From last lecture

- Spatial multiplexing – basic technique for high throughputs
- Transmit precoding – exploiting channel state information
- Channel estimation in MIMO systems
  - Estimate the channel using least squares techniques
- MIMO communication in frequency selective channels
  - Generalize equalization for MIMO frequency selective channels
  - ZF detector in frequency selective channel
  - LS channel estimation
Preview of today’s lecture

◆ The MIMO-OFDM concept

◆ Equalization and detection in MIMO-OFDM
  ✦ ZF and ML detectors

◆ Channel state information in MIMO-OFDM
  ✦ Precoding based on the channel state
  ✦ Channel estimation in MIMO-OFDM

◆ Carrier frequency offset estimation in MIMO-OFDM
MIMO communication in frequency selective channels

Learning objectives

- Objective: Derive MIMO equation for a frequency selective channel
Received signal model in frequency selective channels

- Let \( \{h_{i,j}[\ell]\}_{\ell=0}^{L} \) denote the complex baseband discrete-time equivalent channel between the \( j^{th} \) transmit antenna and the \( i^{kth} \) receive antenna.
- Spatial multiplexing is used at the transmitter where symbol stream \( \{s_{j}[n]\} \) is sent on antenna \( j \).
- The discrete-time baseband received signal is

\[
y_{i}[n] = \sum_{\ell=0}^{L} h_{i,1}[\ell] s_{1}[n - \ell] + h_{i,2}[\ell] s_{2}[n - \ell] + \cdots + h_{i,N_{t}}[\ell] s_{N_{t}}[n - \ell]
\]

**self interference among the transmitted signals**
**MIMO equation in frequency selective channels**

- Define the multivariate impulse response \( \{H[\ell]\}_{\ell=0}^L \) where \[
[H[\ell]]_{i,j} = h_{i,j}[\ell]
\]

- The received signal can be written in matrix form as

\[
y[n] = \sum_{\ell=0}^{L} H[\ell] s[n - \ell] + v[n].
\]
Zero forcing equalizer for frequency selective channels

- Let \( \{G[k]\}_{k=0}^{K} \) denote an order \( K \) multivariate impulse response that corresponds to the equalizer.
- A zero-forcing equalizer would find a \( \{G[k]\}_{k=0}^{K} \) such that

\[
\sum_{k=0}^{K} G[k] H[n - k] \approx \delta[n - n_d] I
\]

- It is possible to find exact solutions with the right choice of \( N_r \).
- Finding the equalizer and equalizing are both high complexity.
ZF equalizer for frequency selective channels

- The zero forcing equalizer can be found using block Toeplitz matrices

\[
\begin{bmatrix}
H_{1,m} & H_{2,m} & \cdots & H_{N_r,m}
\end{bmatrix}
\begin{bmatrix}
g_{1,m} \\
g_{2,m} \\
\vdots \\
g_{N_r,m}
\end{bmatrix}
= e_{n_d}
\]

with \( g_{k,m} = [g_{k,m}[0], g_{k,m}[1], \ldots, g_{k,m}[K]]^T \) and

\[
\begin{bmatrix}
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{bmatrix}
\leftarrow n_d + 1
\]

- Under certain circumstances it is possible to find a perfect FIR matrix inverse for the MIMO equalizer
Definition of the training sequence for LS channel estimation

- Suppose that training sequence \( \{t_j[n]\}_{n=0}^{N_{tr}-1} \) is sent from antenna \( j \)
- In matrix form

\[
T_j = \begin{bmatrix}
    t_j[L] & \cdots & t_j[0] \\
    t_j[L+1] & \ddots & t_j[1] \\
    \vdots & & \vdots \\
    t_j[N_{tr}-1] & \cdots & t_j[N_{tr}-1-L]
\end{bmatrix}
\]
Least square solution

- Writing the observed data as a function of the unknowns in matrix form

\[
y_i = \begin{bmatrix} y_i[L] \\ y_i[L+1] \\ \vdots \\ y_i[N_{tr}-1] \end{bmatrix}, \quad h_{i,j} = \begin{bmatrix} h_{i,j}[0] \\ h_{i,j}[1] \\ \vdots \\ h_{i,j}[L] \end{bmatrix}, \quad \text{and} \quad v_i = \begin{bmatrix} v_i[L] \\ v_i[L+1] \\ \vdots \\ v_i[N_{tr}-1] \end{bmatrix}
\]

\[
y_i = \left[ T_1 \quad T_2 \quad \cdots \quad T_{N_t} \right] \begin{bmatrix} h_{i,1} \\ h_{i,2} \\ \vdots \\ h_{i,N_t} \end{bmatrix} + v_i
\]

