Lecture #3

EE 313 Linear Systems and Signals
Preview of today’s lecture

◆ Continuous-time complex exponential review
  ✦ Explain damping and frequency

◆ Discrete-time complex exponentials
  ✦ Explain damping and frequency (in discrete time)
  ✦ Compute the frequency of a complex sinusoid

◆ Unit-impulse and unit–step functions
  ✦ Define and use the unit-pulse and unit-step functions
CT complex exponential review

Learning objectives

- Determine the key parameters of a complex exponential
- Sketch a complex exponential based on its form
CT complex exponential

- General form

\[ x(t) = Ce^{at} = |C|e^{j\theta}e^{at}, \quad C, a \text{ are complex} \]

\[ C = c_1 + jc_2 \]

**Cartesian to polar**

\[ |C| = \sqrt{c_1^2 + c_2^2} \quad \theta = \tan^{-1}\frac{c_2}{c_1} \]

**Polar to Cartesian**

\[ c_1 = \cos \theta \cdot |C| \]

\[ c_2 = \sin \theta \cdot |C| \]
**CT complex exponential**

- **Rewriting in polar form**

\[
x(t) = C e^{at} = |C| e^{j\theta} e^{at}, \quad C, a \text{ are complex}
\]

\[
a = r + j\omega
\]

\[
x(t) = |C| e^{(\theta + \omega t) + rt}
\]

\[
= |C| e^{rt} e^{j(\theta + \omega t)}
\]

- Increasing or decaying exponential “envelope”
- Complex sinusoid
- \(\omega\) is the frequency

\[
\text{Re}\{x(t)\} = |C| e^{rt} \cdot \cos(\omega t + \theta), \quad T = \frac{2\pi}{\omega}
\]
Example: Visualizing a complex exponential

Consider a typical complex exponential:

\[ x(t) = Ce^{at} \]

\[ C = 3 - j, \quad a = 1 + 10j \]

The Problem:

- Express \( x(t) \) in terms of its envelope and complex sinusoidal parts
- Express the real and imaginary parts of \( x(t) \)
- Express the magnitude of \( x(t) \)
- Plot all of these from time \( t = 0 \) to \( t = 3 \), showing the envelope
Solution and Matlab

- First, convert \( C \) to polar form and then simplify:

\[
x(t) = \sqrt{10}e^{-j \arctan(1/3)}e^{t(1+10j)}
= \sqrt{10}e^{t}e^{j(10t-\arctan(1/3))}
\]

- Apply Euler’s to get Real and Imaginary parts:

\[
Re\{x(t)\} = \sqrt{10}e^{t} \cos(10t - \arctan(1/3))
Im\{x(t)\} = \sqrt{10}e^{t} \sin(10t - \arctan(1/3))
\]

- Magnitude is simply:

\[
|x(t)| = \sqrt{10}e^{t}
\]

- What do we expect these plots to look like? What could this be in the real world?
Real and Imaginary parts are just $\pi/2$ shifts (magnitude is just the positive envelope)

$$Re\{x(t)\} = \sqrt{10}e^t \cos(10t - \arctan(1/3))$$
$$Im\{x(t)\} = \sqrt{10}e^t \sin(10t - \arctan(1/3))$$
Matlab to create these plots

\[ C = 3 - j; \]
\[ a = 1 + 10j; \]
\[ t = 0:0.01:3; \text{; \% equivalent to \textit{linspace} (0,3,101)} \]
\[ x = C*\exp(a*t); \]
\[ \text{env} = \text{abs}(C)*\exp(\text{real}(a) *t); \]

\[ \text{figure(1);} \]
\[ \text{plot}(t,\text{real}(x),t,\text{env},t,-\text{env}); \]
\[ \text{xlabel('time');} \]
\[ \text{ylabel('Real part of } x(t) \text{ and its envelope');} \]

\[ \text{figure(2);} \]
\[ \text{plot}(t,\text{imag}(x),t,\text{env},t,-\text{env}); \]
\[ \text{xlabel('time');} \]
\[ \text{ylabel('Imaginary parts of } x(t) \text{ and its envelope');} \]
DT exponential and sinusoidal signals

