Lecture #24

EE 313 Linear Systems and Signals
Preview of today’s lecture

◆ The inverse Laplace transform
  ✤ Not normally computed directly as it requires a complex integral
  ✤ Use partial fractions and the table to compute the inverse

◆ Laplace transform properties
  ✤ Use properties to compute the Laplace transform and its inverse

◆ Note: several examples are provided with solutions for reference

◆ Relevant sections of Oppenheim and Willsky: 9.3, 9.5-9.6, A.2
Partial fraction expansion and Inverse Laplace

Key points

- Determine the signal in time-domain from its Laplace transform
- Use partial fraction expansion to solve LT problems
Laplace transform

- The Laplace transform for a general signal \( x(t) \) is defined as \( \mathcal{L}\{x(t)\} \)

\[
X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt
\]

- For \( s \) imaginary, i.e., \( s = j\omega \), this integral correspond to the FT
- For general values of the complex variable \( s \), it is referred to as the LT

- The values of \( s \) where the integral converges is the region of convergence
Inverse Laplace transform

- Given a LT \( X(s) \) of the signal \( x(t) \), this signal can be written as

\[
x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} \, ds
\]

- Notes
  - This is a contour integral on the complex plane
  - Direct computation is beyond the scope of this course

- The alternative is to use table of transforms pairs
## Table based approach to inverse Laplace transform

<table>
<thead>
<tr>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
<td>All $s$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\Re{s} &gt; 0$</td>
</tr>
<tr>
<td>$-u(-t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\Re{s} &lt; 0$</td>
</tr>
<tr>
<td>$t^{n-1} \frac{1}{(n-1)!} u(t)$</td>
<td>$\frac{1}{s^n}$</td>
<td>$\Re{s} &gt; 0$</td>
</tr>
<tr>
<td>$-t^{n-1} \frac{1}{(n-1)!} u(-t)$</td>
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<td>$\Re{s} &lt; 0$</td>
</tr>
<tr>
<td>$e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{s + \alpha}$</td>
<td>$\Re{s} &gt; -\alpha$</td>
</tr>
<tr>
<td>$-e^{-\alpha t} u(-t)$</td>
<td>$\frac{1}{s + \alpha}$</td>
<td>$\Re{s} &lt; -\alpha$</td>
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<tr>
<td>$t^{n-1} \frac{1}{(n-1)!} e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{(s + \alpha)^n}$</td>
<td>$\Re{s} &gt; -\alpha$</td>
</tr>
<tr>
<td>$-t^{n-1} \frac{1}{(n-1)!} e^{-\alpha t} u(-t)$</td>
<td>$\frac{1}{(s + \alpha)^n}$</td>
<td>$\Re{s} &lt; -\alpha$</td>
</tr>
</tbody>
</table>

- Laplace transform pairs for certain common functions
- Rewrite given Laplace transform to be linear comb. of table entries
- Use given ROC to find inverse

**Applied most often to $X(s)$ that are the ratio of two polynomials**
Partial fraction expansion (PFE)

- PFE is a technique for expanding ratios of two polynomials

- Example

\[ H(s) = \frac{3s - 5}{s^2 - 2s - 3} = \frac{2}{s + 1} + \frac{1}{s - 3} \]

- Special cases (see appendix A.2 for more details)
  - Unique roots in the denominator
  - Repeated roots in the denominator
  - Improper fractions (numerator order \(\geq\) denominator order)
Example: proper fraction with non-repeated roots

- Consider a proper fraction of two polynomials

\[ H(s) = \frac{F(s)}{G(s)} = \frac{F(s)}{(s + a)(s + b)(s + c)} \]

- It can be written as

\[ H(s) = \frac{A}{s + a} + \frac{B}{s + b} + \frac{C}{s + c} \]

\[
\begin{align*}
A &= H(s)(s + a) \bigg|_{s = -a} = \frac{F(-a)}{(b - a)(c - a)} \\
B &= H(s)(s + b) \bigg|_{s = -b} = \frac{F(-b)}{(a - b)(c - b)} \\
C &= H(s)(s + c) \bigg|_{s = -c} = \frac{F(-c)}{(a - c)(b - c)}
\end{align*}
\]
Example

- Partial fraction expansion with unique roots

\[
H(s) = \frac{3s - 5}{s^2 - 2s - 3} = \frac{3s - 5}{(s + 1)(s - 3)}
\]

\[
H(s)(s + 1) = \frac{A}{s + 1} + \frac{B}{s - 3}
\]

\[
H(s)(s + 1)|_{s=-1} = A + \frac{B}{s - 3}(s + 1)_{s=-1}
\]

\[
\frac{3s - 5}{(s + 1)(s - 3)}(s + 1)_{s=-1} = A
\]

\[
\frac{3s - 5}{s - 3}_{s=-1} = A
\]

\[
3(-1) - 5 = A
\]

\[
A = 2
\]
Example (continued)