The least squares channel estimate follows as

\[
\hat{h}_i = (\bar{T}^*\bar{T})^{-1}\bar{T}^*y_i
\]

estimates all the channels seen by receive antenna

\(i\) at once
Complete MIMO system with spatial multiplexing

Block diagram of a MIMO communication system

**Tx**

- Bits → Symbol mapping → s[n]
- s[n] → Spatial Multiplexer
- Spatial Multiplexer 1 : N_t → M_{tx} → g_{tx}[n] → D/C → √E_s/N_t → RF Upconversion
- RF Upconversion

**Rx**

- RF Downconversion → C/D → g_{rx}[n] → z^{-k} → M_{rx} → Joint Detector
- Joint Detector
- Symbol Sync. → Channel Estimation
- Channel Estimation
- Spatial Demultiplexer N_t : 1
- Spatial Demultiplexer
- s[n]
MIMO communication in frequency selective channels

Learning objectives

- Objective: Derive MIMO equation for a frequency selective channel
Introduction to MIMO-OFDM

Learning objectives
- Objective: Define the system model for MIMO-OFDM
MIMO-OFDM combines:
- Spatial multiplexing and diversity features of MIMO
- Ease of equalization when using OFDM modulation

- It is used in IEEE 802.11n and IEEE 802.11ac
- A variation known as MIMO-OFDMA is used in 3GPP LTE, and 3GPP LTE Advanced, and will probably be used in 5G
MIMO-OFDM with spatial multiplexing: transmitter

Consider a MIMO-OFDM system with spatial multiplexing as illustrated in Fig. 19 and Fig. 20. Denote the symbol stream (assumed to originate in the frequency domain) as $s_r^n$, the $N_t^\hat{1}$ vector symbol as $s_r^n$, and the corresponding subsymbols on antenna $j$ as $s_jr^n$. After the spatial multiplexing operation, each subsymbol stream is passed into a SISO OFDM transmitter operation that consists of a $1:N$ serial-to-parallel operation followed by an $N$-IDFT and the addition of a cyclic prefix addition of length $L_c$. Let $w_r^n$ denote the $N_t^\hat{1}$ time-domain vector output of the cyclic prefix addition block and $w_jr^n$ the samples to be sent on the $j$th transmit antenna.

Figure 19: MIMO-OFDM system block diagram transmitter. The spatial multiplexing operation happens prior to the usual OFDM transmitter operations. The output of each OFDM modulator is possible pulse-shaped, converted to continuous-time, then upconverted.

Figure 20: MIMO-OFDM system block diagram receiver. (power normalization missing from figure)
MIMO-OFDM with spatial multiplexing: receiver

Figure 19: MIMO-OFDM system block diagram transmitter. The spatial multiplexing operation happens prior to the usual OFDM transmitter operations. The output of each OFDM modulator is possible pulse-shaped, converted to continuous-time, then upconverted.

Figure 20: MIMO-OFDM system block diagram receiver.
Received signal per receive antenna

- Consider the signal at the \( r^{th} \) receive antenna assuming perfect synchronization

\[
y_i[n] = \sum_{j=1}^{N_t} \sum_{\ell=0}^{L} h_{i,j}[\ell] w_j[n - \ell] + v_i[n]
\]

- Let

\[
h_{i,j}[k] = \sum_{\ell=0}^{L} h_{i,j}[\ell] e^{-j \frac{2\pi k \ell}{N}}
\]

be the N-DFT of the (zero-padded) channel between the \( i^{th} \) receive antenna and the \( j^{th} \) transmit antenna

- Discarding the first \( L_c \) samples and taking the N-DFT

\[
y_i[n] = \sum_{j=1}^{N_t} h_{i,j}[k] s_m[k] + v_i[n].
\]
Received signal in vector form

- Define the matrix response 
  \[ [H[k]]_{i,j} = h_{i,j}[k] \]

\[
H[k] = \sum_{\ell=0}^{L} H[\ell] e^{-j \frac{2\pi k \ell}{N}}
\]

