal{L}\{x(t)\} \)

\[
X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt
\]

- LT is usually applied to understand systems
  - Helps solving LCCDEs
  - Associated with a region of convergence connected to system stability
- FT transform is usually applied to understand signals
  - Helps to understand the frequency content of signals
  - Determine how a system affects the frequencies in a signal
Decreasing exponential step example

- Find the LT of the signal $x(t) = e^{-at}u(t)$, $a$ is real

- Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-st}dt$$

$$= \int_{0}^{\infty} e^{-(a+s)t}dt$$

$$= \frac{-1}{a+s} \left[ \lim_{t \to \infty} e^{-(a+s)t} - 1 \right]$$

This term will not go to $\infty$ (will converge) when

$$\Re \left\{ s + a \right\} > 0$$

Or equivalently since $a$ is real

$$\Re \left\{ s \right\} > -a$$
Decreasing exponential step example (cont.)

- The Laplace transform is only defined for values of \( s \) in the Region of convergence (ROC)

- To emphasize this point, we write

\[
X(s) = \frac{1}{s + a}, \quad \text{Re}\{s\} > -a
\]

- A Laplace transform is not complete if the ROC is not specified!
Why is only the Re\{s\} important?

- Consider

\[
X(s) = \lim_{t \to \infty} e^{-(s+a)t}
\]

- It can be written as

\[
X(s) = \lim_{t \to \infty} e^{-(Re\{s\}+jIm\{s\}+a)t} \\
= \lim_{t \to \infty} e^{-(Re\{s\}+a)t} e^{-jIm\{s\}t}
\]

- Note that the amplitude does not depend on the imaginary part

\[
\left| e^{-(s+a)t} \right| = \left| e^{-(Re\{s\}+a)t} \right|
\]
Why is only the Re\{s\} important? (cont.)

- As we have \( |e^{-(s+a)t}| = |e^{-(Re\{s\)+a)t}| \)
- If \(- (Re\{s\} + a) < 0\)  the limit converges
- If \(- (Re\{s\} + a) > 0\)  the limit does not exist
- If \( (Re\{s\} + a) = 0\)  the limit does not exist (rotation)
Illustrating the ROC

Dashed line indicates that the edge is not included in the ROC, a solid line indicates that it is included.

Shaded area corresponds to the values of $s$ where $X(s)$ exists i.e., is not infinity.
Types of ROCs considered in this course

- Right hand plane
- Strip
- Left hand plane

(in special cases the whole plane as well)
Connection between Laplace and Fourier

If the ROC includes the imaginary axis then

\[ X(j\omega) = X(s)|_{s=j\omega} \]

You actually put the \( j \) here

Can find FT from LT in many cases

Note: If the ROC does not include the imaginary axis then either the FT does not exist or it is one of the special “defined transforms” like the FT of a periodic function.
Another decreasing exponential step example

- Find the Laplace transform and ROC of the signal with a real

\[ x(t) = -e^{-at}u(-t) \]

- The LT is given by

\[
X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
= \int_{-\infty}^{0} -e^{-at}e^{-st}dt \\
= \int_{-\infty}^{0} -e^{-(a+s)t}dt \\
= \frac{1}{s + a} - \lim_{t \to -\infty} \frac{1}{s + a}e^{-(s+a)t}
\]
Another decreasing exponential step example (cont.)

So,

\[
X(s) = \frac{1}{s + a} - \lim_{t \to -\infty} \frac{1}{s + a} e^{-(s+a)t}
\]

This term converges when

\[
\text{Re}\{s + a\} < 0
\]

Then, we have

\[
-e^{-at} u(-t) \overset{\mathcal{L}}{\rightarrow} \frac{1}{s + a} \quad \text{Re}\{s\} < -a
\]
ROC is often illustrated as part of a pole-zero plot

- If \( x(t) \) is a linear combination of exponentials then

\[
X(s) = \frac{N(s)}{D(s)}
\]

- If order of \( D(s) > N(s) \) then there are “zeros at infinity” similarly if order of \( D(s) < N(s) \) then “poles at infinity”

- Roots are called zeros

- Roots are called poles

- Can never have poles inside the ROC

Example with 1 real pole and 2 complex zeros
Example of creating a pole-zero plot

- Suppose you are given the following Laplace transform

\[ X(s) = \frac{1}{(s - 1)(s - 2)} \]