Learning objectives

- Explain the properties of exponential and sinusoidal signals
- Analyze problems that include exponential and sinusoidal signals
**DT complex exponentials**

- General form is \( x[n] = C\alpha^n \)

- Rewriting using the polar form

\[
x[n] = |C|e^{j\theta} \left( Re^{j\omega_0} \right)^n = |C| R^n e^{j\theta} e^{j\omega_0 n} = |C| R^n e^{j(\omega_0 n + \theta)}
\]

- Note: \(|C|R^n\) is complex envelope, \(\omega_0\) is frequency and \(\theta\) is the phase
Special case: real exponential

\[ x[n] = |C| R^n e^{j(\omega_0 n + \theta)} \]

- **Example**
  - \( \alpha > 1 \)
  - \( 0 < \alpha < 1 \)
  - \( \alpha < 0 \)

- **Example**
  \( (-\frac{1}{2})^n = (-1)^n (\frac{1}{2})^n \)
Example

- Plot the first three samples of

\[ x[n] = (-2)^n \]

\[ = \alpha^n, \quad \alpha = -2 \]

\[ = (-1)^n 2^n, \quad (ab)^n = a^n b^n \]
Special case: complex sinusoid

\[ x[n] = |C| R^n e^{j(\omega_0 n + \theta)} \quad \rightarrow \quad x[n] = e^{j\omega_0 n} \]

- Using Euler’s

\[ e^{j\omega_0 n} = \cos(\omega_0 n) + j \sin(\omega_0 n) \]

- A general discrete-time sinusoid can be written as

\[ A \cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n} \]
DT complex exponentials – high & low frequency

- Observe the following fact for integers \( k \) and \( n \) (\( n \) is the “time”)

\[
e^{j((\omega_0+2\pi k)n+\theta)} = e^{j(\omega_0 n + \theta + 2\pi kn)}
\]

\[
= e^{j(\omega_0 n + \theta)} e^{j2\pi kn}
\]

\[
= e^{j\omega_0 n + \theta} \quad \rightarrow \quad 1
\]

- This means that frequencies \( \omega_0 + 2\pi k \) are equivalent!!
  - We normally report the smallest value as the frequency
  - Use range \( \omega_0 \in [0, 2\pi] \) or \( \omega_0 \in [-\pi, \pi] \)

- \(|\omega_0| \) near 0 is low while \(|\omega_0| \) near \( \pi \) is high

- Equivalently, in Hz, frequency ranges from \([-\frac{1}{2}, \frac{1}{2}]\) multiplied by the sampling frequency \( 1/T_S \) (recall samples taken every \( T_S \) secs)
Illustrations of DT frequency and periodicity

```
figure(1);
n_range = 0:60;
increment = 0.05;
omega = 2*pi*increment;
for k=1:100
    % Note extra formatting below to make fonts and lines bigger
    stem(n_range, cos(omega*k*n_range), 'LineWidth', 3, 'MarkerSize', 9);
    xlabel('n');
    title(['cos(2\pi', num2str(increment*k, 4), 'n)'], 'FontSize', 28);
    set(gca, 'FontSize', 14)
    pause; % used for lecture to show successive plots, press enter
end; %k
```
Periodicity is **not guaranteed** in discrete-time

- $\cos\left(\frac{\pi n}{6}\right)$: periodic
- $\cos\left(\frac{3n}{6}\right)$: not periodic
Matlab code for reference

n_range = 0:60;
omega_0 = pi/6;
omega_1 = 3/6;

figure(2);
stem(n_range, cos(omega_0*n_range),'LineWidth',3, 'MarkerSize',9);
title(['cos(pi n /6 ) '],'FontSize',28);
set(gca,'ytick',[]);
print -f2 -depsc lecture3Fig2;

figure(3);
stem(n_range, cos(omega_1*n_range),'LineWidth',3, 'MarkerSize',9);
title(['cos(3 n /6 ) '],'FontSize',28);
set(gca,'ytick',[]);
print -f3 -depsc lecture3Fig3;
Explaining DT periodicity

