Preview of today’s lecture

◆ Brief review
  ✦ Determine the output of an LTI system in terms of the input
  ✦ Calculate the DT convolution of two signals

◆ Continuous-time convolution
  ✦ Define the impulse response of a continuous-time system
  ✦ Determine the output of an LTI system using the convolution
  ✦ Compute the continuous-time convolution between two signals
Brief review

Learning objectives

- Determine the output of an LTI system in terms of the input
- Calculate the DT convolution of two signals
Impulse response of an LTI system

- If the system has an impulse response $h[n]$, then

\[ \delta[n] \xrightarrow{\text{LTI System}} h[n] \]

- Time shift (follows from time invariance)

\[ \delta[n - 1] \xrightarrow{\text{LTI System}} h[n - 1] \]

- Scaling (follows from linearity)

\[ A\delta[n - 1] \xrightarrow{\text{LTI System}} Ah[n - 1] \]
Output of an LTI system to a simple input

- Consider an LTI system with impulse response $h[n]$, and input

$$x[n] = x[0]\delta[n] + x[1]\delta[n - 1]$$

- Then, the output of the system will be

$$x[0]h[n] + x[1]h[n - 1]$$
DT convolution

- Generally, for any input

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \]

- The system output is given by the convolution

\[ y[n] = x[n] \ast h[n] \]
Important convolution properties

- Convolution is commutative

\[ h[n] \ast x[n] = x[n] \ast h[n] \]

\[ \sum_{k=-\infty}^{\infty} x[k]h[n - k] = \sum_{k=-\infty}^{\infty} h[k]x[n - k] \]

- Can switch the order of the items being summed to simplify

- Remember the length property:
  - If \( h[n] \) has a duration of \( M \) samples, and \( x[n] \) has a duration of \( N \) samples, then \( y[n] \) has a duration of \( M + N - 1 \) samples.
  - Good sanity check for your answer when convolving two finite-duration sequences
One Last Discrete Time Convolution Example

◆ Consider this input and impulse response:

\[ x[n] = \delta[n + 1] + \delta[n] + \delta[n - 1] \]
\[ h[n] = u[n] - u[n - 5] \]

◆ Determine the output \( y[n] \) in two different ways:
  ✦ “Brute force” (considering each input sample)
  ✦ “Flip and slide” graphical method
Example Solution

- **Plot the two signals** (show that $x[n] = u[n+1] - u[n-2]$)

- **Brute Force Method**
  - Each input sample triggers the impulse response
  - Thus $y[n] = h[n+1] + h[n] + h[n-1]$
  - Plot these 3 functions and manually sum them up

- **Flip and slide method**
  - Easy to flip and slide $h[n]$
  - Show each step and get to the same answer as the previous method
  - Builds intuition on what happens when convolving two rectangle functions
Example Solution (brute force)

$h[n+1]$

$h[n]$

$h[n-1]$

$y[n]$
Example Solution (flip and slide)

\[ x[k] \]

\[ h[n-k] \]
MATLAB Demo

- http://dspfirst.gatech.edu/
Continuous-time convolution

Learning objectives
- Define the impulse response of a continuous-time system
- Determine the output of an LTI system using the convolution
- Compute the continuous-time convolution between two signals
Consider the input signal $\delta(t)$

- The output corresponding to this input is the **impulse response**
  - The resulting sequence is usually called $h(t)$

- All systems have an impulse response, but:
  - The impulse response is special **only** for LTI systems
  - Focus on LTI systems throughout this course
Example (typical “first-order” differential system)

\[
\delta(t) \rightarrow \text{System} \rightarrow y(t) = h(t)
\]

delta function input                  Exponential function out

**Linearity**

i. \(a \delta(t) \rightarrow a h(t) = a e^{-t} u(t)\)

ii. \(\delta(t) - \delta(t - T_0) h(t) + h(t - T_0)\)

**A “delta-like” function**

\(x(n\Delta t)\)

\(\Delta t\)

**Figure 3:** Area "Base \(\Delta t\) Height "

\(x(n\Delta t)\)

\(\Delta t\)

Impulse of area 1, for small \(\Delta t\)
Time invariance

Because of time invariance

\[ \delta(t - T_0) \rightarrow h(t) \rightarrow h(t - T_0) \]

Example

\[ \delta(t - T_0) \rightarrow h(t) \rightarrow h(t - T_0) \]

Shifts in the input shift the output

When written like this, implies an LTI system with impulse response \( h(t) \).
Linearity

- Because of the homogenous property

\[ \alpha \delta(t) \rightarrow h(t) \rightarrow \alpha h(t) \]

Scaling the input scales the output

- Because of the additive property

\[ \alpha \delta(t) + \beta \delta(t - T_0) \rightarrow h(t) \rightarrow \alpha h(t) + \beta h(t - T_0) \]

