Lecture #12

EE 471C / EE 381K-17 Wireless Communication Lab

Professor Robert W. Heath Jr.
Preview of today’s lecture

- **Frequency selective channel** (going beyond frequency flat)
  - Explain different impairments caused by the frequency selective wireless communication channel

- Least-squares FIR equalizer from a channel estimate
  - Derive and compute the least-squares equalizer using an estimate of the channel

- Least minimum mean squared error estimation and equalization
  - Derive and compute the LMMSE equalizer

- Single carrier frequency domain equalization
  - Explain important aspects of the DFT
  - Derive the SC-FDE equalizer
Impairments in frequency selective channels

Learning objectives:

- Explain different impairments caused by the frequency selective wireless communication channel
Frequency selective channels

- The channel between TX and RX is not just a delay and scaling
- More generally, the complex baseband equivalent channel is represented by an impulse response

Called frequency selective b/c in general $H(f)$ is not flat (in BW of $x(t)$)
- A reasonable assumption about $h(t)$ is that it is causal and FIR
Propagation mechanisms

- Diffraction
- Reflection
- Scattering
- LoS

ACCESS POINT

CLIENT 1

CLIENT 2
Simplified multipath channel

- Multipath: the transmitted signal reaches the receiver via a number of different paths

\[ y(t) = h_0 x(t - d_0/c) + h_1 x(t - d_1/c) + h_2 x(t - d_2/c) + v(t) \]

- \( h_0, h_1, h_2 \): gains for the existing paths
- \( d_0, d_1, d_2 \): distances travelled by the wave from Tx to Rx
- \( c \): speed of light
- \( v(t) \): noise
Example of multipath channel

\[ h_c(t) = \delta(t) + \alpha \delta(t - \tau) \]

\[ |H_c(f)|^2 = 1 + \alpha^2 + \alpha \cos(2\pi f \tau) \]

- Effect in the received signal
  - 4-PAM signal
  - \( \tau = 3T/2 \)
Model for the received signal

- Consider the received signal after matched filtering and sampling

\[ y(t) = g_{rx}(t) \ast h(t) \ast \sqrt{E_x} \sum_{m=-\infty}^{\infty} s[m]g_{tx}(t - mT) + g_{rx}(t) \ast v(t) \]

- Define the effective channel \( h_{ef}\) = \( g_{rx}(t) \ast g_{tx}(t) \ast \sqrt{E_x}h(t) \)

- Then the received signal model becomes

\[ y(t) = \sum_{m=-\infty}^{\infty} s[m]h_{ef}(t - mT) + g_{rx}(t) \ast v(t) \]

Looks like complex pulse amplitude modulation with a new pulse shape that includes the channel
Model for the received signal

- Sampling at the symbol rate
  \[ y[n] = \sum_{\ell=-\infty}^{\infty} h[\ell] s[n - \ell] + v[n] \]

- Suppose the channel is causal
  - Noncausal filters are implemented with delay, propagation is causal

- Suppose the channel is FIR
  - Signal decays with distance, meaning long reflections are very weak

- The simplified system with FIR
  \[ y[n] = \sum_{\ell=0}^{L} h[\ell] s[n - \ell] + v[n] \]
  Order of the FIR channel, usually assumed known
Example of the impact of frequency selectivity

- Suppose that $h[n] = \sqrt{E_x} \delta[n] + \sqrt{E_x} h_1 \delta[n - 1]$.
  - This is a two-tap discrete-time channel

$$y[n] = \sqrt{E_x} s[n] + \sqrt{E_x} h_1 s[n - 1] + v[n]$$

$$\text{SINR} = \frac{E_x}{E_x |h_1|^2 + N_o}$$
Receiver processing to remove ISI

- **Equalizer**: filter to remove or reduce ISI
  - Find the equalizer assuming the channel is known

