Lecture #19

EE 471C / EE 381K-17 Wireless Communication Lab

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Preview of today’s lecture

- Transmit Diversity for MISO Systems without Channel State Information at the Transmitter
  - Explain how the Alamouti code works
  - Describe the generalization to space-time coding
  - Derive the ML detector for space time codes

- Spatial Multiplexing
  - Review the spatial multiplexing concept
  - Extend formulation to $N_t$ transmit antennas
  - Derive the ML detector for spatial multiplexing
Transmit Diversity for MISO Systems without Channel State Information at the Transmitter

Learning objectives

- Objective: explain how the Alamouti code works
- Objective: describe the generalization to space-time coding
Reminder about channel normalization

- For scalar channels

\[ h = \sqrt{G} h_s \]

- For SIMO channels

\[ h = \sqrt{G} h_s \]

- For MISO and more general MIMO channels

\[ G_{\text{MIMO}} = \frac{E_x}{P_r,\text{lin}(d)N_t} = \frac{G}{N_t} \]

\[ h = \sqrt{G_{\text{MIMO}}} h_s \]

\[ G = \frac{E_x}{P_{rx,\text{lin}}(d)} \]

path-loss, distant dependant in this case

divide power across transmit antennas
Transmit diversity without channel state

- Exploit spatial decorrelation at the transmitter
  - Send “different” streams with the same information from each TX antenna
  - Need space-time coding or other processing
- Use largely spaced antenna arrays or polarization
- Open loop TX diversity is used in the downlink in 3G CDMA and LTE systems
The Alamouti code

- Space time code: mapping symbols to antennas for diversity gain
- Alamouti code is a space-time code for two transmit antennas
  - It appears in 3G (WCDMA) and a variation is in IEEE 802.11n
  - Is the most widely used space-time code

\[
\begin{bmatrix}
\sqrt{\frac{E_x}{2}} & s_1 & -s_2^* \\
\sqrt{\frac{E_x}{2}} & s_1 & s_2 \\
\end{bmatrix}
\]

\[
y[0] = (h_1 s[0] + h_2 s[1]) + v[0] \\
y[1] = (-h_1 s^*[1] + h_2 s^*[0]) + v[1]
\]
Key idea of the Alamouti code

- Vector input-output relationship after taking the conjugate of $y[1]$

\[
\begin{bmatrix}
y[0] \\
y^*[1]
\end{bmatrix} =
\begin{bmatrix}
h_1 & h_2 \\
h_2^* & -h_1^*
\end{bmatrix}
\begin{bmatrix}
s[0] \\
s[1]
\end{bmatrix} +
\begin{bmatrix}
v[0] \\
v^*[1]
\end{bmatrix}
\]

or in matrix form

\[y = Hs + v\]

- Multiplying both sides on the left by $H^*$ gives

\[H^*y = H^*Hs + H^*v\]
Structure of the Alamouti code matrix

- The resulting $H$ has a special structure

$$H^*H = \begin{bmatrix} h_1^* & h_2 \\ h_2^* & -h_1 \end{bmatrix} \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} = \begin{bmatrix} |h_1|^2 + |h_2|^2 & h_1^* h_2 - h_1 h_2^* \\ h_2^* h_1 - h_1 h_2^* & |h_1|^2 + |h_2|^2 \end{bmatrix} = \begin{bmatrix} |h_1|^2 + |h_2|^2 & 0 \\ 0 & |h_1|^2 + |h_2|^2 \end{bmatrix}$$

- The columns of $H$ are orthogonal!
Distribution of signal and noise terms

- Because of the structure of $H$

$$H^{*}y = (|h_1|^2 + |h_2|^2)s + H^{*}v$$

- Furthermore, the entries of $H^{*}v$ are i.i.d. with $\mathcal{CN}(0, N_0|h_1|^2 + |h_2|^2)$

$$\mathbb{E}[H^{*}vv^{*}H] = H^{*}\mathbb{E}[vv^{*}]H = N_0H^{*}H = N_0(|h_1|^2 + |h_2|^2)I.$$
Deriving the ML detector

Because the noise terms are independent, the decoding can proceed independently on each entry.

