Lecture #13

EE 471C / EE 381K-17 Wireless Communication Lab

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Frequency selective channels

- Called frequency selective because in general $H(f)$ is not flat
- The channel includes all frequency selective effects
  - Pulse-shaping filter
  - Transmit signal scaling
  - Propagation effects
  - Receive filter

\[ x(t) \xrightarrow{h(t)} v(t) \xrightarrow{} r(t) \]
LTI ISI channel model

- The simplified system with an LTI channel is

\[ y[n] = \sum_{\ell=0}^{L} h[\ell] s[n - \ell] + v[n] \]

- Results in intersymbol interference (ISI)
  - Example: \( L=1 \)

\[ y[n] = h[0] s[n] + h[1] s[n - 1] + v[n] \]

- Need to estimate and equalize the channel to remove ISI
Equalization of frequency selective channels

- Time domain approaches
  - LS equalizer – design an approximate FIR inverse
  - LMMSE equalizer – minimizes mean squared error in the reconstruction, accounts for noise

- Frequency domain approaches
  - Direct frequency domain equalization is difficult
  - Single carrier frequency domain equalization
    - Use cyclic prefix
    - Leverage principles of the discrete Fourier transform (DFT)
Preview of today’s lecture

- Review of SC-FDE equalization

- Linear equalization in the frequency domain with OFDM
  - The shortest introduction to OFDM ever

- Estimating frequency selective channels
  - LS channel estimation in the frequency domain
  - LS channel estimation in the time domain

- Direct equalization
Orthogonal frequency division multiplexing (OFDM)

Learning objectives:

- Describe the OFDM digital modulation technique
- Derive the OFDM receiver and prove that it works
Review of the SC-FDE transmitter

- Transmitter formats information in blocks and appends cyclic prefix.

Transmitter block diagram:
Review of the SC-FDE receiver

- Discarding the cyclic prefix gives a received signal equation

\[ \bar{y}[n] = \sum_{\ell=0}^{L} h[\ell] s[((n - \ell))_N] \]

- Using the DFT equalization can proceed as

\[ \hat{s}[n] = \mathcal{F}_N^{-1} \left( \frac{y[k]}{h[k]} \right) \]

\[ = \mathcal{F}_N^{-1} \left( \frac{\mathcal{F}_N(\bar{y}[n])}{\mathcal{F}_N(h[n])} \right) \]

- Can use frequency domain equalization with the N-DFT
SC-FDE receiver block diagram

Frame sync determines the start of the cyclic prefix

creates the blocks
Introduction to OFDM

- Orthogonal Frequency Division Multiplex (OFDM) is a type of multicarrier modulation that allows frequency domain equalization
  - Compared to SC-FDE, IDFT operation is moved to the transmitter

Advantages of OFDM

- Allows simple frequency domain equalization at the receiver
- Allows sophisticated types of adaptive modulation where information is adapted to the frequency response of the channel
- Obtains diversity against fading with error control coding
Applications of OFDM

- WiFi - IEEE 802.11g, IEEE 802.11a, IEEE 802.11n, IEEE 802.11ac
- 3GPP LTE, 3GPP LTE Advanced, 5G??
- Digital video broadcast - DVB (used outside of US), DVB-h
- Digital audio broadcast - DAB (used outside of US)
OFDM transmitter operation

- Consider a block of length $N \{s[n]\}_{n=0}^{N-1}$
- Let $L_c$ be the length of the cyclic prefix
- The OFDM transmitter sends the samples

$$w[n] = \frac{1}{N} \sum_{m=0}^{N-1} s[m] e^{j2\pi \frac{m(n-L_c)}{N}} \quad n = 0, \ldots, N + L_c - 1$$

- Note that $w[n] = w[n + N]$ for $n = 0, 1, \ldots, L_c$ (this is a cyclic prefix)
- Note that $\{w[n]\}_{n=L_c}^{N+L_c-1}$ (this is the data) corresponds to $F_N^{-1}(s[n])$
- $N$ is the number of subcarriers or tones
OFDM transmitter block diagram

- For OFDM

  ![OFDM Transmitter Block Diagram](image)