- Linear equalizer (LE)
  - FIR equalizer: \( \{ f[\ell] \}_{\ell=0}^{L_f} \)
  - Ideal equalizer: \( \sum_{\ell=0}^{L_f} f[\ell] \hat{h}[n - \ell] = \delta[n - n_d] \)
  - FIR solution has low complexity with the worst performance

- Decision feedback equalizer (DFE)
  - Takes LE + subtract tentative estimate of symbol
Impairments in frequency selective channels

Learning objectives:

- Explain different impairments caused by the frequency selective wireless communication channel
Least squares equalizer

Learning objectives:

- Develop LS equalizers to remove the effects of intersymbol interference
Computing the equalizer from the channel estimate

- Find the equalizer that removes the effects of the channel given an estimate $\hat{h}[n]$

$$\sum_{\ell=0}^{L_f} f[\ell] \hat{h}[n - \ell] = \delta[n - n_d]$$
Formulation of the LS equalizer

- Ideal equalizer → \( \sum_{\ell=0}^{L_f} f[\ell] \hat{h}[n - \ell] = \delta[n - n_d] \)

- Write as a linear system

\[
\begin{bmatrix}
\hat{h}[0] & 0 & \cdots & \cdots \\
\hat{h}[1] & \hat{h}[0] & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\hat{h}[L] & 0 & \hat{h}[L] & \cdots \\
0 & \hat{h}[L] & \cdots & \cdots \\
\vdots & & & \\
\end{bmatrix}
\begin{bmatrix}
f[0] \\
f[1] \\
\vdots \\
f[L_f] \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0 \\
\end{bmatrix}
\]

- Toeplitz structure

- \( \textbf{H} \) is tall \( L_f + L + 1 \times L_f + 1 \)

- Find LS solution of \( \textbf{Hf}_{n_d} = \textbf{e}_{n_d} \)
Computing the LS equalizer

- Toeplitz structure in \( \mathbf{H} \) leads to efficient algorithms to solve LS
- \( \mathbf{H} \) is full rank as long as at least one coefficient is nonzero
- The LS solution assuming \( \mathbf{H} \) is full rank is

\[
\mathbf{f}_{LS,n_d} = (\mathbf{H}^*\mathbf{H})^{-1}\mathbf{H}^*\mathbf{e}_{n_d}
\]

- with squared error

\[
J[n_d] = \mathbf{e}_{n_d}^* (\mathbf{I} - \mathbf{H}(\mathbf{H}^*\mathbf{H})^{-1}\mathbf{H}^*)\mathbf{e}_{n_d}
\]

- The squared error can be further minimized by choosing \( n_d \)
Example: ISI channel

BPSK $s[k]$

4-PAM $s[k]$

channel $h[k]$

$y[k]$
LS equalizer, order 3

optimal delay $n_d = 4$

$J_f$

$y[k] \rightarrow f_k \rightarrow \hat{s}[k]$

$BPSK \hat{s}[k]$

$4_{\text{PAM}} \hat{s}[k]$

$h[k] * f[k]$
Example: LS equalizer, order 5

- \( J_f \)
- \( y[k] \)
- \( \hat{s}[k] \)
- \( f_k \)
- \( h[k] \ast f[k] \)
- \( n_d \)
- \( n_d = 6 \)

Optimal delay

- BPSK
- \( \hat{s}[k] \)
- \( k \)
- \( 0 \) to 1000

- 4_PAM
- \( \hat{s}[k] \)
- \( k \)
- \( 0 \) to 1000

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Example: LS equalizer, order 10

Optimal delay $n_d = 9$
Receiver with channel estimation and linear equalization

Figure 17: Receiver with channel estimation and linear equalization.

Performance can nonetheless be improved by taking the best sample, especially when the pulse shape has excess bandwidth. An alternative is to implement a fractionally spaced equalizer, which operates on the signal prior to downsampling [30]. This is explored further in the problems at the end of chapter.

