Lecture #4

EE 313 Linear Systems and Signals
Preview of today’s lecture

◆ Review of delta and unit-step functions
  ✦ Explain the properties of unit-impulse and unit-step functions
  ✦ Analyze problems that include unit-impulse and unit-step functions

◆ CT and DT system examples
  ✦ Describe the output of the systems in terms of their inputs
  ✦ Provide CT and DT system examples

◆ System properties
  ✦ Determine if a system is memoryless or has memory
  ✦ Determine if a system is causal
  ✦ Determine if a system is invertible
Review of delta and unit-step functions

Learning objectives

- Explain the properties of unit-impulse and unit-step functions
- Analyze problems that include unit-impulse and unit-step functions
Review of delta and unit-step functions

\[ \delta[n] \]

\[ u[n] \]

\[ \delta(t) \]

\[ u(t) \]
Properties of the delta function

- **Sifting (or sampling) property**
  \[ x(t)\delta(t) = x(0)\delta(t) \]
  \[ x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0) \]

- **Integration property**
  \[ \int x(t)\delta(t)\,dt = x(0) \]
  \[ \int x(t)\delta(t - \tau)\,d\tau = x(t) \]
  \[ \int x(t)y(t)z(t)\delta(t - \tau)\,d\tau = x(t)y(t)z(t) \]

Avoid common error: be sure to leave in delta function!
Example

Consider a periodic signal with a period of 2. On (0,2) it is:

\[ x(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
-2 & 1 \leq t < 2 
\end{cases} \]

Relate the derivative of this signal to the impulse train

\[ \sum_{n} \delta(t - nT) \]
Example solution

- Plot the function

- Use the property

\[ \frac{du(t)}{dt} = \delta(t) \]
Example solution

Key is to be able to deduce that:

\[ x(t) = -2 + \ldots + 3u(t + 2) - 3u(t + 1) + 3u(t) - 3u(t - 1) + \ldots \]

\[ = -2 + \sum_{n=-\infty}^{\infty} 3u(t - 2n) - \sum_{n=-\infty}^{\infty} 3u(t - 2n - 1) \]
Example solution

- Using the property

\[ \frac{du(t)}{dt} = \delta(t) \]

(and assuming we can move the derivative inside the sum)

\[ x(t) = -2 + \sum_{n=-\infty}^{\infty} 3u(t - 2n) - \sum_{n=-\infty}^{\infty} 3u(t - 2n - 1) \]

\[ g(t) = \frac{dx(t)}{dt}, \text{ where} \]

\[ g(t) = \sum_{n=-\infty}^{\infty} 3\delta(t - 2n) - \sum_{n=-\infty}^{\infty} 3\delta(t - 2n - 1) \]
Delta and unit step summary

- Delta function is a building block for other signals
  - Kronecker delta (discrete time) is well defined and easy to understand
  - Dirac delta (continuous time) requires more care and is conceptually more subtle: a limiting case of a “normal” function

- Delta function is also called the “unit impulse function”

- Unit step function is commonly used so that “signals start at 0” (or some other convenient time)
  - Important not to forget the unit step if part of signal or solution
Introduction to Systems

Learning objectives

- Describe the output of the systems in terms of their inputs
- Provide CT and DT system examples
Continuous and discrete time systems

Input signal $x(t)$  \rightarrow \text{CT System} \rightarrow \text{Output signal } y(t)$

Input signal $x[n]$  \rightarrow \text{DT System} \rightarrow \text{Output signal } y[n]$
Amplifier (or all-pass amplifier)

\[ x(t) \xrightarrow{A} y(t) = A \cdot x(t) \]

- A is the gain
- Passes all input frequencies equally
  - In our parlance, it does not filter \( x(t) \)
Amplitude modulation

At the receiver:

\[ y(t) = x(t) \cos(\omega_c t) \]

Amplitude Modulation Transmitter (AM radio)

\[ x(t) \]

\[ \cos(\omega_c t) \]

\[ y(t) \]

Information “rides” the carrier

\[ \cos(2\omega_c t) \]

\[ x(t) \]

\[ \frac{1}{2} (1 + \cos(2\omega_c t)) x(t) \]

\[ x(t) \]

Filter out high frequencies

Note: Only works for certain bandlimited signals \( x(t) \), not true in general
Accumulating credit card balance

- Input $x[n]$ is expenses in month $n$
- Output $y[n]$ is balance in month $n$
- Setup of problem
  - Pay 2% of the balance each month
  - Interest rate is $r$ per month
- What is governing equation and block diagram for this “system”?

\[
y[n] = (1 + r)y[n - 1] - 0.02y[n - 1] + x[n] \\
= (0.98 + r)y[n - 1] + x[n]
\]

Note: assumes balance computed at end of month ($T = 1$ month)

Output depends on current input and past outputs (causal)
RC circuit example

Remember KCL:

\[ v_r(t) = i(t)R \]

\[ v_{out}(t) = \int^{t}_{-\infty} i(t) \frac{1}{C} dt \]

\[ i(t) = C \frac{v_{out}}{dt} \]

\[ v_{out}(t) + v_r(t) = v_{in}(t) \]

\[ v_{out}(t) + i(t)R = v_{in}(t) \]