- Partial fraction expansion unique roots

\[ H(s)(s - 3)|_{s=3} = \frac{A}{s+1}(s - 3)_{s=3} + \frac{B}{s-3}(s - 3)_{s=3} \]

\[ \frac{3s - 5}{(s + 1)(s - 3)}(s - 3)_{s=3} = B \]

\[ \frac{3s - 5}{(s + 1)}_{s=3} = B \]

\[ B = 1 \]
Example: proper fraction with repeated root

\[ Y(s) = \frac{3s + 2}{(s - 1)^2(s + 2)} = H(s)X(s) \]

repeated root \( \rightarrow te^{+t}u(t) \)

\[ Y(s) = (s - 1)^2 + \frac{B}{s - 1} + \frac{C}{s + 2} \]

Can be written as

\[ A = Y(s)(s - 1)^2 \bigg|_{s=1} = \frac{3 + 2}{3} = \frac{5}{3} \]

\[ B = \left[ \frac{d}{ds} \left( Y(s)(s - 1)^2 \right) \right] \bigg|_{s=1} \text{, tricky one} \]

\[ = \frac{d}{ds} \left( \frac{3s + 2}{s + 2} \right) \bigg|_{s=1} = \left( \frac{3}{s + 2} - \frac{3s + 2}{(s + 2)^2} \right) \bigg|_{s=1} \]

\[ = 1 - \frac{5}{3^2} = \frac{4}{9} \]
Example: proper fraction with repeated root (cont.)

\[ Y(s) = \frac{A}{(s - 1)^2} + \frac{B}{(s - 1)} + \frac{C}{s + 2} \]

\[ \begin{align*}
\text{Therefore} \\
A &= Y(s)(s - 1)^2 \bigg|_{s=1} = \frac{3 + 2}{3} = \frac{5}{3} \\
B &= \left[ \frac{d}{ds} (Y(s)(s - 1)^2) \right] \bigg|_{s=1} = \frac{4}{9} \\
C &= Y(s)(s + 2) \bigg|_{s=-2} = \frac{-4}{9}
\end{align*} \]

\[ \text{Giving} \]
\[ Y(s) = \frac{5/3}{(s - 1)^2} + \frac{4/9}{s - 1} - \frac{4/9}{s + 2} \]
Finding the inverse Laplace transform

- After applying the PFE, use Table 9.2 to find the inverse

- Consider again \( Y(s) = \frac{5}{3} \frac{1}{(s - 1)^2} + \frac{4}{9} \frac{1}{s - 1} - \frac{4}{9} \frac{1}{s + 2} \)

- If ROC is \( \text{Re}\{s\} > 1 \) \( y(t) = \frac{5}{3} te^t u(t) + \frac{4}{9} e^t u(t) - \frac{4}{9} e^{-2t} u(t) \)

- If ROC is \( \text{Re}\{s\} < -2 \) \( y(t) = -\frac{5}{3} te^t (-t) - \frac{4}{9} e^t (-t) + \frac{4}{9} e^{-2t} u(-t) \)

- If ROC is \(-2 < \text{Re}\{s\} < 1 \) \( y(t) = -\frac{5}{3} te^t (-t) - \frac{4}{9} e^t (-t) - \frac{4}{9} e^{-2t} u(t) \)
**Example: improper fraction**

- What if the order of $N(s) \geq \text{order of } D(s)$?

- Answer: use long division and apply linearity (discussed next)

- Example

$$H(s) = \frac{3s^2 + s + 1}{s^2 - 2s - 3} = 3 + \frac{10 + 7s}{s^2 - 2s - 3}$$

Apply partial fraction expansion to this term

Use table to find this
Solution depends on ROC

**ROC**

- **ROC** \( \Re\{s\} < -1 \)
  
  \[
  h(t) = -e^t u(-t) - e^{-3t} u(-t)
  \]

- **ROC** \( -1 < \Re\{s\} < 3 \)
  
  \[
  h(t) = e^t u(t) - e^{-3t} u(-t)
  \]

- **ROC** \( \Re\{s\} > 3 \)
  
  \[
  h(t) = e^t u(t) + e^{-3t} u(t)
  \]
Laplace transform properties

Key points
- Use LT properties to simplify calculation & build intuition
- Analyze problems that include LT properties
### Table 9.1: Properties of the Laplace Transform