The symbols can then be found by solving the pair of ML detection problems:

\[
\hat{s}_1 = \arg \min_{s \in \mathcal{C}} \left\| \begin{bmatrix} h_1^* & h_2 \end{bmatrix} y - (|h_1|^2 + |h_2|^2) s \right\|^2
\]

\[
\hat{s}_2 = \arg \min_{s \in \mathcal{C}} \left\| \begin{bmatrix} h_2^* & -h_1 \end{bmatrix} y - (|h_1|^2 + |h_2|^2) s \right\|^2
\]
Performance of the Alamouti code

- **Post-processing SNR**

\[
\text{SNR}_{h}^{\text{Ala}} = \frac{(|h_1|^2 + |h_2|^2)^2}{N_o|h_1|^2 + |h_2|^2}
\]
\[
= \frac{1}{N_o}(|h_1|^2 + |h_2|^2)
\]
\[
= \frac{G_{\text{MIMO}}}{N_o}||h_s||^2
\]
\[
= \frac{G}{N_oN_t}||h_s||^2
\]

- The performance achieved by the Alamouti code is 3 dB worse than that achieved by MRT (since \(N_t = 2\))
Space-time coding

- The Alamouti code is a special case of a space-time code
- Each codeword can be visualized as a $N_t \times N_{\text{code}}$ matrix
  - $N_{\text{code}}$ is the number of temporal symbol periods used by the code
  - Denote this codebook as $S$ and the $k^{th}$ entry of the codebook as $S_k$
- We assume a normalization of the codewords such that

  $$\mathbb{E}[\operatorname{tr}(S_k^*S_k)] = N_t N_{\text{code}}$$

- **Example: Codebook for the Alamouti code when BPSK symbols are used**

  \[
  S = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix} \quad \text{BPSK}
  \]

  \[
  S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{bmatrix}
  \]
Received signal modl with space-time coding

- Consider the received signal assuming that codeword
  \[ S = [s[0], s[1], \ldots, s[N_{\text{code}} - 1]] \] is transmitted over a flat-fading channel
  \[ y[n] = h^*s[n] + v[n] \quad n = 0, 1, \ldots, N_{\text{code}} - 1 \]

- In matrix form
  \[ Y = h^*S + V \]

- Alternative vector form is achieved by stacking all the columns using the \( \text{vec}(Y) \) operator

- The \( \text{vec} \) operator generates a vector by stacking the columns of a matrix on top of each other
Reshaping the signal model

- The vec operator often shows up with the Kronecker product
- The Kronecker product of a $N \times M$ matrix $A$ and a $P \times Q$ matrix $B$ is
  $$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1M}B \\ \vdots & \ddots & \vdots \\ a_{N1}B & \cdots & a_{NM}B \end{bmatrix}$$

- Useful identity
  $$\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)$$

- Using these definitions we can rewrite $y$
  $$y = \text{vec}(Y) \text{ and } s = \text{vec}(S)$$
  $$y = (I \otimes h^*)s + v$$
ML detection for space-time codes

- Consider the ML detector of a sequence of space-time symbols
- Let \( S \) denote the set of all possible sequences of transmitted codewords
- The ML detector is

\[
\hat{S} = \arg\min_{Q \in S} \| y - (I \otimes h^*) \text{vec}(Q) \|^2
\]

- Or more compactly

\[
\hat{S} = \arg\min_{Q \in S} \| Y - h^* Q \|^2_F
\]
Comments on the ML detector

- In general, the ML detector requires a search over all possible transmitted sequences
  - Introducing structure can simplify the complexity of the search

- The output of the ML detector is the most likely transmitted codeword

- Bit sequence generated from the inverse space-time mapping operation

- The performance of a space-time code depends on the channel and on the codebook
Performance of the ML detector

