Lecture #11

EE 471C / EE 381K-17 Wireless Communication Lab

Professor Robert W. Heath Jr.
Announcements

- Reminder that midterm #1 is on Wednesday
  - You can have 1 sheet of regular size paper front / back
  - Must be handwritten by you (this means using a pen and your hand)
  - You may also use the textbook from the course
From last lecture

- Flat channel model  \( h_c(t) = \alpha \delta(t - \tau_d) \)

- \( \tau_d \)
  - Symbol timing mismatch
  - Frame timing mismatch

- \( \alpha \)
  - Amplitude distortion

Symbol synchronization
Frame synchronization

This lecture

Channel estimation and Equalization
Carrier frequency offset
Preview of Today’s Lecture

- Frame synchronization for frequency flat channels
  - Formulate the frame synchronization problem
  - Solve frame synchronization using energy detector or a correlation
- Interlude on least squares
  - Define and compute the least-squares solution for a linear system
- Frequency flat channel estimation
  - Formulate and solve the frequency-flat channel estimation problem
  - Implement detection with channel estimation
- Carrier frequency offset
  - Explain the origin of carrier frequency offset
  - Present a simple carrier frequency offset estimator for flat-fading channels
Frame synchronization

Learning objectives:
- Formulate the frame synchronization problem.
- Solve the frame synchronization problem using a correlator.
Frame synchronization system model

- In a frequency-flat channel, after symbol synchronization

\[ y[n] = h \sum_{m=-\infty}^{\infty} s[m]g((n - m)T - \tau_d) + v[n] \]

Symbol synchronization resolves the part of the delay in \([0,T)\), it is not able to resolve delays that are multiples of \(T\)

- A residual offset of \(d\) symbols will remain

\[ h = \alpha e^{-j2\pi f_c \tau_d} \sqrt{E_x} \]

\[ y[n] = hs[n - \boxed{d}] + v[n] \]

Received signal
Motivation for frame synchronization

To reconstruct the transmitted bit sequence it is necessary to know where the symbol stream starts.

Example: consider a 4-PAM system transmitting ASCII symbols.

<table>
<thead>
<tr>
<th>Bits</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>-3a</td>
</tr>
<tr>
<td>01</td>
<td>-a</td>
</tr>
<tr>
<td>10</td>
<td>+a</td>
</tr>
<tr>
<td>11</td>
<td>+3a</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Character</th>
<th>Binary</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>01 10 00 01</td>
<td>-a  +a  -3a  -a</td>
</tr>
<tr>
<td>b</td>
<td>01 10 00 10</td>
<td>-a  +a  -3a  +a</td>
</tr>
<tr>
<td>c</td>
<td>01 10 00 11</td>
<td>-a  +a  -3a  +3a</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

- Received symbol sequence \( \cdots, -a, -a, +a, -3a, -a, \cdots \)
- Which character was sent, \( X \) or \( a \)?

\[ \cdots, \underbrace{-a, -a, +a, -3a, -a}_X, \cdots \quad \cdots, \underbrace{-a, -a, +a, -3a, -a, \cdots}_a \]
Use training data to help with correlation

- Training sequence is known information inserted into the transmission
- Known information is useful for
  - Frame and carrier synchronization
  - Channel estimation
- Most commercial wireless systems have multiple forms of training
  - Known as training sequences, pilots, reference signals, etc.
Frame synchronization using training sequence

- Suppose \( \{t[n]\}_{n=0}^{N_{tr}-1} \) is the training sequence known at the receiver.

- Correlation based detector correlates with the training sequence to compute

\[
R[n] = \sum_{k=0}^{N_{tr}-1} t^*[k]y[n + k]
\]

and solves for

\[
\hat{d} = \arg \max_n |R[n]|
\]
The frame synchronization algorithm may find a false peak if it happens that the starting point is delayed, then the peaks would shift accordingly.

A block diagram of a receiver including both symbol synchronization and frame synchronization is shown. The frame synchronization happens after the downsampling and prior to the symbol detection. Fixing the frame synchronization operations is illustrated in Fig. 8. The frame synchronization happens delayed, then the peaks would shift accordingly.