- Stacking the observations for \( i = 1, 2, \ldots, N_t \) gives the canonical MIMO-OFDM system equation

\[
y[k] = H[k]s[k] + v[k]
\]

- Many flat-fading results (equalization, detection, and precoding) can be extended to MIMO-OFDM with a suitable change in notation
Main takeaways

- Dealing with frequency selective fading in MIMO systems is challenging

- OFDM is attractive for MIMO systems because it simplifies one dimension of the equalization problem (frequency selectivity) and yet still leaves the interference between spatial streams

- Many flat-fading results (equalization, detection, and precoding) can be extended to MIMO-OFDM with a suitable change in notation, which justifies the study of flat-fading
Introduction to MIMO-OFDM

Learning objectives

- Objective: Define the system model for MIMO-OFDM
Break
Advice from Dr. Amine Mezghani

◆ Why graduate school?

◆ What does it take to be successful in graduate school?
Back to the main event
Equalization and detection in MIMO-OFDM

Learning objectives

- Objective: Generalize the ML and ZF detectors for MIMO-OFDM
ML detector for MIMO OFDM

- A maximum likelihood detector would solve

\[ \hat{s}[k] = \arg \min_{\bar{s} \in S} ||y[k] - H[k]\bar{s}||^2 \]

for \( k = 0, 1, \ldots, N-1 \)

- Brute force search complexity
- The candidate symbol vectors are multiplied by \( H[k] \), which is different for every subcarrier
- Higher storage requirements than in the flat fading case
ZF detector for MIMO OFDM

- The receiver first compute \( z[k] = G[k]y[k] \) with \( G[k] = H[k]^\dagger \)

- Then it applies a separate detector for each substream

\[
\hat{s}_k[n] = \arg \min_{c \in \mathcal{C}} |z_k[n] - c|^2
\]

- Several papers propose different ways to reduce complexity
Main takeaways

- Both ML detection and zero-forcing detection extend directly to MIMO-OFDM from their flat-fading counterparts.

- Complexity is higher due to the fact that the channel changes every subcarrier as $H[k]$, e.g. with ZF more inverses need to be computed.

- Equalization complexity much lower than the time domain approach.
Equalization and detection in MIMO-OFDM

Learning objectives
- Objective: Generalize the ZF and ML detectors for MIMO-OFDM
Precoding in MIMO-OFDM

Learning objectives

- Objective: Generalize precoding to MIMO-OFDM
Precoding per-subcarrier

- Let \( F[k] \) denote a precoder derived for channel \( H[k] \) and suppose now that \( s[k] \) is \( N_s \times 1 \)

- The received signal with precoding is

\[
x[k] = H[k]F[k]s[k] + v[k]
\]

for \( k = 0, 1, \ldots, N-1 \)

- The receiver performs equalization and detection based on the combined channel \( H[k] F[k] \)
Precoding with limited feedback

- The concept of limited feedback can be used to convey quantized precoders from the receiver back to the transmitter
  - Shared codebook at the transmitter and receiver
  - Quantization of each $H[k]$
- Feedback overhead grows with $N$ since each subcarrier needs its own precoder
  - Reduce by using a single precoder for several adjacent subcarriers
- MIMO OFDM precoding is widely deployed in IEEE 802.11n, IEEE 802.11ac, 3GPP LTE, 3GPP LTE Advanced, etc.
Main takeaways

- Precoding in a MIMO-OFDM system is similar to that performed in a flat-fading MIMO systems

- In a MIMO-OFDM system, the precoder varies for each subcarrier making the potential feedback requirements higher than a flat-fading system
Precoding in MIMO-OFDM

Learning objectives

- Objective: Generalize precoding to MIMO-OFDM
Channel Estimation in MIMO-OFDM

Learning objectives

- Objective: Review different approaches for estimating the channel based on training data
Channel estimation in MIMO-OFDM

- Estimation of the channel based on training data
- We focus on training data inserted into a single OFDM symbol
Algorithms exploiting training in the time domain

- Suppose that $\{t[n]\}_{n=0}^{N-1}$ is a known training sequence and that $s[n] = t[n]$

  In this case $N_{tr} = N$

- The time-domain signal after the cyclic prefix addition is $\{w[n]\}_{n=0}^{N+L-1}$