<table>
<thead>
<tr>
<th>Section</th>
<th>Property</th>
<th>Signal</th>
<th>Laplace Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.5.1</td>
<td>Linearity</td>
<td>( x(t) )</td>
<td>( X(s) )</td>
<td>( R )</td>
</tr>
<tr>
<td>9.5.2</td>
<td>Time shifting</td>
<td>( x(t - t_0) )</td>
<td>( e^{-s t_0}X(s) )</td>
<td>( R )</td>
</tr>
<tr>
<td>9.5.3</td>
<td>Shifting in the s-Domain</td>
<td>( e^{s_0t}x(t) )</td>
<td>( X(s - s_0) )</td>
<td>At least ( R_1 \cap R_2 )</td>
</tr>
<tr>
<td>9.5.4</td>
<td>Time scaling</td>
<td>( x(at) )</td>
<td>( \frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td>9.5.5</td>
<td>Conjugation</td>
<td>( x^*(t) )</td>
<td>( X^<em>(s^</em>) )</td>
<td>( R )</td>
</tr>
<tr>
<td>9.5.6</td>
<td>Convolution</td>
<td>( x_1(t) * x_2(t) )</td>
<td>( X_1(s)X_2(s) )</td>
<td>At least ( R_1 \cap R_2 )</td>
</tr>
<tr>
<td>9.5.7</td>
<td>Differentiation in the Time Domain</td>
<td>( \frac{d}{dt}x(t) )</td>
<td>( sX(s) )</td>
<td>At least ( R )</td>
</tr>
<tr>
<td>9.5.8</td>
<td>Differentiation in the s-Domain</td>
<td>( -tx(t) )</td>
<td>( \frac{d}{ds}X(s) )</td>
<td>( R )</td>
</tr>
<tr>
<td>9.5.9</td>
<td>Integration in the Time Domain</td>
<td>( \int_{-\infty}^{t}x(\tau)d(\tau) )</td>
<td>( \frac{1}{s}X(s) )</td>
<td>At least ( R \cap {\Re{s} &gt; 0} )</td>
</tr>
</tbody>
</table>

Similar to FT, but we need to be careful about the ROC.
**LT properties - Linearity**

- **If**
  
  \[ x_1(t) \overset{\mathcal{L}}{\rightarrow} X_1(s) \quad \text{with ROC } R_1 \]
  
  \[ x_2(t) \overset{\mathcal{L}}{\rightarrow} X_2(s) \quad \text{with ROC } R_2 \]

- **Then**
  
  \[ ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\rightarrow} aX_1(s) + bX_2(s) \quad \text{with ROC containing } R_1 \cap R_2 \]
LT properties – Shifting in the time and s-domain

- Shifting in the time domain
  - If
    \[ x(t) \xrightarrow{\mathcal{L}} X(s) \quad \text{with ROC R} \]
  - Then
    \[ x(t - t_0) \xrightarrow{\mathcal{L}} e^{-st_0} X(s) \quad \text{with ROC R} \]

- Shifting in the s-domain
  - If
    \[ x(t) \xrightarrow{\mathcal{L}} X(s) \quad \text{with ROC R} \]
  - Then
    \[ e^{s_0 t} x(t) \xrightarrow{\mathcal{L}} X(s - s_0) \quad \text{with shifted version of R (i.e., s is in the ROC if s-s_0 is in R)} \]
LT properties – Time scaling

- If
  \[ x(t) \overset{\mathcal{L}}{\rightarrow} X(s) \]
  with ROC R

- Then
  \[ x(at) \overset{\mathcal{L}}{\rightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \]
  with scaled ROC (i.e., s is in the ROC if \( s/a \) is in R)

![Diagram showing ROCs for different values of a](image)
LT properties – Conjugation and differentiation

- If
  \[ x(t) \xrightarrow{\mathcal{L}} X(s) \]
  with ROC R

- Then
  \[ x^*(t) \xrightarrow{\mathcal{L}} X^*(s^*) \]
  with ROC R

- If
  \[ x(t) \xrightarrow{\mathcal{L}} X(s) \]
  with ROC R

- Then
  \[ \frac{dx(t)}{dt} \xrightarrow{\mathcal{L}} sX(s) \]
  with ROC at least R
LT properties – Convolution property

◆ If

\[ x_1(t) \overset{\mathcal{L}}{\longrightarrow} X_1(s) \quad \text{with ROC } R_1 \]
\[ x_2(t) \overset{\mathcal{L}}{\longrightarrow} X_2(s) \quad \text{with ROC } R_2 \]

◆ Then

\[ x_1(t) * x_2(t) \overset{\mathcal{L}}{\longrightarrow} X_1(s)X_2(s) \quad \text{with ROC containing } R_1 \cap R_2 \]
Example using properties

- Determine the Laplace transform of q(t)

\[ x(t) = e^{-3t}u(t) \]
\[ q(t) = e^{-t}x(2t) \]
\[ X(s) = \frac{1}{s + 3} \quad \text{ROC Re}\{s\} > -3 \]
\[ x_1(t) = x(2t) \]
\[ X_1(s) = \frac{1}{2} X(s/2) \]
\[ = \frac{1}{2} \frac{1}{s/2 + 3} \]
\[ = \frac{1}{s + 6} \]
\[ q(t) = e^{-t}x_1(t) \]
\[ Q(s) = X_1(s - (-1)) = \frac{1}{s + 7} \]
\[ q(t) = e^{-t}e^{-3(2t)}u(2t) = e^{-7t}u(t) \]
Example of using the table to find an inverse LT

