Lecture #8

EE 313 Linear Systems and Signals

Convolution! Yeah!
Preview of today’s lecture

◆ Continuous-time convolution (continued)
  ✦ Review: Impulse response of a CT LTI system
  ✦ Two more examples

◆ LTI systems properties in terms of the impulse response
  ✦ Determine system properties from inspecting the impulse response
  ✦ Step response of LTI systems

◆ Relevant sections of Oppenheim and Willsky: 2.2 and 2.3
Impulse response of a CT LTI system

- If the system has an impulse response $h(t)$, then

$$\delta(t) \rightarrow h(t) \rightarrow h(t)$$

- The output of an LTI system is determined by the convolution

$$x(t) \rightarrow h(t) \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$
Convolution with an impulse function is a “No-Op”

Convolution with $\delta(t)$

\[
f(t) \ast \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau
\]

\[
= \int_{-\infty}^{\infty} f(t)\delta(t - \tau)d\tau
\]

\[
= f(t) \int_{-\infty}^{\infty} \delta(t - \tau)d\tau
\]

\[
= f(t)
\]

Similarly, \( f(t) \ast \delta(t - t_0) = f(t - t_0) \)
Convolutions example #1

Determine and sketch the convolution of the following input and impulse response:

\[ x(t) = \delta(t) + 2\delta(t - 1) \]

\[ h(t) = \begin{cases} 
  t, & 0 \leq t \leq 1, \\
  2 - t, & 1 < t \leq 2, \\
  0, & \text{elsewhere}
\end{cases} \]
Solution sketch

\[ x(t) \]

\[ y(t) \]

\[ h(t) \]

\[ 2h(t-1) \]
Convolution example #1 – solution sketch

- Plot $h(t)$, which is a unit height and area triangle from $t = 0$ to $2$.
- $Y(t) = h(t) + 2h(t-1)$
  - $t$, from $t = 0$ to $2$
  - $6 - 2t$, from $t = 2$ to $3$
- Verify it meets the length property ($2 + 1 = 3$)
Convolution example #2

- Determine and sketch \( y(t) = x(t) \ast h(t) \), where

\[
x(t) = \begin{cases} 
1, & 0 \leq t \leq 1 \\
0, & \text{elsewhere}
\end{cases}
\]

\[
h(t) = x(t/\alpha) \quad 0 < \alpha \leq 1
\]
Convolution example #2 – solution sketch

Therefore,

\[ y(t) = \begin{cases} 
  t, & 0 \leq t \leq \alpha \\
  \alpha, & \alpha \leq t \leq 1 \\
  1 + \alpha - t, & 1 \leq t \leq (1 + \alpha) \\
  0, & \text{otherwise}
\end{cases} \]
LTI systems properties in terms of the impulse response

Learning objectives

- Relate system properties to impulse response characteristics
- Determine system properties from the impulse response
Memoryless

◆ A memoryless system must have an impulse response that is simply a scaled impulse

    For discrete-time systems

\[ h[n] = k\delta[n] \]
\[ y[n] = h[n] \ast x[n] = kx[n] \]

    For continuous-time systems

\[ h(t) = k\delta(t) \]
\[ y(t) = kx(t) \]

◆ Can’t do very much with a memoryless system!
Invertibility

- An LTI system is invertible if there exists $g[n]$ such that

$$h[n] \ast g[n] = \delta[n]$$

- The definition is similar for continuous-time
- Finding the correct $g[n]$ or $g(t)$ can be a challenge
- Question: If $h[n]$ is causal and has memory, will $g[n]$ be causal?
**Invertibility example 1**

- Is this system defined by the impulse response below invertible?
  
  \[ h(t) = \delta(t - \epsilon), \quad \epsilon > 0 \]

- Yes, it is invertible because
  
  \[ g(t) = \delta(t + \epsilon), \quad \text{(shift back)} \]

  satisfies

  \[ h(t) * g(t) = \delta(t) \]
Invertibility example 2

- Consider the system defined by the impulse response \( h[n] = u[n] \)
- The output of the system is given by

\[
y[n] = h[n] \ast x[n]
\]

\[
= \sum_{k=-\infty}^{\infty} x[k] h[n - k], \quad (\text{definition of conv})
\]

\[
= \sum_{k=-\infty}^{\infty} x[k] u[n - k]
\]

\[
= \sum_{k=-\infty}^{n} x[k]
\]
Invertibility example 2 (continued)

\[ y[n] = \sum_{k=-\infty}^{n} x[k] \]

- This function is called an “accumulator”
- Key is to note that the output can be written recursively as
  \[ y[n] = x[n] + x[n - 1] + x[n - 2] \cdots \]
  \[ \underbrace{= y[n - 1]} \]
- Therefore, \( x[n] \) can be recovered from \( y[n] \) as
  \[ x[n] = y[n] - y[n - 1] \]
Invertibility example 2 (continued)

- The inverting system impulse response is

\[ g[n] = \delta[n] - \delta[n - 1] \]

- Check:

\[ h[n] \ast g[n] = \delta[n] \]

\[ u[n] \ast (\delta[n] - \delta[n - 1]) = u[n] - u[n - 1] = \delta[n] \]
Causality in discrete-time