- The performance of the ML detector is measured through the probability of codeword error.
- The probability of codeword error can be upper-bounded using the pairwise error probability.
- Probability of codeword error given a channel realization: \( \mathbb{P}(E_x/N_o|\mathbf{h}) \)
- Assuming all symbols are equally likely and taking \( G = E_x \):

  \[
  \mathbb{P}(Q_k \rightarrow Q_\ell | \mathbf{h}) = Q \left( \sqrt{\frac{1}{2N_o} \| \mathbf{h}^*(Q_k - Q_\ell) \|_F^2} \right)
  \]

- Looking for the worst error event and inserting into the union bound:

  \[
  \mathbb{P}(E_x/N_o|\mathbf{h}) \leq (N_{\text{code}} - 1)Q \left( \sqrt{\frac{1}{2N_o} \min_{Q_k \neq Q_\ell \in \mathcal{S}} \| \mathbf{h}^*(Q_k - Q_\ell) \|_F^2} \right)
  \]
Probability of vector symbol error in Gaussian channels

- The average probability of error can be used to devise good codebooks

\[
\mathbb{E}_h \left[ \mathbb{P}(Q_k \rightarrow Q_\ell | h) \right] \leq \frac{(N_{\text{code}} - 1)}{2} \mathbb{E}_h \left[ e^{-\frac{1}{4N_0} ||h^*(Q_k - Q_\ell)||_F^2} \right]
\]

\[
= \frac{(N_{\text{code}} - 1)}{2} \mathbb{E}_h \left[ e^{-\frac{1}{4N_0} h^*(Q_k - Q_\ell)(Q_k - Q_\ell)^* h} \right]
\]

- Defining the error covariance matrix \( R_{k,\ell} = (Q_k - Q_\ell)(Q_k - Q_\ell)^* \), using properties of multi-variate Gaussians

\[
\mathbb{E}_h \left[ \mathbb{P}(Q_k \rightarrow Q_\ell | h) \right] \leq \frac{(N_{\text{code}} - 1)}{2} \frac{1}{\left| I + \frac{E_x}{4N_t N_0} R_{k,\ell} \right|}
\]

\[
= \frac{(N_{\text{code}} - 1)}{2} \frac{1}{\prod_{m=1}^{\text{rank}(R_{k,\ell})} \left( 1 + \frac{E_x}{4N_t N_0} \lambda_m(R_{k,\ell}) \right)}
\]
Probability of vector symbol error in Gaussian channels

- Worst case error is used to upper bound the average probability of error

\[ \mathbb{E}_h \left[ \mathbb{P}(E_x/N_0|h) \right] \leq \frac{(N_{\text{code}} - 1)}{2} \max_{k, \ell, k \neq \ell} \frac{N_{\text{code}} - 1}{\prod_{m=1}^{\text{rank}(R_{k, \ell})} \left( 1 + \frac{E_x}{4N_tN_o} \lambda_m(R_{k, \ell}) \right)} \]

(104)

\[ \approx \frac{(N_{\text{code}} - 1)}{2} \max_{k, \ell, k \neq \ell} \frac{N_{\text{code}} - 1}{\left( \frac{E_x}{4N_tN_o} \right)^{\text{rank}(R_{k, \ell})} \prod_{m=1}^{\text{rank}(R_{k, \ell})} \lambda_m(R_{k, \ell})} \]

(105)

- A space-time code with a full-rank error covariance for all possible error pairs is known as a full-rank space-time code.

The probability of codeword error depends on the worst case error covariance rank.
Design of the codebook

- **Diversity gain** of a space-time codebook: minimum value of the rank $L$ of the difference matrix over all pairs of codewords
  - Full diversity gain: $L = N_t$

- **Design objective**: construct the largest possible codebook with full diversity gain and the maximum possible coding gain

- **Properties of the codebook that gives good performance**:
  - If all codeword difference matrices have full rank
    - The system achieves $N_t$ order diversity
    - It is a full diversity system
Transmit Diversity for MISO Systems without Channel State Information at the Transmitter

Learning objectives

- Objective: explain how the Alamouti code works
- Objective: describe the generalization to space-time coding
Break
Verizon using drones!

Flying around two mobile phones to test coverage at the Formula 1 track in Austin!