- Compare with SC-FDE

  ![SC-FDE Transmitter Block Diagram](image)

Usually this is assumed to be a rectangle function.
OFDM receiver operation I

- Suppose that synchronization has been performed

- The observed signal after matched filtering and downsampling is

\[ y[n] = \sum_{\ell=0}^{L} h[\ell]w[n - \ell] + v[n] \]

- The receiver discards the first \( L_c \) samples to form

\[ \bar{y}[n] = y[n + L_c] \quad n = 0, 1, \ldots, N - 1 \]
OFDM receiver operation 2

Substituting for the transmit signal and neglecting noise

\[ \hat{y}[n] = \sum_{\ell=0}^{L} h[\ell] w[n + L_c - \ell] \]

\[ = \frac{1}{N} \sum_{\ell=0}^{L} h[\ell] \sum_{m=0}^{N-1} s[m] e^{i2\pi \frac{m(n + L_c - L_c - \ell)}{N}} \]

\[ = \frac{1}{N} \sum_{\ell=0}^{L} h[\ell] \sum_{m=0}^{N-1} s[m] e^{i2\pi \frac{mn}{N}} e^{-j2\pi \frac{m\ell}{N}} \]

\[ = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{\ell=0}^{L} h[\ell] e^{-j2\pi \frac{m\ell}{N}} s[m] e^{i2\pi \frac{mn}{N}} \]

\[ = \mathcal{F}_N^{-1}(h[m]s[m]). \]

\[ \hat{s}[n] = \frac{\mathcal{F}_N(\hat{y}[n])}{\mathcal{F}_N(h[n])} \]
OFDM receiver block diagram

Figure 22: Block diagram of an OFDM receiver, inserting (154) for $w_r n_s$ and interchanging summations gives

$$\hat{y_r}[n] = \sum_{m=0}^{N-1} h_r[m] \sum_{l=0}^{L-1} s_r[l] e^{j2\pi mn/L}$$

Therefore taking the DFT gives

$$y_r[n] = \sum_{m=0}^{N-1} h_r[m] s_r[n]$$

for $n = 0, 1, ..., N-1$. And equalization simply involves multiplying by $h_1^r[n]$; low SNR performance could be improved by multiplying by $p|h_r[n]|^2$ instead of by $h_1^r[n]$. In terms of time domain quantities

$$s_r[n] \equiv \hat{s}[n]$$

The effective channel experienced by $s_r[n]$ is $h_r[n]$, which is a flat fading channel.

OFDM effectively converts a problem of equalizing a frequency-selective channel into...
Equalization comparison

<table>
<thead>
<tr>
<th>Equalization strategy</th>
<th>Time-domain</th>
<th>SC-FDE</th>
<th>OFDM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equalization strategy</td>
<td>Order $K$ filter</td>
<td>One-tap filter</td>
<td>One-tap filter</td>
</tr>
<tr>
<td>Receiver complexity per $N$ symbols</td>
<td>$KN$ multiplies and adds</td>
<td>$2N \log_2 N$</td>
<td>$N \log_2 N$</td>
</tr>
<tr>
<td>Changes to transmitter</td>
<td>None</td>
<td>Insert cyclic prefix</td>
<td>IFFT and cyclic prefix</td>
</tr>
<tr>
<td>Complexity scales with $L$?</td>
<td>Yes since $L \uparrow$ means $K \uparrow$</td>
<td>No (only $L_c$ gets longer)</td>
<td>No (only $L_c$ gets longer)</td>
</tr>
<tr>
<td>Peak-to-average power ratio (PAPR)</td>
<td>Low</td>
<td>Low</td>
<td>High</td>
</tr>
<tr>
<td>Impact on other receiver functions</td>
<td>No</td>
<td>No</td>
<td>Yes (may require different estimation and synchronization algorithms)</td>
</tr>
</tbody>
</table>
OFDM terminology