A final comment on complexity. The length of the equalizer \( L_f \) is a design parameter that depends on \( L \). The parameter \( L \) is the extent of the multi-path in the channel and is determined by the bandwidth of the signal as well as the maximum delay spread derived from propagation channel measurements. The equalizer is an FIR inverse of an FIR filter. As a consequence the performance will improve if \( L_f \) is large, assuming perfect channel knowledge. The complexity required per symbol, however, also grows with \( L_f \). Thus there is a tradeoff between choosing large \( L_f \) to have better equalizer performance and smaller \( L_f \) to have more efficient receiver implementation. A rule-of-thumb is to take \( L_f \) to be at least \( 4L \).

2.3 Linear equalization in the frequency domain with SC-FDE

Both the direct and indirect equalizers require a convolution on the received signal to remove the effects of the channel. In practice this can be done with a direct implementation using the overlap-and-add or overlap-and-save methods for efficiently computing convolutions in the frequency domain. An alternative to FIR in the time domain is to perform equalization completely in the frequency domain. This has the advantage of allowing an ideal inverse of the channel to be computed. Application of frequency domain equalization, though, requires additional mathematical structure in the transmitted waveform as we now explain.

In this section, we describe a technique known as SC-FDE [31]. At the transmitter, SC-FDE divides the symbols into blocks and adds redundancy in the form of a cyclic prefix. The receiver can exploit this extra information to permit equalization using the DFT. The result is an equalization strategy that is capable of perfectly equalizing the channel, in the absence of noise. SC-FDE is supported in IEEE 802.11ad and a variation is used the uplink of 3GPP LTE.
Least squares equalizer

Learning objectives:

- Develop LS equalizers to remove the effects of intersymbol interference
Break
AI and the Mobile Device

http://texaswirelesssummit.org

See also Professor Heath’s related research talk
https://www.youtube.com/watch?v=xslrovhTWec&t=4s
Back to the main event
Background on linear minimum mean squared error (LMMSE) estimation

Learning objectives:
- Derive the LMMSE estimator
- Use the LMMSE estimator
Background on LMMSE estimator

- Suppose we want to estimate $x$ from an observation $y$
  - Unknown vector $x$ of size $M \times 1$ with zero mean and covariance $C_{xx}$
  - Observation vector $y$ with zero mean and covariance $C_{yy}$
  - $x$ and $y$ are jointly correlated with covariance matrix $C_{yx}$
- The objective of the LMMSE estimator is to determine a linear transformation such that

$$G_{\text{MMSE}} = \arg \min_G \mathbb{E} \left[ \| x - G^* y \|^2 \right]$$

- Equivalently

$$G_{\text{MMSE}} = \arg \min_G \mathbb{E} \left[ \sum_{m=1}^M \| x_m - g_m^* y \|^2 \right]$$
**LMMSE estimator**

- **Solving for one column of** $G_{\text{MMSE}}$

  \[
  \frac{d}{dg_k} \mathbb{E} \left[ \sum_{m=1}^{M} |x_m - g_m^* y|^2 \right] = \mathbb{E} \left[ \frac{d}{dg_k} \sum_{m=1}^{M} |x_m - g_m^* y|^2 \right]
  \]
  \[
  = \mathbb{E} \left[ \frac{d}{dg_k} |x_k - g_k^* y|^2 \right]
  \]
  \[
  = \mathbb{E} [y (y^* g_k - x_k^*)].
  \]

- **MMSE orthogonality equation**
  
  ✦ Taking the expectation and setting the result to zero

  \[
  C_{yy} g_k = [C_{yx}] {:, k}.
  \]

  ✦ Solution is

  \[
  g_{k, \text{MMSE}} = C_{yy}^{-1} [C_{yx}] {:, k}
  \]


LMMSE estimator

- Reassembling the column of $G$ and combining the results together

\[ G_{\text{MMSE}} = C_{yy}^{-1}C_{yx} \]

- The MMSE estimate of $x$ is

\[ x_{\text{MMSE}} = G_{\text{MMSE}}^*y = C_{yx}^*C_{yy}^{-1}y. \]