- 2 zeros at infinity
- Poles at 1 and 2
- And you are told the ROC has the form of a right half plane
Extra ROC example

For each of the following integrals, specify the values of the real parameter \( \sigma \) which ensure that the integral converges

A. \[
\int_{-\infty}^{\infty} e^{-5|t|} e^{-(\sigma+j\omega)t} \, dt
\]

B. \[
\int_{-5}^{5} e^{-5t} e^{-(\sigma+j\omega)t} \, dt
\]
Extra ROC example solution (part A)

The given integral may be written as

$$\int_{-\infty}^{0} e^{-(5+\sigma)t} e^{j\omega t} dt + \int_{0}^{\infty} e^{-(5+\sigma)t} e^{j\omega t} dt.$$  

The first integral converges for $\sigma < 5$. The second integral converges if $\sigma > -5$. Therefore, the given integral converges when $|\sigma| < 5$. 
Extra ROC example solution (part B)

The given integral may be written as

$$\int_{-5}^{5} e^{-(5+\sigma)t} e^{j\omega t} dt.$$

Clearly this integral has a finite value for all finite values of $\sigma$. 

Laplace transform pairs

Key points
- Derive common Laplace transform pairs
Laplace of the unit delta

Find the Laplace transform and ROC of

\[ x(t) = \delta(t) \]

Solution:

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} \, dt \]

\[ X(s) = \int_{-\infty}^{\infty} \delta(t)e^{-st} \, dt \]

\[ X(s) = e^{s\cdot0} = 1 \]

ROC is all \( s \) in the complex plane
Laplace of the unit step

- Find the Laplace transform and ROC of

\[ x(t) = u(t) \]

- Solution:
  - LT is
  
  \[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt \]
  
  \[ = \int_{-\infty}^{\infty} u(t)e^{-st} dt \]
  
  \[ = \frac{-1}{s} e^{-st} \bigg|_{t=0}^{\infty} \]
  
  \[ = \lim_{t \to \infty} \frac{-1}{s} e^{-st} + \frac{1}{s} = \frac{1}{s} \]

\[ \text{ROC: } \text{Re}\{s\} > 0 \]
## Common Laplace transform pairs

<table>
<thead>
<tr>
<th>Time domain signal</th>
<th>Laplace transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(t) = e^{-at}u(t)$</td>
<td>$X(s) = \frac{1}{s + a}$</td>
<td>$\text{Re}{s} &gt; -a$</td>
</tr>
<tr>
<td>$x(t) = -e^{-at}u(-t)$</td>
<td>$X(s) = \frac{1}{s + a}$</td>
<td>$\text{Re}{s} &lt; -a$</td>
</tr>
<tr>
<td>$x(t) = u(t)$</td>
<td>$X(s) = \frac{1}{s}$</td>
<td>$\text{Re}{s} &gt; 0$</td>
</tr>
<tr>
<td>$x(t) = \delta(t)$</td>
<td>$X(s) = 1$</td>
<td>Entire plane</td>
</tr>
</tbody>
</table>
Complete table of transform pairs 1/2

<table>
<thead>
<tr>
<th>Transform pair</th>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta(t)$</td>
<td>1</td>
<td>All $s$</td>
</tr>
<tr>
<td>2</td>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\Re{s} &gt; 0$</td>
</tr>
<tr>
<td>3</td>
<td>$-u(-t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\Re{s} &lt; 0$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{t^{n-1}}{(n-1)!} u(t)$</td>
<td>$\frac{1}{s^n}$</td>
<td>$\Re{s} &gt; 0$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{t^{n-1}}{(n-1)!} u(-t)$</td>
<td>$\frac{1}{s^n}$</td>
<td>$\Re{s} &lt; 0$</td>
</tr>
<tr>
<td>6</td>
<td>$e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{s + \alpha}$</td>
<td>$\Re{s} &gt; -\alpha$</td>
</tr>
<tr>
<td>7</td>
<td>$-e^{-\alpha t} u(-t)$</td>
<td>$\frac{1}{s + \alpha}$</td>
<td>$\Re{s} &lt; -\alpha$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{(s + \alpha)^n}$</td>
<td>$\Re{s} &gt; -\alpha$</td>
</tr>
</tbody>
</table>
### Complete table of transform pairs 2/2