A sum of inputs leads to a sum of outputs
Stair step approximation of an input signal

- Consider the rectangle function

- Suppose that we approximate a signal using this function

\[
x(3\Delta)\delta(t - 3\Delta) = x\left(\frac{3T}{10}\right)\delta_{\frac{T}{10}}\left(t - \frac{3T}{10}\right)
\]

\[
\Delta = \frac{T}{10}
\]
Stair step approximation of an input signal

- Write the stair case approximation of a function as

\[
\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \Delta
\]

- Recall from the derivation of the delta function

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} \delta_\Delta(t) = \delta(t)
\]
Stair step approximation of an input signal

- Now taking the limit

\[
\lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau = x(t)
\]

Easy to see via sampling property that this must be true.

Any signal can be written as an integral of itself with shifted deltas.
Back to the LTI system

◆ What if we put in

\[ x(\tau)\delta(t - \tau) \rightarrow h(t) \rightarrow x(\tau)h(t - \tau) \]

◆ How about

\[ x(\tau_1)\delta(t - \tau_1) + x(\tau_2)\delta(t - \tau_2) \rightarrow h(t) \rightarrow x(\tau_1)h(t - \tau_1) + x(\tau_2)h(t - \tau_2) \]
Uncovering the convolution

◆ Now putting \( x(t) \) in the integral format, can see that:

\[
\int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \quad \rightarrow \quad h(t) \quad \rightarrow \quad \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
\]

◆ Thus, the input and output of an LTI system are related via the convolution integral:

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau
\]
Basic convolution properties

◆ Commutative

\[ y(t) = x(t) \ast h(t) = \int x(\tau) h(t - \tau) d\tau \]

\[ = h(t) \ast x(t) = \int h(\tau) x(t - \tau) d\tau \]

◆ Associative

\[ f(t) \ast [g(t) \ast h(t)] = [f(t) \ast g(t)] \ast h(t) \]

◆ Distributive

\[ f(t) \ast (h(t) + g(t)) = f(t) \ast h(t) + f(t) \ast g(t) \]

Same properties hold in DT case as well
Example #1

- Find the output of a system with impulse response

\[ h(t) = (1 - t)[u(t) - u(t - 1)] \]

for the input

\[ x(t) = u(t) - u(t - 1) \]
Example #1: Graphical solution

- Plot one signal versus $\tau$
- **Reverse** the second signal and shift it by $t$
  - Here, plot it to the left of $h(\tau)$
  - So plotted $t$ has a negative value, usually

There are 4 intervals (why?):

- $t < 0$
- $0 \leq t \leq 1$
- $1 \leq t \leq 2$
- $2 < t$
Example #1: First Interval

First interval: there is no overlap!

\[ y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) \, d\tau \]

- \[ t < 0, \quad h(\tau) x(t - \tau) = 0 \]
- \[ y(t) = 0 \]
Example #1: Second Interval

- Second interval \(0 \leq t \leq 1\)

\[
y(t) = \int_0^t (1 - \tau) d\tau = \tau - \frac{\tau^2}{2}\bigg|_0^t = t - \frac{t^2}{2}
\]
Example #1: Third Interval

- Third interval \(1 \leq t \leq 2\)

\[
y(t) = \int_{t-1}^{1} (1 - \tau) d\tau
\]

\[
= \tau - \frac{\tau^2}{2} \bigg|_{t-1}^{1}
\]

\[
= 1 - \frac{1}{2} - \left(t - 1 - \frac{(t - 1)^2}{2}\right)
\]

\[
= \frac{t^2}{2} - 2t + 2
\]

Integrating the same function, but with different integration limits
Example #1: Fourth and Final Interval

- Fourth interval \(2 < t\)

\[y(t) = 0\]

- Sanity checks
  - Check at \(t=1\), the output should be the same for intervals 2 & 3 in order for it to be continuous
  - Same for \(t = 0\) and \(t = 2\) (should be zero there)
  - Duration of output should be \(T_1 + T_2 = 1+1 = 2\)
Example #1: Plot of $y(t)$
Length/Duration of a convolution in continuous time

$z(t)$ Length is $T_1$

$h(t)$ Length is $T_2$

$z(t) * h(t)$ Length is $T_1 + T_2$
Length of a convolution in continuous time

- $z(t)$: Length is $T_0 + T_1$
- $h(t)$: Length is $T_2$
- $z(t) \ast h(t)$: Length is $T_0 + T_1 + T_2$
Animation example

http://www.cse.yorku.ca/~asif/spc/ConvolutionIntegral_Final3.swf
Summary of lecture

- Output of an LTI system is completely characterized by the impulse response of the system

\[ \delta(t) \xrightarrow{\text{LTI system}} h(t) \]

- Input and output of an LTI system are related through convolution

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]