Of our training data, the peaks happen at 21 and 42. For example, the Barker code of length 7 is given by \( \{1, 1, 1, -1, -1, 1, -1\} \). The SNR is 5dB. We consider a frame snippet that consists of data symbols, training symbols, and data symbols. In Fig. 1.3, there are two peaks that correspond to locations 21 and 42.

Peaks at 21 and 42

SNR of 5dB

Consider a system as described by (42). There are two peaks that correspond to locations 21 and 42. For example, the analog hardware may have a carrier sense feature where it can determine when there is a significant signal of interest. Then the digital hardware can determine the first value of the training sequence and start evaluating the correlation and looking for the peak. A threshold can also be used to select the starting point, i.e. finding the first value of the training sequence. There are several approaches to avoid this problem. First training sequences can be selected that have good correlation properties.
Improving robustness using periodic insertion

- Periodic insertion of the training symbols can improve performance and resistance to finding the synchronization sequence in the data.

- Correlate over multiple periods

\[
\hat{d} = \arg \max_n \left[ \sum_{p=0}^{P-1} \left| \sum_{k=0}^{N_{tr}-1} t^*[k] y[n + k + pN_{tot}] \right| \right]
\]

\(N_{tot}\) is the length of the frame in symbols, \(P\) is the number of periods.
Design of training sequences

- Training sequences are generally designed to have special properties that fit the application.

- Often a combination of properties are used so that the training sequence may serve multiple purposes.

- For frame synchronization, a training sequence should have good autocorrelation properties.
**Zadoff-Chu training sequence**

\[
p[n] = e^{i \frac{M\pi n^2}{N_p}} \quad \text{for } N_p \text{ even}
\]

\[
p[n] = e^{i \frac{M\pi (n(n+1))}{N_p}} \quad \text{for } N_p \text{ odd}
\]

- Where \( M \) is an integer that is co-prime with \( N_p \)
- Zadoff-Chu have perfect periodic autocorrelation and good aperiodic correlation properties
- The DFT of a Zadoff-Chu sequence is scaled Zadoff-Chu sequence
- The magnitude of a Zadoff-Chu sequence is constant in time and frequency domains (unimodular)

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Frank training sequence

\[ p[mq + k] = e^{i2\pi \frac{rkm}{q}} \quad 0 \leq k, m < q \]

- Where \( 0 \leq n \leq q^2 - 1 \), \( q \) is any integer, and \( r \) is co-prime with \( q \)
- Take \( q = 4 \) and \( r = 3 \) to obtain a length 4 PSK sequence
- Frank sequences have the same properties as Zadoff-Chu sequences and a part of a general family called Frank-Zadoff-Chu

Example

Frank sequence with $q=4$ and $r=3$

Good autocorrelation properties

Perfect periodic correlation
Example

Frank sequence with $q=4$ and $r=3$

Good autocorrelation properties when surrounded by zeros but this is not efficient

correlation for training surrounded by zeros

Zeros Training Zeros
Example

Frank sequence with \( q=4 \) and \( r=3 \)

Good autocorrelation properties when surrounded by data as well
Receiver block diagram

- Receiver including symbol synchronization and frame synchronization

Complex baseband received signal

Still need to deal with effects caused by the unknown channel $h$
Frame synchronization

Learning objectives:

- Formulate the frame synchronization problem.
- Solve the frame synchronization problem using a correlator.
Interlude on Least Squares and Maximum Likelihood

Learning objectives:
- Define and compute the least-squares solution for a linear system
- Connect least squares and maximum likelihood parameter estimation
System of linear equations

Consider a system of linear equations written in matrix form

\[ Ax = b \]

- \( A: N \times M \) is the known matrix of coefficients called the data
- \( x: M \times 1 \) is a vector of unknowns
- \( b: N \times 1 \) is a known vector often called the observation vector
Visualization the product