Exploits the joint correlation to extract information about $x$ from $y$

Decorrelates $y$
Mean squared error performance

- The minimum mean squared error achieved is

\[
\sum_{m=1}^{M} \mathbb{E} \left[ |x_m - g_{m,\text{MMSE}}^* y|^2 \right] = \sum_{m=1}^{M} \mathbb{E} \left[ |x_m x_m^* - g_{m,\text{MMSE}}^* y x_m^*|^2 \right]
\]

\[
= \sum_{m=1}^{M} [C_{xx}]_{m,m} - g_{m,\text{MMSE}}^* [C_{yx}]_{:,m}
\]

\[
= \sum_{m=1}^{M} [C_{xx}]_{m,m} - [C_{yx}]^*_{:,m} C_{yy}^{-1} [C_{yx}]_{:,m}
\]

\[
= \text{tr} [C_{xx}] - \text{tr} [C_{yx}^* C_{yy}^{-1} C_{yx}]
\]
Summarizing LMMSE

- Given zero mean vectors \( \mathbf{x} \) and \( \mathbf{y} \) with

\[
\begin{align*}
C_{yy} &= \mathbb{E}[\mathbf{yy}^*] \\
C_{xx} &= \mathbb{E}[\mathbf{xx}^*] \\
C_{yx} &= \mathbb{E}[\mathbf{yx}^*]
\end{align*}
\]

- The LMMSE objective is to find a matrix \( \mathbf{G} \) to estimate \( \mathbf{x} \) from \( \mathbf{y} \)

\[
\mathbf{G}_{\text{MMSE}} = \arg\min_{\mathbf{G}} \mathbb{E} [\| \mathbf{x} - \mathbf{G}^* \mathbf{y} \|^2]
\]

- The solution is

\[
\mathbf{G}_{\text{MMSE}} = C_{yy}^{-1} C_{yx}
\]

\[
\mathbf{x}_{\text{MMSE}} = \mathbf{G}_{\text{MMSE}}^* \mathbf{y} = C_{yx}^* C_{yy}^{-1} \mathbf{y}.
\]

- The error in the estimate is

\[
\text{tr}[C_{xx}] - \text{tr} \left[ C_{yx}^* C_{yy}^{-1} C_{yx} \right]
\]
Background on the linear minimum mean squared error (LMMSE) estimation

Learning objectives:
- Derive the LMMSE estimator
- Use the LMMSE estimator
LMMSE equalizer

Learning objectives:
- Derive and compute the LMMSE equalizer
Reformulating the equalization problem

- Consider the received signal after the equalizer

\[ \hat{s}[n - n_d] = \sum_{\ell=0}^{L_f} f_{n_d}[\ell]y[n - \ell] \rightarrow \hat{s}[n - n_d] = f_{n_d}^* y[n] \]

where

\[ f_{n_d} = [f_{n_d}[0], f_{n_d}[1], \ldots f_{n_d}[L_f]]^* \]

\[ y^T[n] = [y[n], y[n-1], \ldots, y[n-L_f]^T \]

\[ y[n] = H^T s[n] + v[n] \]

\[ s^T[n] = [s[n], s[n-1], \ldots, s[n-\ell]]^T \]

- The LMMSE equalizer minimizes the error

\[ \mathbb{E} \left| s[n - n_d] - f_{n_d}^* y[n] \right|^2 \]
Reformulating the equalization problem

- Consider the received signal after the equalizer

\[
\hat{s}[n - n_d] = \sum_{\ell=0}^{L_f} f_{n_d}[\ell] y[n - \ell] \quad \Rightarrow \quad \hat{s}[n - n_d] = f_{n_d}^T y[n]
\]

where

\[
y^T[n] = [y[n], y[n - 1], \ldots, y[n - L_f]]^T
\]

- The LMMSE equalizer minimizes the error

\[
\mathbb{E} \left[ |s[n - n_d] - f_{n_d}^T y[n]|^2 \right]
\]
Computing the auto and cross correlations