- Consider \( C = 1, \quad \alpha = e^{j\omega_0} (= Re^{j\omega_0}, \quad R = 1) \)
  \[
  x[n] = (e^{j\omega_0})^n = e^{j\omega_0 n}
  \]

- When is this signal periodic?
  - Need to find \( N \) such that \( x[n + N] = x[n] \)
    \[
    x[n + N] = e^{j\omega_0(n+N)} = e^{j\omega_0 n} e^{j\omega_0 N}
    \]
  - Must have \( \omega_0 N = 2\pi k \) for some positive integer \( k \)

For arbitrary \( \omega_0 \) a discrete-time sinusoid is not periodic
Example

Determine the fundamental period of the periodic signal

\[ x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5} \]
Example

- Determine the fundamental period of the periodic signal

\[ x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5} \]

- Solution

periods

\[
\begin{align*}
    k \frac{2\pi}{4\pi/7} &= k \frac{1}{2/7} \\
    k \frac{2\pi}{2\pi/5} &= k \frac{1}{1/5}
\end{align*}
\]

\[ \rightarrow 7 \]

\[ \rightarrow 5 \]

Overall period is the least common multiple of the periods \( N = 35 \)

Use: \( N = 2\pi k/\omega_0 \) where \( k \) is smallest integer
Unit–step and unit-impulse functions

Learning objectives
- Explain the properties of unit-impulse and unit-step functions
- Understand how they can be used
- Analyze problems that include unit-impulse and unit-step functions
DT unit-impulse function

\[
\delta[n] = \begin{cases} 
1, & n = 0 \\
0, & n \neq 0 
\end{cases}
\]

Also known as the Kronecker delta function

\[
\sum_{n=-\infty}^{\infty} \delta[n] = 1
\]
Can build any DT sequence using the unit-impulse

- Example

\[ 5\delta[n - 3] \]

\[ 3\delta[n] - 2\delta[n + 1] + 2\delta[n - 1] \]
Sifting property of the impulse function

Consider

\[ x[n] \delta[n] = x[0] \delta[n] \]
Sifting property of the impulse function

What about?

In general, sifting property is:

\[ x[n] \delta[n - n_0] = x[n_0] \delta[n - n_0] \]
Examples

- Simplify the following expressions

\[ x[n] \delta[n + 1] \]
\[
\sum_{n=-\infty}^{\infty} x[n] \delta[n + 1] = \sum_{n=-\infty}^{\infty} x[-1] \delta[n + 1] = x[-1] \]
\[ (\cos(\pi n/4) + 1) \delta[n - 1] \]
\[
\sum_{n=-\infty}^{\infty} (\cos(\pi n/4) + 1) \delta[n - 1] = \sum_{n=-\infty}^{\infty} (\cos(\pi 1/4) + 1) \delta[n - 1] = (\cos(\pi 1/4) + 1) \]

\[ x[n] \delta[n - n_0] = x[n_0] \delta[n - n_0] \]

- Signal

- Value
DT unit-step function

\[ u[n] = \begin{cases} 
1, & n \geq 0 \\
0, & n < 0 
\end{cases} \]
DT unit-step function

**Example**

\[ 3u[n - 2] \]

\[ u[n] - u[n - 3] \]

**Note:** 
\[ u[n] - u[n - 3] = \delta[n] + \delta[n - 1] + \delta[n - 2] \]
Using the sifting property with unit step functions

- Consider

\[ x[n] = \alpha^n u[n], \quad \alpha = 2 \]

\[ = 2^n u[n] \]

