Lecture #5

EE 471C / EE 381K-17 Wireless Communication Lab

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Preview of today’s lecture

◆ Discrete-time processing of bandlimited continuous-time signals
  ✤ Determine equivalent combination of sampling, digital filtering, and reconstruction to process a bandlimited continuous-time signal with discrete-time signal processing

◆ Defining the frequency response of a random signal
  ✤ Define and calculate the power spectrum of a wide sense stationary stochastic process

◆ Bandwidth of a signal
  ✤ Define and calculate various real measures of bandwidth

◆ Complex envelope representation of wireless passband signals
  ✤ Use the complex envelope to represent a passband signal at baseband
Discrete-time Processing of Bandlimited Continuous-Time Signals

Learning objective:

- Determine equivalent combination of sampling, digital filtering, and reconstruction to process a bandlimited continuous-time signal with discrete-time signal processing
Filtering a bandlimited signal (1)

Consider

If $x(t)$ is bandlimited, then we can generate it in discrete-time

Because $y(t)$ is bandlimited can sample it to obtain discrete-time

Remember:

\[ y(f) = h_c(f)x(f) \]
Filtering a bandlimited signal (2)

- Is it possible to replace

\[
x[n] \xrightarrow{D/C} x(t) \xrightarrow{h_c(\tau)} y(t) \xrightarrow{C/D} y[n]
\]

- With the following for some suitable \( h[n] \)?

\[
x[n] \xrightarrow{h[\ell]} y[n]
\]

Yes!
Then suppose that

Substituting for

Now suppose that

Substituting for

These facts can be used with additional results on sampling to obtain an equivalent converter operating at an appropriate sampling frequency.

With these definitions, it should be clear that

1. The signal

Suppose that

\[ y = \text{rect}(f/W)h_c(f) \]

2. In the time domain

\[ h_{\text{low}}(t) = W \int \text{sinc}(\tau W)h_c(t - \tau)d\tau \]
Explaining the process (2)

- The spectrum of $y_c(t)$

\[ y_c(f) = h_{\text{low}}(f)x_c(f) \]

- The spectrum of $y[n]$

\[ y(e^{j2\pi f}) = \frac{1}{T}y_c\left(\frac{f}{T}\right) \quad \text{for} \quad f \in [-1/2, 1/2) \]

\[ = \frac{1}{T}h_{\text{low}}\left(\frac{f}{T}\right)x_c\left(\frac{f}{T}\right) \quad \text{for} \quad f \in [-1/2, 1/2) \]

Note: $T \times (e^{j2\pi f}) = x_c\left(\frac{f}{T}\right) \quad f \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

recall

\[ x(e^{j2\pi f}) = \frac{1}{T} \sum_k x_c\left(\frac{f}{T} - \frac{k}{T}\right) \]
Explaining the process (3)

- Substituting $T x (e^{j2\pi f}) = x_c \left( \frac{f}{T} \right)$ for $f \in \left( -\frac{1}{2}, \frac{1}{2} \right)$

$$y (e^{j2\pi f}) = h_{\text{low}} \left( \frac{f}{T} \right) \times (e^{j2\pi f}) \quad \text{for } f \in [-1/2, 1/2)$$

$$= \frac{1}{T} T h_{\text{low}} \left( \frac{f}{T} \right) \times (e^{j2\pi f}) \quad \text{for } f \in [-1/2, 1/2)$$

- Now suppose that

$$h(f) = T h_{\text{low}}(f)$$

$$y (e^{j2\pi f}) = \frac{1}{T} h \left( \frac{f}{T} \right) \times (e^{j2\pi f}) \quad \text{for } f \in [-1/2, 1/2)$$

$$y (e^{j2\pi f}) = h (e^{j2\pi f}) \times (e^{j2\pi f}) \quad \text{for } f \in [-1/2, 1/2)$$
Discrete-time equivalent system

Equivalent system is obtained by scaling, lowpass filtering, and sampling the continuous-time system.

\[ h[n] = T h_{\text{low}}(nT) \]
\[ = T W \int \text{sinc}(\tau W) h_c(nT - \tau) \, d\tau \]
\[ = \int \text{sinc}(\tau W) h_c(nT - \tau) \, d\tau. \]

\( TW = 1 \) if sampling at exactly Nyquist.
Example

Example 34 Suppose that an LTI system delays the input by an amount $\tau_d$. Determine the impulse response of this system and its discrete-time equivalent assuming that the input signal has a bandwidth of $B/2$.