<table>
<thead>
<tr>
<th></th>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>(-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t))</td>
<td>(\frac{1}{(s+\alpha)^n})</td>
<td>(\Re{s} &lt; -\alpha)</td>
</tr>
<tr>
<td>10</td>
<td>(\delta(t - T))</td>
<td>(e^{-sT})</td>
<td>All (s)</td>
</tr>
<tr>
<td>11</td>
<td>([\cos \omega_0 t]u(t))</td>
<td>(\frac{\omega_0}{s^2 + \omega_0^2})</td>
<td>(\Re{s} &gt; 0)</td>
</tr>
<tr>
<td>12</td>
<td>([\sin \omega_0 t]u(t))</td>
<td>(\frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2})</td>
<td>(\Re{s} &gt; -\alpha)</td>
</tr>
<tr>
<td>13</td>
<td>([e^{-\alpha t} \cos \omega_0 t]u(t))</td>
<td>(\frac{\omega_0}{(s + \alpha)^2 + \omega_0^2})</td>
<td>(\Re{s} &gt; -\alpha)</td>
</tr>
<tr>
<td>14</td>
<td>([e^{-\alpha t} \sin \omega_0 t]u(t))</td>
<td>(\frac{\omega_0}{(s + \alpha)^2 + \omega_0^2})</td>
<td>(\Re{s} &gt; -\alpha)</td>
</tr>
<tr>
<td>15</td>
<td>(u_n(t) = \frac{d^n \delta(t)}{dt^n})</td>
<td>(s^n)</td>
<td>All (s)</td>
</tr>
<tr>
<td>16</td>
<td>(u_{-n}(t) = u(t) \ast \cdots \ast u(t))</td>
<td>(\frac{1}{s^n})</td>
<td>(\Re{s} &gt; 0)</td>
</tr>
</tbody>
</table>
Extra Laplace transform pair example 1

Consider the signal

\[ x(t) = e^{-5t}u(t) + e^{-\beta t}u(t) \]

and denote its Laplace transform by \( X(s) \). What are the constraints placed on the real and imaginary parts of \( \beta \) if the ROC of \( X(s) \) is given by \( \text{Re}\{s\} > -3 \)
Extra Laplace transform pair example 1 (cont.)

Using an analysis similar to that used in Example 9.3, we know that the given signal has a Laplace transform of the form

\[ X(s) = \frac{1}{s + 5} + \frac{1}{s + \beta} \]

The corresponding ROC is \( \text{Re}\{s\} > \max(-5, \text{Re}\{\beta\}) \). Since we are given that the ROC is \( \text{Re}\{s\} > -3 \), we know that \( \text{Re}\{\beta\} = 3 \). There are no constraints on the imaginary part of \( \beta \).
Extra Laplace transform pair example 2

◆ Determine the Laplace transform and ROC of the signal

\[ x(t) = e^{-4t}u(t) + e^{-5t}(\sin 5t)u(t) \]

◆ Solution
Using an approach similar to that shown in part (a), we have

\[ e^{-4t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+4}, \quad \Re\{s\} > -4. \]

Also,

\[ e^{-5t}e^{jt}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+5-j5}, \quad \Re\{s\} > -5. \]
Extra Laplace transform pair example 1 (cont.)

and

\[ e^{-5t}e^{-j5t}u(t) \leftrightarrow \frac{1}{s + 5 + j5}, \quad \Re\{s\} > -5. \]

From this we obtain

\[ e^{-5t}\sin(5t)u(t) = \frac{1}{2j} [e^{-5t}e^{j5t} - e^{-5t}e^{-j5t}]u(t) \leftrightarrow \frac{5}{(s + 5)^2 + 25}, \]

where \( \Re\{s\} > -5. \) Therefore,

\[ e^{-4t}u(t) + e^{-5t}\sin(5t)u(t) \leftrightarrow \frac{s^2 + 15s + 70}{s^3 + 14s^2 + 90s + 100}, \quad \Re\{s\} > -5. \]
Summary

- Laplace transform is used for the analysis of systems
  - Similar in operation to the Fourier transform but is associated with a region of convergence

- Region of convergence
  - Determines where the Laplace transform works
  - Often plotted in association with the poles and zeros of the transfer function

- Transform pairs
  - Transforms have been provided for several common functions used in the course, along with their ROCs