- Given the following LT and ROC, find the time domain signal \( x(t) \)

\[
X(s) = \frac{s}{s^2 + 9}, \quad \text{Re}\{s\} < 0
\]

From Table 9.2 we know that

\[
\cos(3t)u(t) \overset{\mathcal{L}}{\rightarrow} \frac{s}{s^2 + 9}, \quad \text{Re}\{s\} > 0.
\]

Using the time scaling property, we obtain

\[
\cos(3t)u(-t) \overset{\mathcal{L}}{\rightarrow} -\frac{s}{s^2 + 9}, \quad \text{Re}\{s\} < 0.
\]

Therefore, the inverse Laplace transform of \( X(s) \) is

\[
x(t) = -\cos(3t)u(-t).
\]
Example where ROC changes

Consider the following problem

\[
X_1(s) = \frac{1}{(s-1)(s+1)} \quad \text{ROC Re}\{s\} > 1
\]

\[
X_2(s) = \frac{(s-1)}{(s+1)(s+3)} \quad \text{ROC Re}\{s\} > -1
\]

\[
X(s) = X_1(s)X_2(s)
\]

\[
= \frac{1}{(s+1)^2(s+3)} \quad \text{ROC Re}\{s\} > -1
\]
Summary

- Inverse Laplace transform is not directly computed in this course
  - Use Laplace transform table to find the inverse
  - Often involves partial fraction expansion

- Key to PFE is the Heaviside coverup technique
  - Separate ratio of two polynomials into a sum of polynomials

- Laplace transform properties
  - Properties are very similar to the Fourier transform
  - Need to pay attention to what happens with the ROC
Additional Examples
Example

- Determine the function of time $x(t)$ for each of the following Laplace transforms and their associated ROC

$$X(s) = \frac{s + 1}{s^2 + 5s + 6} \quad -3 < \text{Re}\{s\} < -2$$
Using partial fraction expansion on $X(s)$, we obtain

$$X(s) = \frac{2}{s + 3} - \frac{1}{s + 2}.$$

From the given ROC, we know that $x(t)$ must be a two-sided signal. Therefore,

$$x(t) = 2e^{-3t}u(t) + e^{-2t}u(-t).$$
Example

Consider the signal \( x(t) \) and denote its LT by \( X(s) \)

\[
x(t) = e^{-5t}u(t - 1)
\]

A - Find \( X(s) \) and determine its ROC

B - Determine the values of the finite numbers \( A \) and \( t_0 \) such that the LT of \( g(t) \) has the same algebraic form as \( G(s) \)

\[
g(t) = Ae^{-5t}u(-t - t_0)
\]

What is the ROC of \( G(s) \)
(a) \[ X(s) = \int_{-\infty}^{\infty} e^{-5t}u(t-1)e^{-st}dt \]
\[ = \int_{1}^{\infty} e^{-(5+s)t}dt \]
\[ = \frac{e^{-(5+s)}}{s+5} \]

As shown in Example 9.1, the ROC will be \( Re\{s\} > -5 \).

(b) By using eq. (9.3), we can easily show that \( g(t) = Ae^{-5t}u(-t-t_0) \) has the Laplace transform
\[ G(s) = \frac{Ae^{(s+5)t_0}}{s+5} \]

The ROC is specified as \( Re\{s\} < -5 \). Therefore, \( A = 1 \) and \( t_0 = -1 \).
Example (with some important terms)

- Let $x(t)$ be a signal that has a rational Laplace transform with exactly two poles located at $s = -1$ and $s = -3$.

- If $g(t) = e^{2t}x(t)$ and $G(j\omega)$ [FT of $g(t)$] converges, determine whether $x(t)$ is left sided, right sided, or two-sided?
Solution

From Table 9.1, we know that

\[ g(t) = e^{2t}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} G(s) = X(s - 2). \]

The ROC of \( G(s) \) is the ROC of \( X(s) \) shifted to the right by 2.

We are also given that \( X(s) \) has exactly 2 poles, located at \( s = -1 \) and \( s = -3 \). Since \( G(s) = X(s - 2) \), \( G(s) \) also has exactly two poles, located at \( s = -1 + 2 = 1 \) and \( s = -3 + 2 = -1 \). Since we are given \( G(j\omega) \) exists, we may infer that the \( j\omega \)-axis lies in the ROC of \( G(s) \). Given this fact and the locations of the poles, we may conclude that \( g(t) \) is a two sided sequence. Obviously \( x(t) = e^{-2t}g(t) \) will also be two sided.