Answer: This system corresponds to a delay, therefore $h_c(t) = \delta(t - \tau_d)$. Applying (3.107), the discrete-time equivalent system is

$$h[n] = \int \text{sinc}(\tau B) h_c(nT - \tau) d\tau$$  \hspace{1cm} (3.108)

$$= \int \text{sinc}(\tau B) \delta(nT - \tau_d - \tau) d\tau$$  \hspace{1cm} (3.109)

$$= \text{sinc}(nBT - B\tau_d)$$  \hspace{1cm} (3.110)

$$= \text{sinc}(n - B\tau_d).$$  \hspace{1cm} (3.111)

It may seem surprising that this is not a delta function, but recall that $\tau_d$ can take any value; if $\tau_d$ is an integer fraction of $T$ then $B\tau_d$ will be an integer and $h[n]$ will become a dirac delta function.
Discrete-time Processing of Bandlimited Continuous-Time Signals

Learning objective:

- Determine equivalent combination of sampling, digital filtering, and reconstruction to process a bandlimited continuous-time signal with discrete-time signal processing.
Defining the frequency response of a random signal

Learning objective:

- Define and calculate the power spectrum of a wide sense stationary stochastic process.
Random processes in the frequency domain

- Random signals do not have a conventional definition of a spectrum
  - If $x(t)$ is a random process,
    - What does the Fourier transform of $x(t)$ mean?
    - Is it well defined?

- The power spectrum is the Fourier transform of the autocovariance function of a WSS random process
  - Use power spectrum to define **bandwidth** of a random signal
  - Use power spectrum to compute **power** of a random signal
The power spectrum for random processes

- For continuous-time signals

\[ P_x(f) = \int_{-\infty}^{\infty} C_{xx}(t)e^{-j2\pi ft} dt \]

- For discrete-time signals

\[ P_x(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} C_{xx}[n]e^{-j2\pi fn} \]

- Power spectrum is real and non-negative due to conjugate symmetry
Example 42  Find the power spectrum for a zero-mean WSS random process $x(t)$ with an exponential correlation function with parameter $\beta > 0$

$$R_{xx}(t) = e^{-2\beta |t|} \quad t \in (-\infty, \infty).$$

**Answer:** Because the process is zero-mean, computing the power spectrum involves taking the Fourier transform of the auto-correlation function. The answer follows from calculus and simplification.

$$P_x(f) = \int_{-\infty}^{\infty} R_{xx}(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} e^{-2\beta |t|} e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{0} e^{(2\beta-j2\pi f)t} dt + \int_{0}^{\infty} e^{-(2\beta-j2\pi f)t} dt$$

$$= \frac{1}{2\beta - j2\pi f} + \frac{1}{2\beta + j2\pi f}$$

$$= \frac{4\beta}{4\beta^2 + 4\pi^2 f^2}$$

$$= \frac{\beta}{\beta^2 + \pi^2 f^2}. \quad (3.215)$$
Defining the frequency response of a random signal

Learning objective:

- Define and calculate the power spectrum of a wide sense stationary stochastic process.
Break
Bandwidth of a signal

Learning objective:

- Define and calculate various real measures of bandwidth
Bandwidth of a signal

Bandwidth is determined from its Fourier transform or power spectrum.

Bandwidth is some measure of the extent of frequency content.

This is a **baseband** signal.
Spectrum allocation in wireless

Wireless services occupy small slices of larger frequency spectrum

FYI: Cellular is land mobile
Passband signals

- **Passband signals** have energy that is nonzero for a frequency band concentrated about $\pm f_c$, where $f_c$ is the carrier frequency and $f_c \gg 0$.

- Bandwidth is determined by signal extent around the carrier
  - All wireless signals are passband
  - Some variations of this definition that place the carrier differently
Measuring the bandwidth

Several different criteria for measuring bandwidth

Definition depends on whether baseband or passband
Some common measures

**Half-power bandwidth** The half-power bandwidth, or 3dB bandwidth, is defined as the value of the frequency over which the power spectrum is at least 50% of its maximum value, i.e.

\[ B = f^{(h)}_{3\text{dB}} - f^{(l)}_{3\text{dB}}, \]  

(3.67)

where \( f^{(h)}_{3\text{dB}} \) and \( f^{(l)}_{3\text{dB}} \) are as shown in Fig. 3.3.