- The resulting $T_i$ matrix (constructed in this case from $\{w[n]\}_{n=0}^{N+L-1}$) will be $N \times (L + 1)$ and $y_i$ will be $N \times 1$

- Time-domain least squares channel estimate is $\hat{h}_i = (\bar{T}^* \bar{T})^{-1} \bar{T}^* y_i$. 

$$\hat{h}_i = (\bar{T}^* \bar{T})^{-1} \bar{T}^* y_i.$$
Algorithms exploiting the training in the freq. domain

- The received signal is

\[ y[k] = H[k]t[k] + v[k] \quad k = 0, 1, \ldots, N - 1 \]

- Rewrite \( H[k] \) as a function of \( H[\ell] \) in matrix form

\[
H[k] = \sum_{\ell=0}^{L} H[\ell] e^{-j \frac{2\pi k \ell}{N}} \\
= \begin{bmatrix} H[0] & H[1] & \cdots & H[L] \end{bmatrix} \begin{bmatrix} I_{N_t} \\ e^{-j \frac{2\pi k}{N} I_{N_t}} \\ \vdots \\ e^{-j \frac{2\pi k L}{N} I_{N_t}} \end{bmatrix}
\]
Vectorizing the channel matrix

- Define the vector

\[ e^T[k] = \begin{bmatrix} 1 & e^{-j \frac{2\pi k}{N}} & \cdots & e^{-j \frac{2\pi k L}{N}} \end{bmatrix} \]

- Rewrite the channel using the Kronecker product

\[ H[k] = [H[0] \ H[1] \ \cdots \ H[L]] (e^T[k] \otimes I_{N_t}) \]

- Compute

\[
\text{vec}(H[k]) = \left( (e^T[k] \otimes I_{N_t}) \otimes I_{N_r} \right) \begin{bmatrix} \text{vec}(H[0]) \\ \text{vec}(H[1]) \\ \vdots \\ \text{vec}(H[L]) \end{bmatrix}
\]
Simplifying the formulation

- Using Kronecker product identities

\[
\begin{align*}
y[k] &= \text{vec}(y[k]) \\
&= \text{vec}(H[k]t[k]) + v[k] \\
&= (t[k]^T \otimes I_{N_r}) \text{vec}(H[k]) + v[k] \\
&= (t[k]^T \otimes I_{N_r}) ((e[k]^T \otimes I_{N_t}) \otimes I_{N_r}) h + v[k] \\
&= \underbrace{(e[k]^T \otimes t[k]^T \otimes I_{N_r})}_T[k] h + v[k].
\end{align*}
\]
Building a linear system from multiple observations

- Known values of the training can be used to build a least squares estimator.
- Suppose that pilot subcarriers are used: \( \mathcal{K} = \{k_1, k_2, \ldots, k_t\} \)
- Training is known only at subcarriers.
- Stacking the observations gives

\[
\begin{bmatrix}
y[k_1] \\
y[k_2] \\
\vdots \\
y[k_t]
\end{bmatrix} =
\begin{bmatrix}
T[k_1] \\
T[k_2] \\
\vdots \\
T[k_t]
\end{bmatrix} h
\begin{bmatrix}
v[k_1] \\
v[k_2] \\
\vdots \\
v[k_t]
\end{bmatrix} +
\begin{bmatrix}
y \end{bmatrix}
\]
Least squares solution of MIMO-OFDM channel

- The dimensions of $\bar{T}$ are $|\mathcal{K}|N_r \times N_t N_r (L + 1)$.
- With enough pilot subcarriers, $|\mathcal{K}|$ can be made large enough to ensure that $\bar{T}$ is square or tall.
- Assuming a good training sequence design so that $\bar{T}$ is full rank, the LS estimate is computed as
  \[
  \hat{h} = (\bar{T}^* \bar{T})^{-1} \bar{T}^* \bar{y}
  \]
- The final step is to reform $\{H[\ell]\}_{\ell=0}^L$ from $h$ and then to take the N-DFT to find $\{H[k]\}_{k=0}^{N-1}$.
- While the complexity of the LS solution seems high, $(\bar{T}^* \bar{T})^{-1} \bar{T}^*$ can be precomputed.
  - Only a matrix multiplication is required to generate the estimate.
Main takeaways