- Need to relate observations, symbols and noise

\[
y[n] = [h[0], h[1], \ldots, h[L]] \begin{bmatrix} s[n] \\ s[n-1] \\ \vdots \\ s[n-L] \end{bmatrix} + v[n]
\]

\[
\begin{bmatrix} y[n] \\ y[n-1] \\ \vdots \\ y[n-L_f] \end{bmatrix} = \begin{bmatrix} h[0] & h[1] & \ldots & h[L] & 0 & \ldots & 0 \\ 0 & h[0] & h[1] & \ldots & h[L] & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 0 & h[0] & h[1] & \ldots & h[L] \end{bmatrix} \begin{bmatrix} s[n] \\ s[n-1] \\ \vdots \\ s[n-L-L_f] \end{bmatrix} + \begin{bmatrix} v[n] \\ v[n-1] \\ \vdots \\ v[n-L_f] \end{bmatrix}
\]

\[
y[n] = H^T (L_f + 1) \times (L + L_f + 1) \begin{bmatrix} s[n] \\ s[n] + v[n] \end{bmatrix}
\]
Solving for the LMMSE equalizer

- Assuming that
  - \( s[n] \) is IID with zero mean and unit variance,
  - \( v[n] \) is IID with variance \( \sigma_v^2 \),
  - \( s[n] \) and \( v[n] \) are independent

- Then
  \[
  C_{yy} = \mathbb{E} \left[ y[n]y^*[n] \right] = H^T H^c + \sigma_v^2 I \\
  C_{ys} = \mathbb{E} \left[ y[n]s^*[n-n_d] \right] = H^T e_{n_d}.
  \]

- The LMMSE equalizer is computed obtaining the MMSE estimate that minimizes
  \[
  \mathbb{E} \left[ \left| s[n-n_d] - f_{n_d}^T y[n] \right|^2 \right]
  \]

  \[
  f_{n_d,MMSE} = C_{yy}^{-c} C_{ys}^c \\
  = (H^*H + \sigma_v^2 I)^{-1} H^* e_{n_d}
  \]
Linear equalization discussion

- Connections between LMMSE and LS solutions

\[ f_{n_d, \text{MMSE}} = \left( H^*H + \sigma_v^2 I \right)^{-1} H^*e_{n_d} \]

- Complexity of the equalizer depends on the choice of \( L_f \)
  - Larger values give better performance but higher complexity
  - Generally take \( L_f = 4L \) is a reasonable compromise

\[ f_{LS, n_d} = (H^*H)^{-1}H^*e_{n_d} \]
A final comment on complexity. The length of the equalizer $L_f$ is a design parameter that depends on $L$. The parameter $L$ is the extent of the multi-path in the channel and is determined by the bandwidth of the signal as well as the maximum delay spread derived from propagation channel measurements. The equalizer is an FIR inverse of an FIR filter. As a consequence the performance will improve if $L_f$ is large, assuming perfect channel knowledge. The complexity required per symbol, however, also grows with $L_f$. Thus there is a tradeoff between choosing large $L_f$ to have better equalizer performance and smaller $L_f$ to have more efficient receiver implementation. A rule-of-thumb is to take $L_f$ to be at least $4L$.

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LMMSE equalizer

Learning objectives:
- Derive and compute the LMMSE equalizer
Background on the discrete Fourier transform

Learning objectives:

- Calculate the DFT and IDFT of a sequence
- Calculate the circular convolution of two sequences.
Discrete Fourier Transform (DFT)

- The DFT is a basis expansion for finite-length signals

Analysis \[ x[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \quad k = 0, 1, \ldots, N - 1 \]

Synthesis \[ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{j\frac{2\pi}{N}kn} \quad n = 0, 1, \ldots, N - 1 \]

- The DFT can be computed efficiently with the Fast Fourier transform for \( N \) a power of 2 and certain other special cases
Discrete Fourier Transform (DFT)