- Expanding

\[
Ax = [a_1, a_2, \ldots, a_M] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_M \end{bmatrix} = \sum_{m=1}^{M} a_m x_m = b
\]

- This shows that an exact solution is a linear combination of the columns of \( A \)
Minimizing the squared error

- Suppose that $\mathbf{A}$ is square $N = M$ and that $\mathbf{A}$ is invertible (or full rank) then
  $$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- Suppose that $N > M$ a.k.a. $\mathbf{A}$ is “tall” and $\mathbf{A}$ is full rank (implies that the columns are linearly independent)
  - System is overdetermined
  - Propose to find the solution to minimize the squared error

$$\min \| \mathbf{A}\mathbf{x} - \mathbf{b} \|^2$$

Suppose that $\mathbf{A}$ is not square $N \neq M$ and that $\mathbf{A}$ is invertible (or full rank) then$$\mathbf{x} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{b}$$
Finding the minimizer

- Using matrix calculus (differentiating w.r.t. complex conjugate)

\[
\frac{d}{dx^c} \|Ax - b\|^2 = \frac{d}{dx^c} x^* A^* Ax - \frac{d}{dx^c} x^* A^* b - \frac{d}{dx^c} b^* Ax + \frac{d}{dx^c} b^* b
\]

\[= A^* Ax - A^* b.\]

- Setting equal to zero leads to the orthogonality condition

\[A^* (Ax - b) = 0\]

- Simplifying leads to the normal equations

\[A^* Ax = A^* b\]
Least squares solution

- $A^*A$ is a square matrix and is invertible because $A$ is full rank, thus

  $$x_{LS} = (A^*A)^{-1}A^*b$$

- The squared error is

  $$J(x_{LS}) = \Arrowvert Ax_{LS} - b\Arrowvert^2 = x_{LS}^*A^*(Ax_{LS} - b) - b^*(Ax_{LS} - b) = b^*b - b^*Ax_{LS}$$

Measures the performance of the estimator
Geometric interpretation

- \( A(A^*A)^{-1}A^*b \) is the projection of \( b \) onto the space spanned by the columns of \( A \)
- The error is \((A(A^*A)^{-1}A^* - I)b\) is the projection of \( b \) onto the orthogonal complement of the rows of \( A \)
Maximum likelihood estimation

- Suppose there is measurement noise, modeled as $N(0, I)$
  \[ y = Ax + v. \]
- Use maximum likelihood to estimate $x$
- Give a value of $x$ called $\bar{x}$ the likelihood function is $(A$ known also)
  \[ f_{y|A,x}(y|A, \bar{x}) = \frac{1}{\pi^N} e^{-(y-A\bar{x})^*(y-A\bar{x})} \]
- Differentiating and setting the result equal to zero gives
  \[ A^*(y - A\bar{x}) = 0 \]
  \[ x_{ML} = (A^*A)^{-1}A^*y. \]

Maximum likelihood estimate corresponds to the least squares solution neglecting noise.
Interlude on Least Squares and Maximum Likelihood

Learning objectives:
- Define and compute the least-squares solution for a linear system
- Connect least squares and maximum likelihood parameter estimation
Break
Connected vs. autonomous vs. automated cars

**CONNECTED**
- V2X communication capabilities

**AUTOMATED**
- Some safety-critical control functions **without direct driver input**

**AUTONOMOUS**
- Self driving capabilities **without connectivity**

May or may not be connected

May or may not be self driving

M. Parent, "Automated Vehicles: Autonomous or Connected", IEEE 14th International Conference in Mobile Data Management (MDM), pp.2-2, 3-6 June 2013
Back to the main event
Frequency flat channel estimation

Learning objectives:

- Formulate and solve the frequency-flat channel estimation problem
- Implement detection with channel estimation
Frequency flat channel estimation

- Suppose that symbol and frame synchronization are solved then

\[ y[n] = h s[n] + v[n] \]

- Sampling and removing delay leaves distortion due to \( h \)
- Some modulation schemes are robust to such distortion (e.g., differential QPSK)
- For general linear modulation, need channel estimation and equalization
Problem formulation