**X dB bandwidth** The X dB bandwidth is defined as the difference between the largest frequency that suffers X dB of attenuation and the smallest frequency. Similar to the half-power bandwidth, this can be expressed as,

\[ B = f^{(h)}_{X\text{dB}} - f^{(l)}_{X\text{dB}}, \]  

(3.68)

where \( f^{(h)}_{X\text{dB}} \) and \( f^{(l)}_{X\text{dB}} \) are defined in the same manner as \( f^{(h)}_{3\text{dB}} \) and \( f^{(l)}_{3\text{dB}} \).

**Fractional power containment bandwidth** One of the most useful notions of bandwidth, often employed by the FCC, is the fractional power containment bandwidth. The fractional containment bandwidth is

\[ \int_0^{B/2} P_x(f)\,df = \alpha \int_{-\infty}^{\infty} P_x(f)\,df \]  

(3.70)

where \( \alpha \) is the fraction of containment. For example, if \( \alpha = 0.99 \) then the bandwidth \( B \) would be defined such that the signal has \( \alpha \% \) of its bandwidth.

**Noise equivalent bandwidth** The noise equivalent bandwidth is defined as

\[ B = \frac{1}{P_x(f_c)} \int_f P_x(f)\,df. \]  

(3.69)
Baseband illustration

\[ S_x(f) \]

\[ S_{\text{max}} \]

\[ 0.5 \cdot S_{\text{max}} \]

\[ B_{3\text{dB}} \]

\[ B_{1\text{st \ null}} \]

\[ B_{99\%} \]

Area = 0.5% of the total
Passband illustration

\[ S_x(f) \]

- \[ S_{\text{max}} \]
- \[ 0.5 \cdot S_{\text{max}} \]
- \[ B_{99\%} \]
- \[ B_{3\text{dB}} \]
Example

**Example 29** Consider a simplified PSD as shown in Fig. 3.4, and compute the half power bandwidth, noise equivalent bandwidth, and the fractional power containment bandwidth with $\alpha = 0.9$.

**Answer:** From Fig. 3.4, we have $f_{3dB}^{(e)} = f_c - 1.5 \text{ MHz}$, and $f_{3dB}^{(h)} = f_c + 1.5 \text{ MHz}$, thus

$$B_{3dB} = (f_c + 1.5) - (f_c - 1.5) = 3 \text{ MHz}. \quad (3.71)$$

We have $P_x(f_c) = 1$, and the integral can be computed as the area of the trapezoid

$$B_{\text{noise equivalent}} = \frac{1}{P(f_c)} \int_f P(f) \, df = 3 \text{ MHz}. \quad (3.72)$$

The total area of the PSD is 3, so for $\alpha = 0.9$, the area that should be left out is 0.3 on both side or 0.15 for one side of $f_c$. This corresponds to the point $f_c - 2 + \sqrt{0.3}$ on the left. Therefore

$$B_{\text{fractional}} = (f_c + 2 - \sqrt{0.3}) - (f_c - 2 + \sqrt{0.3}) = 2.90 \text{ MHz}. \quad (3.73)$$

Figure 3.4: An example PSD for purposes of bandwidth calculations in Example 29.
Bandwidth of a signal

Learning objective:

- Define and calculate various real measures of bandwidth
Complex envelope representation of wireless passband signals

Learning objective:

- Use the complex envelope to represent a passband signal at baseband
Wireless passband signals

- **Passband signals** have energy that is nonzero for a frequency band concentrated about the carrier $f_c$
- Most passband wireless signals are also narrowband

$$W \ll f_c$$

Caution: the term narrowband is used in several different ways in wireless
Passband signal and simplifications

- General passband signal can be written

\[ x_p(t) = A(t) \cos(2\pi f_c t + \phi(t)) \]

- \( A(t) \) amplitude function
- \( \phi(t) \) phase function

Amplitude and/or phase contain the information bits

Carrier frequency in a digital modulated signal
Inphase and quadrature signals

- **Apply the trig identity** \( \cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B) \)

\[
x_p(t) = A(t) \cos(\phi(t)) \cos(2\pi f_c t) - A(t) \sin(\phi(t)) \sin(2\pi f_c t)
\]

\[
\begin{align*}
x_i(t) & \quad \text{inphase} \\
x_q(t) & \quad \text{quadrature}
\end{align*}
\]

- **Complex envelope** \( x(t) = x_i(t) + jx_q(t) \)