Testing drones for emergency cellular coverage
Back to the main event
Spatial Multiplexing

Learning objectives

• Objective: Extend the Spatial Multiplexing Concept to $M_t$ Transmit Antennas
Spatial multiplexing concept

- Multiplex data in space using the same spectrum and with the same total power as in a SISO system
- Each receiver sees a different combination of signals from all the TXs
- Joint processing can be used to separate out the signals
- Increases capacity by min ( #TX ant., # RX ant.)
- Requires large array separation at both the TX and the RX
  ✦ Sufficiently “rich” scattering in the environment
Spatial multiplexer

- Extension to $N_t$ transmit antennas

- The spatial multiplexer operates as a $1 : N_t$ serial to-parallel converter
- The outputs of the spatial multiplexer are the symbol substreams
Signal model for spatial multiplexing

- Symbols are pulse shaped and scaled for transmission to generate a signal

\[ x_j(t) = \sqrt{\frac{E_x}{N_t}} \sum_{n=-\infty}^{\infty} s_j[n]g_{tx}(t - nT) \]

- Symbols scaled by \( \sqrt{E_x/N_t} \) to keep the total transmitted power constant

- Let \( h_{k,m} \) denote the complex baseband equivalent flat-fading channel between the \( mth \) transmit antenna and the \( kth \) receive antenna

- Baseband received signal assuming perfect synchronization and sampling is

\[ y_i[n] = (h_{i,1}s_1[n] + h_{i,2}s_2[n] + \cdots + h_{i,N_t}s_{N_t}[n]) + v_i[n] \]

where \( v_k[n] \) is the usual AWGN with

\[ \mathcal{N}_C(0, N_0). \]
Spatial multiplexing in matrix form

- More common to write in vector-matrix form

\[
\begin{bmatrix}
y_1[n] \\
y_2[n] \\
\vdots \\
y_{N_r}[n]
\end{bmatrix} =
\begin{bmatrix}
h_{1,1} & h_{1,2} & \cdots & h_{1,N_t} \\
h_{2,1} & h_{2,2} & \cdots & h_{2,N_t} \\
\vdots & \vdots & \ddots & \vdots \\
h_{N_r,1} & h_{N_r,2} & \cdots & h_{N_r,N_t}
\end{bmatrix}
\begin{bmatrix}
s_1[n] \\
s_2[n] \\
\vdots \\
s_{N_t}[n]
\end{bmatrix} +
\begin{bmatrix}
v_1[n] \\
v_2[n] \\
\vdots \\
v_{N_r}[n]
\end{bmatrix}
\]

- Classical MIMO equation

\[ y[n] = Hs[n] + v[n]. \]
Block diagram of a MIMO communication system

**Tx**
- Bits → symbol mapping
- $s[n]$ → Spatial Multiplexer
  - $1 : N_t$
  - $g_{tx}[n]$ → RF Upconversion
  - $\sqrt{E_x/N_t}$

**Rx**
- RF Downconversion
- $g_{rx}[n]$ → RF Upconversion
- $\sqrt{E_x/N_t}$
- Joint Detector
- $z^{-k}$
- Symbol Sync.
- Channel Estimation
- Spatial Demultiplexer
  - $N_t : 1$
- $s[n]$ → Observation $y[n]$

The receiver has to detect $s[n]$ from the observation $y[n]$. 
Spatial Multiplexing

Learning objectives

• Objective: Extend the Spatial Multiplexing Concept to $N_t$ Transmit Antennas
ML Detector for Spatial Multiplexing

Learning objectives

• Objective: Derive the ML detector for Spatial Multiplexing
Deriving the ML detector for spatial multiplexing

- Suppose that all the symbols $s_k[n]$ come from the same constellation $C$
- Vector constellation: constellation formed by the resulting $s[n]$
  - It is denoted as $S$
  - Example: with BPSK, $C = \{-1, 1\}$. Forming all combinations of symbols

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ \end{bmatrix} \right\}$$

- The cardinality of $S$ is
  $$|S| = |C|^{N_t}$$

- The size of the vector constellation grows exponentially with $N_t$
Using the multivariate distribution

- The ML detection problem is to determine the best $\hat{s}[n] \in S$ that maximizes the likelihood of $y[n]$ given a candidate vector symbol $s[n]$ and $H$.