- Let complex scalar $h$ be the unknown channel
  \[ y[n] = hs[n] + v[n] \]

- Channel estimation
  - Exploit known training $\{t[n]\}_{n=0}^{N_{tr}-1}$ to estimate the channel

- Write received signal in vector form using known training
  - Noise is IID therefore the least squares estimator is the maximum likelihood estimator
Channel estimation

- Signal model

\[
\begin{bmatrix}
y[0] \\
y[1] \\
\vdots \\
y[N_{tr} - 1]
\end{bmatrix}_y = \begin{bmatrix}
t[0] \\
t[1] \\
\vdots \\
t[N_{tr} - 1]
\end{bmatrix}_t \cdot h + \begin{bmatrix}
v[0] \\
v[1] \\
\vdots \\
v[N_{tr} - 1]
\end{bmatrix}_v \Rightarrow \mathbf{y} = \mathbf{t}h + \mathbf{v}
\]

- LS solution is

\[
\hat{h} = (\mathbf{t}^*\mathbf{t})^{-1}\mathbf{t}^*\mathbf{y}
\]

\[
= \frac{\sum_{n=0}^{N_{tr} - 1} \mathbf{t}^*[n]\mathbf{y}[n]}{\sum_{n=0}^{N_{tr} - 1} \mathbf{t}^*[n]\mathbf{t}[n]}
\]

Correlator – can jointly perform frame synchronization and channel estimation

Corrects for scaling
ML detector

- ML detector with channel estimation
  - To correct for $h$ simply include estimated channel in ML detector
    \[
    \hat{s}[n] = \arg\min_{s \in \mathcal{C}} |y[n] - \hat{h}s|^2
    \]
  - Channel estimate becomes an input into the detector

- Using estimated $\hat{h}$ introduces an additional source of errors
Equalization

- Remove the effect of the channel prior to detection

\[
\arg \min_{s \in \mathcal{C}} |y[n] - \hat{h}s|^2 = \arg \min_{s \in \mathcal{C}} |\hat{h}|^{-1} \frac{|y[n]|}{\hat{h}} - s^2 = \arg \min_{s \in \mathcal{C}} \left| \frac{y[n]}{\hat{h}} - s \right|^2
\]

- A standard detector can be applied to the equalized signal
Equalization

- The probability of symbol error can be computed treating channel estimation error as noise
- Equalized received signal

\[ \hat{s}[n] = \frac{1}{\hat{h}}(\hat{h} + \hat{h}_e)s[n] + \frac{1}{\hat{h}}v[n] \]

\[ = s[n] + \frac{\hat{h}_e}{\hat{h}}s[n] + \frac{1}{\hat{h}}v[n]. \]

Creates interference, treat as noise

- Impact of channel estimation in SINR

\[ \text{SINR}_{\hat{h}_e} = \frac{|\hat{h}|^2}{|\hat{h}_e|^2 + \sigma_v^2} \]

or

\[ \text{SINR} = \frac{\mathbb{E}[|\hat{h}_e|^2]}{|\hat{h}|^2 + \sigma_v^2}. \]
Receiver block diagram

- Receiver with symbol synchronization, frame synchronization, and channel estimation
Frequency flat channel estimation

Learning objectives:

- Formulate and solve the frequency-flat channel estimation problem
- Implement detection with channel estimation
Learning objectives:

- Explain origin of frequency offset and discuss nature of the impairment
- Analyze a simple offset estimation algorithm based on sending a sinusoid
Chapter 5 Dealing with Impairments 18

Consider the received signal after downconversion

What if only $f'_c$ not $f_c$ is known at the receiver?