- Relationship between the DTFT, \( x(e^{j\omega}) \), and the DFT

\[
x[k] = x \left( e^{j\omega} \right) \bigg|_{\omega = \frac{2\pi k}{N}} \quad k = 0, 1, \ldots, N - 1.
\]

- The DFT can be computed efficiently with the Fast Fourier transform for \( N \) a power of 2 and certain other special cases.
## Table 5: Table of Discrete Fourier Transform Relationships

<table>
<thead>
<tr>
<th>Finite-Length Sequence (Length $N$)</th>
<th>N-point DFT (Length $N$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>$x[k]$</td>
</tr>
<tr>
<td>$x_1[n], x_2[n]$</td>
<td>$x_1[k], x_2[k]$</td>
</tr>
<tr>
<td>$ax_1[n] + bx_2[n]$</td>
<td>$ax_1[k] + bx_2[k]$</td>
</tr>
<tr>
<td>$x[n]$</td>
<td>$N_x[(-k)]$</td>
</tr>
<tr>
<td>$x[((n - m)N]_N$</td>
<td>$W_N^{km} x[k]$</td>
</tr>
<tr>
<td>$W_N^{-\ell n} x[n]$</td>
<td>$x[((k - \ell)N]_N$</td>
</tr>
<tr>
<td>$\sum_{m=0}^{N-1} x_1[m] x_2[[(n - m)N]_N$</td>
<td>$x_1[k] x_2[k]$</td>
</tr>
<tr>
<td>$x_1[n] x_2[n]$</td>
<td>$\frac{1}{N} \sum_{\ell=0}^{N-1} x_1[\ell] x_2[[(k - \ell)N]_N$</td>
</tr>
<tr>
<td>$x^*[n]$</td>
<td>$x^*[((-k)N]_N$</td>
</tr>
<tr>
<td>$x^*[((-n)N]_N$</td>
<td>$x^*[k]$</td>
</tr>
<tr>
<td>Re{$x[n]$}</td>
<td>$X_{ep}[k] = \frac{1}{2} { x[((k)N]_N + x^*[((-k)N]_N }</td>
</tr>
<tr>
<td>jIm{$x[n]$}</td>
<td>$X_{op}[k] = \frac{1}{2} { x[((k)N]_N - x^*[((-k)N]_N }</td>
</tr>
<tr>
<td>$x_{ep}[k] = \frac{1}{2} { x[n] + x^*[((-n)N]_N }$</td>
<td>Re{$x[k]$}</td>
</tr>
<tr>
<td>$x_{op}[k] = \frac{1}{2} { x[n] - x^*[((-n)N]_N }$</td>
<td>jIm{$X[k]$}</td>
</tr>
</tbody>
</table>

† $x[n]$ real

### Symmetry Properties

- $x[k] = x^*[((-k)N]_N$
- Re{$x[k]$} = Re{$x[((-k)N]_N$}
- Im{$x[k]$} = -Im{$x[((-k)N]_N$}
- $|x[k]| = |x[((-k)N]_N$}
- $\angle{x[k]} = -\angle{x[((-k)N]_N$}

$x_{ep}[k] = \frac{1}{2} \{ x[n] + x[((-n)N]_N \}$  Re{$x[k]$}

$x_{op}[k] = \frac{1}{2} \{ x[n] - x[((-n)N]_N \}$  jIm{$x[k]$}
Circular convolution

- Products in the frequency domain become circular convolutions in discrete-time

\[ x_1[n] \ast x_2[n] \quad \iff \quad x_1[k]x_2[k] \]

\[ x_1[n] \ast x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[(n - m)_N] \]

✦ Example of circular shift

\[ x[n] \quad x[n + 2] \quad x[n] \quad \tilde{x}[n] \quad x[(n + 2)_5] \]
Circular convolution with a zero-padded sequence