- $T$: sample period
- $T(N + L_c)$: OFDM symbol period
- Passband bandwidth is $1/T$ (assumes sinc pulse shaping filter)
- Subcarrier spacing: $\Delta_c = \frac{BW}{N} = \frac{1}{NT}$
  - The larger the $N$, the smaller the subcarrier spacing
  - The subcarrier spacing determines the sensitivity to Doppler and residual carrier frequency offset
- Guard interval or cyclic prefix duration: $L_cT$
  - It serves to separate different OFDM symbols
  - In practice the guard interval is determined by the maximum delay spread
  - As the bandwidth increases, $L_c$ must increase to compensate
Visualizing the OFDM spectrum

Bandwidth = $1/T$

Subcarrier spacing = $1/NT$

Can visualize OFDM signal as each symbol rides an equivalent analog carrier with overlapping but orthogonal spectra.

$\text{DC subcarrier}$

$f_c + (-N/2)/NT$

$f_c$

$f_c + (N/2-1)/NT$
Example

- Consider an OFDM system where the OFDM symbol period is 3.2 $\mu$s, the cyclic prefix has length 64, and the number of subcarriers is 256. Find the sample period, the passband bandwidth, the subcarrier spacing and the guard interval.
Orthogonal frequency division multiplexing (OFDM)

Learning objectives:
- Describe the OFDM digital modulation scheme
- Derive the OFDM receiver and prove that it works
Least squares estimation of intersymbol interference channels

Learning objectives:

- Estimate the channel coefficients using the least-squares method
Estimating frequency selective channels

- Channel equalization (usually) requires an estimate of the channel

- Channel estimates are useful for many other purposes
  - Determine the post-processing SNR
  - Predicting the coding and modulation that can be supported
  - Configuring the transmit and receive antennas (in a MIMO system)

- Channel estimation can be done in the time and frequency domains
  - LS channel estimation in the time domain
  - LS channel estimation in the frequency domain
LS estimation of intersymbol interference channels

- Suppose that \( \{t[n]\}_{n=0}^{N_{\text{tr}}-1} \) is a known training sequence
  - Inserted at the beginning of the data

- General output of the system is
  \[
y[n] = \sum_{\ell=0}^{L} h[\ell] s[n - \ell] + v[n]
  \]

- Looking at the first sample
  \[
y[0] = \sum_{\ell=0}^{L} h[\ell] s[n - \ell] + v[n]
\]
  
  \[
  \]

unknown symbols
LS estimation of intersymbol interference channels

- Received signal for \( n = L, \ldots, N_t - 1 \)  

\[
y[n] = \sum_{\ell=0}^{L} h[\ell] t[n - \ell] + v[n]
\]

- Stacking into vector form, this becomes a linear estimation problem

\[
\begin{bmatrix}
y[L] \\
y[L + 1] \\
\vdots \\
y[N - 1]
\end{bmatrix} = 
\begin{bmatrix}
t[L] & t[L - 1] & \cdots & t[0] \\
\vdots & \vdots & \ddots & \vdots \\
t[N_{tr} - 1] & t[N_{t} - 2] & \cdots & t[N_{tr} - 1 - L]
\end{bmatrix} 
\begin{bmatrix}
h[0] \\
h[1] \\
\vdots \\
h[L]
\end{bmatrix} + 
\begin{bmatrix}
v[L] \\
v[L + 1] \\
\vdots \\
v[N_{tr} - 1]
\end{bmatrix}
\]

\[
y = Th + v.
\]
Solve for the channel estimate using least squares

- Least squares solution, assuming $T^*T$ is invertible, is
  \[ \hat{h}_{LS} = (T^*T)^{-1}T^*y \]

- Need $T$ to be square or tall
  \[ N_{tr} - L \geq L + 1 \]
  or equivalently
  \[ N_{tr} \geq 2L + 1 \]

- Need training to be “persistently exciting” to ensure full rank
  ✦ Sequences that have good correlation properties satisfy this requirement
Example of a good training sequence design

- Consider a sequence with good or perfect periodic correlation

\[
\{p[n]\}_{n=0}^{N_p-1} \quad R_p[k] = \sum_{n=0}^{N_p-1} p[n]p^*[((n + k))_{N_p}] \\
|R_p[k]| \approx N_p \delta[k] \quad \text{(equality if perfect)}
\]