- The result is carrier frequency offset (CFO)
### Downconversion

- **Let**  \( f_e = f'_c - f_c \)

\[
\begin{align*}
    r_p(t) \cos(2\pi f'_c t) &= \frac{1}{2} r_i(t) \cos(2\pi f_e t) \\
    &\quad - \frac{1}{2} r_i(t) \cos(2\pi (f_c + f'_c) t) - \frac{1}{2} r_q(t) \sin(2\pi (f_c + f'_c) t) \\
    - r_p(t) \sin(2\pi f'_c t) &= \frac{1}{2} r_q(t) \sin(2\pi f_e t) \\
    &\quad + \frac{1}{2} r_q(t) \cos(2\pi (f_c + f'_c) t) + \frac{1}{2} r_i(t) \sin(2\pi (f_c + f'_c) t)
\end{align*}
\]

- **Low pass filtering removes high frequency components**

\[
r'(t) = e^{j2\pi f_e t} (r_i(t) + jr_q(t)) = e^{j2\pi f_e t} r(t).
\]

- **Correction of**  \( f_e \) **is carrier frequency synchronization**

**derotate to remove**
Mathematical relationship

- Assuming the offset is small, the front-end bandwidth is sufficiently wide, and the pulse shape is concentrated over a few symbols then

\[ y(t) = e^{j2\pi f_e t} \int r(t-\tau)g_{rx}(\tau) d\tau. \]

- In discrete time, including noise and with \( \epsilon = f_e T \)

\[ y[n] = e^{j2\pi \epsilon n} h \sum_{m=-\infty}^{\infty} s[m] g((n-m)T-\tau_d) + v[n] \]

- Rotation occurs after the convolution
- Impacts channel estimation and thus equalization
Visualizing the frequency offset effect

- Special case of flat fading channel

\[ y[n] = e^{j2\pi \epsilon n} h_s[n] + v[n] \]

- \( \epsilon \) is generally small but unknown
- Rotates constellation by \( e^{j2\pi \epsilon n} \)
Frequency offset synchronization

- Frequency offset is a severe impairment
  - Even a small amount of offset leads to significant degradation

- Frequency offset synchronization is challenging
  - Offset occurs after the convolution with the unknown channel
  - Impacts channel estimation and frame synchronization

- Methods for offset correction
  - Exploit structure in the received signal
  - Create and exploit structure using a known training signal
Frequency offset estimation using a sinusoid

- Suppose that frame synchronization has been solved
- Choose as a training sequence a complex sinusoid
  \[ t[n] = \exp(j2\pi f_t n) \text{ for } n = 0, 1, \ldots, N_{tr} - 1 \]
- The received signal during training is
  \[ y[n] = e^{j2\pi \epsilon_n} e^{j2\pi f_t n} h + v[n] \]
- Removing the known offset
  \[ e^{-j2\pi f_t n} y[n] = e^{j2\pi \epsilon_n} h + e^{-j2\pi f_t n} v[n] \]

An example that is relevant also for frequency selective channels since sinusoids are eigenfunctions of LTI systems

This becomes a classic problem of estimating a complex sinusoid with unknown amplitude, frequency and phase (amplitude and phase from the channel)
Approximate least squares solution

- Approximate additive noise as phase noise
  \[ e^{-j2\pi f_t n} y[n] \approx |h|e^{j2\pi \epsilon n + \theta + \nu[n]} \]

  - Phase of \( h \)
  - Gaussian noise in phase term

- Looking at the phase difference between two adjacent samples
  \[ \text{phase}(e^{j2\pi f_t n} y^*[n] e^{-j2\pi f_t (n+1)} y[n+1]) = 2\pi \epsilon + \nu[n+1] - \nu[n]. \]

- Aggregating observations from \( n=1, ..., N_{tr}-1 \)
  \[ p = 2\pi \epsilon \mathbf{1} + \nu \]

- LS solution
  \[ \hat{\epsilon} = \frac{1}{2\pi} (p^*p)^{-1} p \]
  \[ = \frac{1}{2\pi (N_{tr} - 1)} \sum_{n=1}^{N_{tr}-1} \text{phase}(e^{j2\pi f_t} y^*[n] y[n+1]) \]
  Just the average phase difference
Learning objectives:

- Explain origin of frequency offset and discuss nature of the impairment
- Analyze a simple offset estimation algorithm based on sending a sinusoid