Solution involves solving the DE

\[ v_{out}(t) + RC \frac{dv_{out}}{dt} = v_{in}(t) \]
Inter-connected systems – “systems-of-systems”

- **Serial**

  \[ x(t) \rightarrow \text{System 1} \rightarrow \text{System 2} \rightarrow y(t) \]

- **Parallel**

  \[ x(t) \rightarrow \text{System 1} \rightarrow \text{System 2} \rightarrow \text{System 3} \rightarrow y(t) \]
Inter-connected systems – systems-of-systems

Feedback

Widely used in control systems
System properties

Learning objectives
- Classify systems based on their properties
- Distinguish between different system properties
System properties

- Simple properties
  ✦ Memory
  ✦ Causality
  ✦ Invertibility

- Stability
- Time invariance
- Linearity

Today’s lecture
Memoryless systems

- A system is **memoryless** if the output depends **only** on the value of the current input – Otherwise, it is said to have **memory**

- Examples: AM radio

\[
y(t) = x(t) \cos \omega_c t \\
\rightarrow \text{memoryless}
\]

\[
y(t) = x(t) \cos \omega_c (t - 3) \\
\rightarrow \text{memoryless}
\]

\[
y(t) = x(t - 3) \cos \omega_c (t - 3) \\
\rightarrow \text{memory}
\]
Other examples

- Integral

\[ y(t) = \int_{-\infty}^{t} x(\tau) d\tau \]

has memory (depends on all prior values of input)

- RC Circuit (tricky one)

\[ v_{out} + RC \frac{dv_{out}}{dt} = v_{in}(t) \]

derivative is memoryless, but

\[ v_{out} = \frac{1}{C} \int i(t) dt \]

has memory from the capacitor, so this system has memory. (Can also see the feedback from the output in its system block diagram)
Causality

- A system is causal if it only depends on the current and past inputs.
  ✦ Otherwise, it is noncausal.
  ✦ Very important property: if response depends on future inputs, good luck building that system!

- Examples
  - $y(t) = (t + 3)x(t - 3)$ Causal
  - $y(t) = x(-t + 1)$ Noncausal (why?)
Invertibility

- Informally, can the input be recovered from the output?
  - Yes → the system is invertible
- More formally: distinct inputs lead to distinct outputs
- Examples

\[
y(t) = \int_{-\infty}^{t} x(\tau) d\tau \quad \Rightarrow \quad x(t) \quad \Rightarrow \quad \int(.) dt \quad \Rightarrow \quad \frac{d}{dt}(.) \quad \Rightarrow \quad x(t)
\]

Invertible
Example

- Downsampler
  \[ y[n] = x[2n] \]

- Is it
  - Memoryless?
  - Causal?
  - Invertible?
Example

- Downsampler

\[ y[n] = x[2n] \]

- Is it
  - Memoryless? No
  - Causal? No
  - Invertible? No
Example

\[ x(t) \xrightarrow{\text{System}} \log |x(t)| \]

- Is it
  - Memoryless?
  - Causal?
  - Invertible?
Example

\[ x(t) \xrightarrow{\text{System}} \log |x(t)| \]

- Is it
  - Memoryless? Yes
  - Causal? Yes
  - Invertible? No
    - Lose the sign of \( x(t) \)
    - Lose the real and imaginary parts
Examples

\[ y(t) = x(t/3) \]

- Is it
  - Memoryless?
  - Causal?
  - Invertible?
Examples

\[ y(t) = x\left(\frac{t}{3}\right) \]

- Is it
  - Memoryless? No
  - Causal? No
  - Invertible? Yes
Example

\[ y(t) = \cos(3t)x(t) \]

- Is it Memoryless?
- Causal?
- Invertible?
Example

\[ y(t) = \cos(3t)x(t) \]

◆ Is it
  ✦ Memoryless? Yes
  ✦ Causal? Yes

✦ Invertible?
  • No, in general.
  • Can’t divide by \( \cos(3t) \) since you’ll divide by zero twice a period.
  • Can’t usually filter this out either (that integrates \( x(t) \) as well, and/or we can’t separate positive and negative frequency components)
System properties

- **Memory**
  - Does a system have memory or not?
  - Check if there is any dependence on $t$ or $n$ other than the current instant of time

- **Causality**
  - Does the system depend on future inputs?
  - Check if there is dependence on $t$ or $n$ ahead of the current instant of time

- **Invertibility**
  - Can the input be computed from the output?
  - Check the mathematical description of the system
Summary of lecture

- Review of delta and unit-step functions
  - Explain the properties of unit-impulse and unit-step functions
  - Analyze problems that include unit-impulse and unit-step functions
- CT and DT system examples
  - Describe the output of the systems in terms of their inputs
  - Provide CT and DT system examples
- System properties
  - Determine if a system is memoryless or has memory
  - Determine if a system is causal
  - Determine if a system is invertible