- Cyclically prefix \(\{p[n]\}_{n=0}^{N_p-1}\) with \(\{p[n]\}_{n=N_p-L-1}^{N_p-1}\)

\[
T = \begin{bmatrix}
p[0] & p[N_p - 1] & \cdots & p[N_p - L] \\
\vdots & \vdots & \ddots & \vdots \\
p[N_p - 1] & p[N_p - 2] & \cdots & p[N_p - L - 1]
\end{bmatrix}
\]

\[
[T^*T]_{k,\ell} = R_p[k - \ell]
\]

With perfect periodic correlation

\[
T^*T = N_p I \\
\hat{h}_{LS} = T^*y (N_p)^{-1}
\]
Least squares estimation of intersymbol interference channels

Learning objectives:

- Estimate the channel coefficients using the least-squares method
Break
Enabling emergency cellular networks with UAVs
Back to the main event
Least squares channel estimation in the frequency domain

Learning objectives:

- Derive the least-squares channel estimation in the frequency domain
Formulating the LS problem with pilots

- Assume an OFDM system with $N$ subcarriers
- Training is sent in a subset of all the carriers in an OFDM symbol
  - Subcarriers containing training are known as pilots
  - Let $\mathcal{P} = \{p_1, p_2, \ldots, p_P\}$ index the pilot subcarriers
- Write the observations as a function of the unknown channel coefficients
  \[
y[p_1] = h[p_1]t[p_1] + v[p_1]
  \]
  \[
  = \begin{bmatrix} 1 & e^{-j2\pi \frac{p_1}{N}} & \cdots & e^{-j2\pi \frac{p_1L}{N}} \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[L] \end{bmatrix} t[p_1] + v[p_1]
  \]
Collecting data from different pilots together

\[
\begin{bmatrix}
    y[p_1] \\
y[p_2] \\
    \vdots \\
y[p_P]
\end{bmatrix} = \begin{bmatrix}
    t[p_1] & 0 & \cdots & 0 \\
    0 & t[p_2] & 0 & \cdots \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & t[p_P]
\end{bmatrix}
\begin{bmatrix}
    1 & e^{-j2\pi \frac{p_1}{N}} & \cdots & e^{-j2\pi \frac{p_{1L}}{N}} \\
    1 & e^{-j2\pi \frac{p_{2}}{N}} & \cdots & e^{-j2\pi \frac{p_{2L}}{N}} \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & e^{-j2\pi \frac{p_{M}}{N}} & \cdots & e^{-j2\pi \frac{p_{ML}}{N}}
\end{bmatrix}
\begin{bmatrix}
    h[0] \\
h[1] \\
    \vdots \\
h[L]
\end{bmatrix} + \begin{bmatrix}
    v[p_1] \\
v[p_2] \\
    \vdots \\
v[p_M]
\end{bmatrix}
\]

\[y = P E h + v.\]
LS channel estimation in the frequency domain

\[ y = P \hat{E} h + v. \]

- **\( P \)** has the training pilots on its diagonal
  - It is invertible for any non-zero pilots
- **\( E \)** is constructed from samples of DFT vectors
  - It is tall and full rank if \( P \geq L + 1 \)

- With enough pilots, the LS solution is

\[ \hat{h}_{LS} = (E^* P^* \hat{E} P^{-1} E^* P^* y \]

- Can be extended to multiple OFDM symbols & different locations
Least squares channel estimation in the frequency domain

Learning objectives:
- Derive the least-squares channel estimation in the frequency domain
Direct least-square equalizer

Learning objectives:

- Derive and compute the least-squares equalizer without first estimating the channel
LS equalizer from the observed data

- Consider the received signal

\[ y[n] = \sum_{\ell=0}^{L} h[\ell] s[n - \ell] + v[n] \]

where \( s[n] = t[n] \) for \( n = 0, 1, \ldots, N_t - 1 \)

- After equalization with delay \( n_d \)

\[ \hat{s}[n - n_d] = \sum_{\ell=0}^{L_f} f_{n_d}[\ell] y[n - \ell] \]
Least-square equalizer directly from the observed data

- Suppose that \( s[n] = t[n] \) for \( n = 0, 1, \ldots, N_t - 1 \) is the training

\[
\hat{s}[n - n_d] = t[n - n_d] \quad \text{for } n = n_d, n_d + 1, \ldots, n_d + N_t
\]

- Rewriting with knowledge of the training data then (no noise)

\[
t[n] = \sum_{\ell=0}^{L_f} f_{n_d}[\ell] y[n + n_d - \ell]
\]

for \( n = 0, 1, \ldots, N_t \)
Motivation for frequency domain linear equalization

• Objective: Explain why frequency domain equalization is difficult.