- Simplify

\[ y[n] = x[n] \delta[n - 10] \]

\[ = x[10] \delta[n - 10] \]

\[ = 2^{10} u[10] \delta[n - 10] \]

\[ = 1024 \cdot \delta[n - 10] \]
Connections between impulse and step functions

- Important relations

\[
\delta[n] = u[n] - u[n - 1]
\]

\[
\delta[n - n_0] = u[n - n_0] - u[n - (n_0 + 1)]
\]

\[
u[n] = \sum_{m=-\infty}^{\infty} \delta[m]
\]

\[
u[n] = \sum_{m=0}^{n} \delta[n - m]
\]

Also

\[
\delta[n]\delta[n] = \delta[n]
\]

\[
u[n]u[n] = u[n]
\]
CT unit step function

\[ u(t) = \begin{cases} 
0 & t < 0 \\
1 & t > 1 
\end{cases} \]

- At \( t=0 \), \( u(t) \) may be either 0, 1, or \( \frac{1}{2} \) depending on the book
  - The specific choice is only important in a mathematical analysis class
CT unit step function examples

- Examples

\[ 2u(t - 3) \]

\[ u(t + \frac{1}{2}) - u(t - \frac{1}{2}) = \text{rect}(t) \]

This is another common function
CT unit-impulse “delta” function

Unit area rectangle function of width D and height 1/D
As D goes to 0 this becomes $\delta(t)$

Think of the delta function as an extremely short burst of energy or as a disturbance, like hitting a table

Putting a delta into a system leads to the impulse response, which is an important way to characterize a system
Important relationships

Unit area

\[ \frac{du(t)}{dt} = \delta(t) \]

\[ u(t) = \int_{-\infty}^{t} \delta(t) \, dt \]

[Graph]

\[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

\[ x(t) = \delta(t) \]

[Graph]

\[ x(t) = u(t) \]

[Graph]
Key use of impulse function is sifting

\[ x(t) \delta(t) = x(0) \delta(t) \]
\[ x(t) \delta(t - t_0) = x(t_0) \delta(t - t_0) \]

Example

\[ x(t) = 2t \]
\[ y(t) = x(t) (\delta(t - 2) + \delta(t - 4)) \]
\[ y(t) = 4\delta(t - 2) + 8\delta(t - 4) \]
**Examples**

- Simplify the following expressions

\[
\int_{-\infty}^{\infty} \cos \left( \frac{\pi t^2}{2} \right) \delta(t + 2) \, dt
\]

\[
\int_{-\infty}^{\infty} \delta(t - 2) \delta(t + 2) \, dt
\]

\[
\int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) \, d\tau
\]

\[
\int_{-\infty}^{\infty} \cos \left( \pi \frac{(-2)^2}{2} \right) \delta(t + 2) \, dt = \cos (2\pi) = 1
\]

0

\[
x(t)
\]
Cautionary notes on the delta function

- The continuous time unit-impulse function, also called the Direct delta function, is really a generalized function
  - It does not really behave like a normal function
  - It is either zero or infinity
  - Note also that unlike discrete time case, here $\delta(t)\delta(t) = \infty$.
- We should technically only be using $\delta(t)$ under the integral sign
  - It is well defined in the integral sign
  - Some Professors will complain if not in the integral sign (but not me)
  - Take real analysis in the math department for further enlightenment
- Despite these quirks, the delta function is extremely useful for modeling and understanding signals and systems
  - “All models are wrong, but some are useful” – George Box (famous statistician)
Summary of today’s lecture

◆ Complex exponential review
  ✦ Determine the period and fundamental frequency
  ✦ Explained the differences between continuous and discrete-time

◆ Unit-impulse (aka “delta”) and unit-step functions
  ✦ Define and use the unit-impulse function in discrete and continuous time, understanding differences between them
  ✦ Define and use the unit-step function in discrete and continuous time, understand its relation to the delta function