Complex signals are widely used in communications as a way to package the inphase and quadrature components at baseband
Complex envelope to passband signal

- Multiplying

\[ x(t)e^{j2\pi f_c t} = x_i(t)\cos(2\pi f_c t) - x_q(t)\sin(2\pi f_c t) + j(x_i(t)\sin(2\pi f_c t) + x_q(t)\cos(2\pi f_c t)) \]

- Therefore

\[ x_p(t) = \text{Re} \left[ x(t)e^{j2\pi f_c t} \right] \]

Real passband signal

Complex baseband signal

Carrier

upconversion
Upconversion

quadrature

inphase

Process of going from complex envelope to passband
In the time domain

\[ x_p(t) = \text{Re} \left[ x(t) e^{j2\pi f_c t} \right] \]

or

\[ x_p(t) = \frac{1}{2} x(t) e^{j2\pi f_c t} + \frac{1}{2} x^*(t) e^{-j2\pi f_c t} \]

In the frequency domain

\[ x_p(f) = \frac{1}{2} (x(f - f_c) + x^*(-f - f_c)) \]
Passband signal to baseband

- Consider a passband signal $y_p(t)$
- How to obtain the complex envelope from a passband signal?

$$y_p(t) = \text{Re} \left[ y(t)e^{j2\pi f_c t} \right]$$

DOWNCONVERSION
Using trig identities

Consider the identities

\[
\sin u \sin v = \frac{1}{2} [\cos(u - v) - \cos(u + v)]
\]

\[
\cos u \cos v = \frac{1}{2} [\cos(u - v) + \cos(u + v)]
\]

\[
\sin u \cos v = \frac{1}{2} [\sin(u - v) + \sin(u + v)].
\]

Applying the identities

\[
y_p(t) \cos(2\pi f_c t) = \frac{1}{2} y_i(t) + \frac{1}{2} y_i(t) \cos(4\pi f_c t) - \frac{1}{2} y_q(t) \sin(4\pi f_c t)
\]

\[
y_p(t) \sin(2\pi f_c t) = -\frac{1}{2} y_q(t) + \frac{1}{2} y_q(t) \cos(4\pi f_c t) + \frac{1}{2} y_i(t) \sin(4\pi f_c t)
\]

Filter out
Downconversion

To understand how downconversion works, recall the following trigonometric identities:

\[
\cos(2\pi f_c t) \quad \text{and} \quad -\sin(2\pi f_c t)
\]

where \( y_p(t) \) is a passband signal. Because it is passband, there exists an equivalent baseband signal. The real analog may not perform a direct conversion but rather may have a series of upconversions and downconversions. Additionally, there is typically a low-noise amplifier and a bandlimiting filter just after the antenna and a series of gain control circuits.

Using (319)-(321) and (322), we can explain downconversion mathematically in a compact fashion. An ideal low-pass filter with unity gain and cutoff frequency \( f_c \) is given by

\[
y(t) = \int_{-\infty}^{\infty} x(u) \text{rect}(t-u/f_c) \, du
\]

where \( \text{rect}(x) = 1 \) for \( |x| \leq 1 \) and 0 otherwise. The downconverted signal can be written in the time domain from Table 2. The downconverted signal can be written as

\[
y(t) = \cos(2\pi f_c t) y_p(t) + j \sin(2\pi f_c t) y_p(t)
\]

After filtering with an ideal low-pass filter, the baseband component is obtained. To extract the baseband components, therefore, it suffices to apply a low-pass filter to the outputs of (322) and (323) and correct for the scaling factor.
In the frequency domain
\[ y(f) = 2 \text{rect} \left( \frac{f}{W} \right) y_p(f + f_c) \]

In the time domain
\[ y(t) = 2W \int \text{sinc}((t - \tau)W) y_p(t)e^{-j2\pi f_c \tau} d\tau. \]
Connection to the lab

transmitter
Source \rightarrow \text{Channel Coding} \rightarrow \text{Modulation} \rightarrow \text{D/A} \rightarrow \text{RF Upconversion}

receiver
Sink \rightarrow \text{Channel Decoding} \rightarrow \text{Demodulation} \rightarrow \text{A/D} \rightarrow \text{RF Downconversion}

Laptop with MATLAB
(all digital signal processing)

NI USRP 2921

Real world

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Complex envelope representation of wireless passband signals

Learning objective:

○ Use the complex envelope to represent a passband signal at baseband