Direct versus indirect methods

- Least squares solution, under the assumption that noise is additive and zero-mean.
- Optimize order based on squared error.

Figure 30: QAM receiver with direct equalizer estimation and linear equalization.

- Direct method avoids chain of estimation error.
- Indirect method uses channel estimate, which may anyway be useful.

Building a linear equation

\[ J_f[n_d] = \sum_{n=0}^{N_t-1} |t[n] - \sum_{\ell=0}^{L_f} f_{n_d}[\ell]y[n + n_d - \ell]|^2 \]

- Squared error

Building a linear equation

\[ \begin{bmatrix} t[0] \\ t[1] \\ \vdots \\ t[N_t - 1] \end{bmatrix} = \begin{bmatrix} y[n_d] & \cdots & y[n_d - L_f] \\ y[n_d + 1] & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ y[n_d + N_t - 1] & \cdots & s[n_d + N_t - L_f] \end{bmatrix} \begin{bmatrix} f_{n_d}[0] \\ f_{n_d}[1] \\ \vdots \\ f_{n_d}[L_f] \end{bmatrix} \]

- Must have \( N_t \geq L_f + 1 \) to ensure this becomes a LS problem.
**LS equalizer from the observed data**

- Least squares solution to \( \| t - Y_{nd} \hat{f}_{nd} \|^2 \) under the assumption that \( Y \) is full rank

\[
\hat{f}_{nd} = \left( Y_{nd}^* Y_{nd} \right)^{-1} Y_{nd}^* t.
\]

- Optimize order based on squared error

\[
J_f[n_d] = \| t - \hat{Y}_{nd} \hat{f}_{nd} \|^2 = \| t - Y_{nd} \left( Y_{nd}^* Y_{nd} \right)^{-1} Y_{nd}^* t \|^2
\]
Direct versus indirect methods

- **Direct** method avoids chain of estimation error
  - Design the equalizer directly, no channel inverse required
  - Is more robust to interference and other impairments

- **Indirect** method can design equalizers of arbitrary order
  - $N_t$ and $L_f$ are decoupled in the indirect method
  - May anyways need the channel for other functions
QAM receiver with direct equalizer estimation and linear equalization

![Diagram of QAM receiver with direct equalizer estimation and linear equalization](image-url)

In this section, we describe the carrier frequency offset impairment for frequency-selective channels. We present a discrete-time received signal model that includes frequency offset. Then we examine several carrier frequency offset estimation algorithms. We also remark how each facilitates frame synchronization.

### 4.1 Model for frequency offset in frequency-selective channels

In Section 1.6, we introduced the carrier frequency offset problem. In brief, carrier frequency offset occurs when the carrier used for upconversion and the carrier used for downconversion are different. Even a small difference can create a significant distortion in the received signal.

We have all the ingredients to develop a signal model for frequency offset in frequency-selective channels. Our starting point is to recall (69), which essentially says that carrier frequency offset multiplies the matched filtered received signal by $e^{j2\pi f_c t}$. Sampling at the symbol rate, and using our FIR model for the received signal in (89), we obtain

$$y_{Rx}[n] = \sum_{k=0}^{L} h_r[k] s_r[n-k].$$

(212)

It is possible to further generalize (212) to include frame synchronization by including a delay of $d$. Correction of carrier frequency offset therefore amounts to estimating $\hat{\epsilon}$ and then derotating the received signals $e^{j2\pi \hat{\epsilon} n}$. 

Direct least-square equalizer

Learning objectives:
- Derive and compute the least-squares equalizer without first estimating the channel