◆ Consider the output of a discrete-time LTI system

\[ \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k] \]

depends on \( x[n+1], x[n+2], \text{etc} \) (future values)

depends on \( x[n], x[n-1], \text{etc} \) (past values)

◆ A discrete-time system is causal if \( h[n]=0 \) for \( n < 0 \)
  
  ✫ Intuitive: an impulse at time 0 should not cause any response before it actually happens!
Causality in continuous-time

- Same idea: a continuous-time system is **causal** if \( h(t) = 0 \) for \( t < 0 \)
- An impulse at time 0 should not cause a system response to occur before the impulse actually happens!
Stability

- LTI system is BIBO stable if and only if its impulse response is absolutely summable (or integrable) to a finite value

For discrete-time systems

\[
\sum_{k=-\infty}^{\infty} |h[k]| < \infty
\]

For continuous-time systems

\[
\int_{-\infty}^{\infty} |h(t)| \, dt < \infty
\]
Why absolute summability for stability?

- Condition ensures that the output of a bounded input is bounded
  - Consider $|x[n]| < B, \forall n$, where $B$ is a constant
  - The output, using the Cauchy-Schwarz inequality, will be
    \[
    |y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n - k] \right| 
    \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n - k]| 
    \]
    \[
    |y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| 
    \]
    If this term is bounded then, the output is bounded

Note we have proven here that if the impulse response is bounded then it is stable but can also show that if at LTI system is BIBO stable then the impulse response is absolutely summable.
Step response: DT systems

- Step response $s[n]$ is the output of the system when $x[n] = u[n]$

\[
 u[n] \rightarrow \text{System} \rightarrow s[n] \triangleq h[n] * u[n]
\]

\[
 \sum_{k=-\infty}^{\infty} h[k]u[n - k] = \sum_{k=-\infty}^{n} h[k]
\]

- Important response conceptually, what happens when I “turn the input on to a constant value and leave it on”?
- Note: $h[n] = s[n] - s[n - 1]$
Example: DT step response calculation

Determine the step response of the system with impulse response

\[ h[n] = \alpha^n u[n] \]

Convolve the impulse response with the step function

\[ s[n] = \sum_{k=-\infty}^{n} \alpha^k u[k] = \sum_{k=0}^{n} \alpha^k \]

\[ = \frac{1 - \alpha^{n+1}}{1 - \alpha} \]

Typically \( s[n] \) either approaches an asymptote such as \( (1 - \alpha)^{-1} \) here, or it goes to infinity (for an unstable system)
Step response – CT systems

- The output of the system when the input is a unit-step function

\[ u(t) \xrightarrow{\text{System}} s(t) : \]

\[ s(t) \triangleq h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau \]

\[ = \int_{-\infty}^{t} h(\tau) d\tau \]

- Note: \[ \frac{d}{dt} s(t) = \frac{d}{dt} \int_{-\infty}^{t} h(\tau) d\tau = h(t) \]
Example: CT step response calculation

- What is the impulse response of the system with the step response

\[ s(t) = t^2u(t) \]

- Impulse response:

\[ h(t) = \frac{d}{dt}s(t) \]

\[ = 2tu(t) = 2r(t) \]

Unit-ramp function
Example

- Determine if the following system is (a) causal and/or (b) stable

\[ h[n] = n \left( \frac{1}{3} \right)^n u[n - 1] \]

- Solution:
  - Causal?
  - Stable?

this formula might be useful:

\[
\sum_{n=1}^{\infty} na^{n-1} = \frac{1}{(1 - a)^2} \text{ for } |a| < 1
\]
Example – sketch of solution

Determine if the following system is (a) causal and/or (b) stable

Solution:

- Causal?
  Causal because $h[n] = 0$ for $n < 0$.

- Stable?
  Stable because $\sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty$. 

$$h[n] = n \left(\frac{1}{3}\right)^n u[n - 1]$$
Example

- Determine and plot the step response of a system with a real impulse response given by

\[ h(t) = Ae^{-bt}u(t), \ A > 0, \ b > 0 \]
Example – sketch of the solution

Determine and plot the step response of a system with a real impulse response given by

\[ h(t) = Ae^{-bt}u(t), \quad A > 0, \ b > 0 \]

Recall that

\[ s(t) \triangleq u(t) \ast h(t) \]

\[ = \int_{-\infty}^{\infty} h(\tau)u(t - \tau)d\tau \]

\[ = \int_{-\infty}^{t} h(\tau)d\tau \]

Now compute

\[ s(t) = \int_{0}^{t} Ae^{-b\tau}d\tau \]

\[ = \frac{A}{b} (1 - e^{-bt}) \]
Summary of Lecture

- Convolution is fun and easy
  - If you disagree, fear not, we will soon discover cool tools and ideas that allow us to avoid computing convolutions (most of the time)
  - But you still need to learn how to do it in the time domain
  - Convolution is nevertheless a fact of life: all real world LTI systems generate outputs in the time domain given by the convolution sum or integral (yes RLC circuits are doing convolution for you)

- Can infer much about the system properties from inspecting the impulse response