- Since $y[n]$ is multivariate Gaussian with distribution $CN(0, N_oI)$ the likelihood function is given by

$$f_{y|H,s}(y[n]|s[n] = \bar{s}, H) = \frac{1}{\pi^{N_r} N_o^{N_r}} e^{-\frac{1}{N_o} (y[n] - H\bar{s})^*(y[n] - H\bar{s})}$$

- The maximum likelihood solution solves

$$\hat{s}[n] = \arg\max_{s \in S} f_{y|H,s}(y[n]|s[n] = \bar{s}, H)$$

$$= \arg\max_{s \in S} \frac{1}{\pi^{N_r} N_o^{N_r}} e^{-\frac{1}{N_o} (y[n] - H\bar{s})^*(y[n] - H\bar{s})}$$
ML detector for spatial multiplexing

- This is equivalent to minimizing the exponent of the function
\[
\hat{s}[n] = \arg \max_{\mathbf{s} \in S} - \frac{1}{N_0} (\mathbf{y}[n] - \mathbf{H}\bar{s})^* (\mathbf{y}[n] - \mathbf{H}\bar{s})
= \arg \min_{\mathbf{s} \in S} (\mathbf{y}[n] - \mathbf{H}\bar{s})^* (\mathbf{y}[n] - \mathbf{H}\bar{s}).
\]

- The ML decoder for spatial multiplexing performs a brute-force search over all possible vector symbols \(|C|^{N_t}\)
\[
\hat{s}[n] = \arg \min_{\mathbf{s} \in S} \|\mathbf{y}[n] - \mathbf{H}\bar{s}\|^2
\]

- There are several low complexity algorithms that provide an approximate solution, e.g. sphere decoder or lattice decoder
Performance of spatial multiplexing in AWGN

- Evaluated using the pairwise error probability
- The pairwise error probability in the AWGN channel is

\[
P(s^{(k)} \rightarrow s^{(\ell)} | H) = Q\left( \sqrt{\frac{E_x}{N_oN_t} \| Hs^{(k)} - Hs^{(\ell)} \|^2} \right)
\]

- The probability of vector symbol error is bounded as

\[
P(E_x/N_o | H) \leq (|S| - 1) \max_{k,\ell,k \neq \ell} Q\left( \sqrt{\frac{E_x}{2N_oN_t} \| Hs^{(k)} - Hs^{(\ell)} \|^2} \right)
\]

\[
\leq (|S| - 1) Q\left( \sqrt{\frac{E_x}{2N_oN_t} \min_{k,\ell,k \neq \ell} \| Hs^{(k)} - Hs^{(\ell)} \|^2} \right)
\]

The performance of spatial multiplexing depends on the minimum distance of the distorted vector constellation.
Performance of spatial multiplexing in fading

- Let $e^{(k,\ell)} = s^{(k)} - s^{(\ell)}$

- Let the error covariance matrix for spatial multiplexing be defined as $R_{k,\ell} = e^{(k,\ell)} c e^{(k,\ell)} T$

- Upper bound on the probability of error for spatial multiplexing with a ML receiver generalizing the space-time coding result

$$\mathbb{E}_H \left[ P(s^{(k)} \rightarrow s^{(\ell)} \mid H) \right] \leq \frac{1}{2} \frac{1}{\left| I + \frac{E_x}{4N_0 N_t} R_{k,\ell} \otimes I \right|}.$$  

$$= \frac{1}{2} \left( 1 + \frac{E_x}{4N_0 N_t} \| e^{(k,\ell)} \|^2 \right)^{N_r}.$$

IID Rayleigh channel

Spatial multiplexing with a maximum likelihood receiver obtains at most a diversity gain of $N_r$. 

Defining $\Theta$.
ML Detector for Spatial Multiplexing

Learning objectives

- Objective: Derive the ML detector for Spatial Multiplexing