- To illustrate, let length of $h[n]$ be $L+1$, length of $s[n]$ be $N>L$
- Zero-pad $h[n]$ with $N-L-L$ zeros

$$
y[n] = \sum_{\ell=0}^{N-1} h[\ell] s \left[ \left( n - \ell \right) \right]_N
= \sum_{\ell=0}^{L} h[\ell] s \left[ \left( n - \ell \right) \right]_N
= \begin{cases} 
\sum_{\ell=0}^{n} h[\ell] s[n - \ell] + \sum_{\ell=N+1}^{L} h[\ell] s[n + n - \ell] & 0 \leq n < L \\
\sum_{\ell=0}^{L} h[\ell] s[n - \ell] & n \geq L
\end{cases}
$$

Circular wrapping effect

Looks linear
Background on the discrete Fourier transform

Learning objectives:
- Calculate the DFT and IDFT of a sequence
- Calculate the circular convolution of two sequences
Single carrier frequency domain equalization

Learning objectives:

- Use cyclic prefix to convert a linear convolution to a circular convolution and equalize in frequency domain
Cyclic prefix

- Organize data into blocks of N symbols
- Assemble a block of N + Lc symbols where
  - Prefixed part \( w[n] = s[n + N - L_c] \) \( n = 0, 1, \ldots, L_c - 1 \)
  - Data part \( w[n] = s[n - L_c] \) \( n = L_c, L_c + 1, \ldots, L_c + N - 1 \)
- Neglect the first \( L_c \) terms of the convolution

\[
\bar{y}[n] = y[n + L_c] \quad n = 0, 1, \ldots, N - 1
\]

\[
= \sum_{\ell=0}^{L} h[\ell] w[n + L_c - \ell].
\]
Obtaining a circular convolution

- Provided that $L_c \geq L$ then

$$\bar{y}[n] = \sum_{\ell=0}^{L} h[\ell] w[n + L_c - \ell]$$

$$= \sum_{\ell=0}^{L} h[\ell] s[((n - \ell))_N]$$

- Can use frequency domain equalization!

$$\hat{s}[n] = \mathcal{F}_N^{-1} \left( \frac{Y[k]}{H[k]} \right)$$
Connection to frequency domain equalization

- Conclude that the received signal is equivalently

\[ \bar{y}[n] = \sum_{\ell=0}^{L} h[\ell] s[(n - \ell)N] \]

- Using the DFT equalization can proceed as

\[ \hat{s}[n] = \mathcal{F}_N^{-1} \left( \frac{y[k]}{h[k]} \right) \]

\[ = \mathcal{F}_N^{-1} \left( \frac{\mathcal{F}_N(y[n])}{\mathcal{F}_N(h[n])} \right) \]

- Can use frequency domain equalization!
SC-FDE transmitter

SC-FDE: Single carrier frequency domain equalization

Diagram showing the process:
- QAM
- Serial to Parallel (1:N)
- Add CP
- Parallel to Serial (N+Lc):1
- Channel Estimation
- D/C
- Antenna

Mathematical expressions:
- \[ \hat{d} \]
- \[ \hat{m} \]
- \[ g_{tx}[n] \]
- \[ L \]
- \[ T \]
- \[ L \]

Description:
- The main parameters to select in SC-FE are \( N \) and \( L_c \).
- To minimize complexity, it makes sense to take \( N \) to be small.
- The amount of overhead, though, is \( L_c \cdot \frac{N'}{N} \).
- Consequently taking \( N \) to be large reduces the system overhead incurred by redundancy in the cyclic prefix.
- Too large of an \( N \), however, may mean that the channel varies over the \( N \) symbols, violating the LTI assumption.
- In general, \( L_c \) is selected to be large enough that \( L \) for most channel realizations, throughout the use cases of the wireless system.
SC-FDE receiver

Frame sync determines the start of the cyclic prefix

creates the blocks

Channel Estimation

Parallel to Serial $N:1$

QAM Detection

Symbol Sync

Frame Sync

Serial to Parallel $1: N + L_c$

IDFT

DFT

S/P

$H^{-1}[k]$
Single carrier frequency domain equalization

Learning objectives:

- Use cyclic prefix to convert a linear convolution to a circular convolution and equalize in frequency domain