- Channel estimation in MIMO-OFDM generalizes the approach taken in MIMO systems

- Main challenge is writing an equation in terms of the unknown time-domain coefficients for all the antennas
Channel Estimation in MIMO-OFDM

Learning objectives

- **Objective**: Review different approaches for estimating the channel based on training data
Carrier Frequency Offset Estimation in MIMO-OFDM

Learning objectives

- Objective: Generalize the Schmidl-Cox algorithm for MIMO-OFDM
Carrier frequency offset estimation in MIMO-OFDM

- The main approach to CFO estimation is to generalize the Moose and Schmidl-Cox algorithm.
- Consider a system where each RF chain independently generates an imperfect version of $f_c$.
- The transmit signal generated by the $j^{th}$ transmit antenna is created by upconversion using a carrier frequency $f_{c,j}^{(tx)}$.
- The down converted signal at the $i^{th}$ receive antenna is demodulated using carrier frequency $f_{c,i}^{(rx)}$.
- Let the normalized frequency offset be $\epsilon_{i,j} = (f_{c,i}^{(rx)} - f_{c,j}^{(tx)})T$. 

Synchronization is an important operation for MIMO-OFDM communication systems. There are many generalizations and extensions of least squares channel estimators from this derivation is that the least squares estimator in Chapter 5 are powerful and can be applied to a variety of sophisticated settings in the context of MIMO-OFDM.
Received signal model with CFO

- The received signal with carrier frequency offset becomes

\[
y_i[n] = \sum_{j=1}^{N_t} e^{j2\pi \epsilon_{i,j} n} \sum_{\ell=0}^{L} h_{i,j}[\ell] w_j[n - \ell] + v_i[n]
\]

- Even if the offsets \( \{\epsilon_{i,j}\} \) can be estimated, they cannot be easily removed with a multiplication by \( e^{-j2\pi \epsilon_{i,j} n} \)

- If a common reference is used at each of the transmitter and receiver, then for all \( i \) and \( j \),

\[
\epsilon_{i,j} = \epsilon \quad \text{There is only a single offset}
\]
CFO model for MIMO-OFDM with shared oscillators

- If the offset does not depend on $i$, a matrix equation can be written as

$$ y[n] = e^{j2\pi \epsilon n} \sum_{\ell=0}^{L} H[\ell] w[n - \ell] + v[n] $$

- And correction of the offset can be done generating the signal

$$ \exp(-j2\pi \hat{\epsilon} n) y[n]. $$
Generalization of the Schmidl-Cox algorithm

- The simplest approach is to send the same training sequence simultaneously from every antenna.
- The receive signal simplifies as

$$y[n] = e^{j2\pi n} \sum_{\ell=0}^{L} \mathbf{H}[:\ell] \mathbf{1} w[n - \ell] + \mathbf{v}[n]$$

- Each received signal has the form

$$y_i[n] = e^{j2\pi n} \sum_{\ell=0}^{L} \tilde{h}_i[\ell] w[n - \ell] + v_i[n]$$
**LS estimator with same training per antenna**

- Assuming for simplicity of exposition that the odd subcarriers are zeroed such that

  \[ w[n + L_c] = w[n + L_c + N/2] \]

  then with \( \bar{y}_i[n] = y_i[n + L_c] \)

  \[ \bar{y}_i[n + N/2] = e^{j2\pi e N/2} \bar{y}_i[n] \]

  for \( n = 0, 1, \ldots, N/2-1 \) and \( i = 1, 2, \ldots, N_r \).

- A simple estimator using LS follows as

  \[ \hat{e} = \frac{\text{phase} \left( \sum_{i=1}^{N_r} \sum_{n=0}^{N/2-1} \bar{y}_i^*[n + N/2] \bar{y}_i[n] \right)}{\pi N} \]
Main takeaways

- Carrier frequency offset estimation in MIMO systems is normally performed assuming the transmit antennas are tied to the same oscillator and the receive antennas are tied to the same oscillator.

- Sending signals with multiple repetitions allows the Moose method to be employed for carrier frequency offset estimation.

- There are ways to further optimize the construction of the transmitted signals to send different good signals on each antenna.
Carrier Frequency Estimation in MIMO-OFDM

Learning objectives

- Objective: Generalize the Schmidl-Cox algorithm for MIMO-OFDM