Lecture #7

471C / 381V Wireless Communications Lab
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Preview of today’s lecture

- Computing the bandwidth, energy, and power of \( x(t) \)
  - Calculate bandwidth, energy, and power of a general complex pulse amplitude modulated signal

- Additive white Gaussian noise (AWGN) channel
  - Define AWGN including the distribution and properties

- Overview of AWGN transmitter and receiver
  - Describe key components of the transmitter and receiver

- Nyquist pulse shape design
  - Determine properties of the optimal receiver pulse-shape for the AWGN channel
Bandwidth, energy, and power for linear complex pulse amplitude modulation

Learning objectives:

- Calculate bandwidth, energy, and power of a general complex pulse amplitude modulated signal
Linear complex pulse amplitude modulation

![Diagram of signal processing flow]

\[ x(t) = \sqrt{E_x} \sum_{n=\infty}^{\infty} s[n] g_{tx}(t - nT) \]

- **Assumptions so far**
  - \( s[n] \) is an IID random process
  - Constellation normalized so that \( \mathbb{E}[s[n]] = 0 \) and \( \mathbb{E}[|s[n]|^2] = 1 \)

How to compute the bandwidth, symbol energy, and power of \( x(t) \)?
Additive white Gaussian noise (AWGN) channel

Learning objectives:
- Define AWGN including the distribution and properties
Computing the bandwidth

◆ Complex baseband signal

\[ x(t) = \sqrt{E_x} \sum_{n=-\infty}^{\infty} s[n] g_{tx}(t - nT) \]

◆ Since \( s[n] \) is IID (and WSS) \( x(t) \) is a cyclostationary random process

◆ We can define the power spectral density of signals like this as

\[ P_x(f) = \frac{E_x}{T} \mathbb{E} \left[ |s[n]|^2 \right] |G_{tx}(f)|^2 \]

✧ With a unit energy constellation

\[ P_x(f) = \frac{E_x}{T} |G_{tx}(f)|^2 \]

✧ \( |G_{tx}(f)|^2 \) determines the bandwidth of \( x(t) \)
Energy and power

- We always assume $\int_{-\infty}^{\infty} |G_{tx}(f)|^2 df = 1$

- Only want to add any energy to the signal through $E_x$

- The transmit energy is in Joules (energy per symbol) $\int_{-\infty}^{\infty} P_x(f) df = E_x$

- The transmit power is in Watts $E_x/T$
Bandwidth, energy, and power for linear complex pulse amplitude modulation

Learning objectives:
- Calculate bandwidth, energy, and power of a general complex pulse amplitude modulated signal
Additive white Gaussian noise (AWGN) channel (I)

- Thermal noise is present in any wireless communication system
- AWGN is a good model for the impairments due to thermal noise

\[ y(t) = x(t) + v(t) \]

- Other impairments like interference and RF distortion are often modeled as additional Gaussian noise
Additive white Gaussian noise (AWGN) channel (2)

- Noise is additive (multiplicative Gaussian shows up with fading)
- \( v(t) \) is an IID complex Gaussian random variable
  - \( v(t) \sim \mathcal{N}_c(0, \sigma^2) \)
  - \( \text{Re}\{v(t)\} \sim \mathcal{N}(0, \frac{\sigma^2}{2}) \)
  - \( \text{Im}\{v(t)\} \sim \mathcal{N}(0, \frac{\sigma^2}{2}) \)
  - \( \mathbb{E}[\text{Re}\{v(t)\} \cdot \text{Im}\{v(t)\}] = 0 \)

\[
f_v(x) = \frac{1}{\pi \sigma^2} e^{\frac{|x-m|^2}{\sigma^2}}
\]
Additive white Gaussian noise (AWGN) channel (3)

- The noise variance is $\sigma_v^2 = N_o = kT_e$
  - $k$ is Boltzmann’s constant and $k = 1.38 \times 10^{-23} \text{J/K}$
  - the effective noise temperature of the device is $T_e$ in Kelvins.
  - Assume $T_e = 290K$ in the absence of other information*

- The noise is White because it is IID

$$R_{vv}(\tau) = \sigma^2 \delta(\tau) \quad \iff \quad P_v(f) = \sigma^2$$

*You would normally compute the noise figure of the device which depends on the RF components and modifies the effective temperature.
Although AWGN is IID in theory, noise PSD in a communication system is limited by the filters in the analog front-end.

Only care about the contribution of the noise in the signal of interest (further filtering as will be clear later).

Noise is an important impairment in wireless communications.
Signal-to-noise ratio (SNR)

- Main performance measure

\[ \text{SNR} = \frac{\text{Signal power}}{\text{Noise power}} \]

The importance of SNR comes from detection theory; under other assumptions a different conclusion would result.
Additive white Gaussian noise (AWGN) channel

Learning objectives:
- Define AWGN including the distribution and properties
Break
News on wireless

- http://www.wirelessweek.com/
- http://www.fiercewireless.com
- http://www.rcrwireless.com
Back to the main event
Overview of AWGN transmitter and receiver

Learning objectives:
- Describe key components of the transmitter and receiver
Overview of AWGN transmitter and receiver

**Tx** Bits → **Symbol Mapping** → Pulse train \( T \sum_{n=-\infty}^{\infty} s[n] \delta(t - nT) \) → **TX Pulse Shaping Filter** \( g_{tx}(t) \) → **Gain** \( \sqrt{E_x} \)

**Rx** \( y(t) \) → **RX Pulse Shaping Filter** \( g_{rx}(t) \) → **C/D** \( y[n] \) → **Detection** → **Inverse Symbol Mapping** → **Detected Bits**

\[ 1/T \text{ symbol rate} \]

**Problem:** how to design \( g_{tx}(t) \) and \( g_{rx}(t) \) to maximize \( SINR \)

\[ SINR = \frac{S}{I+N} \]

- \( S \): signal power
- \( I \): intersymbol interference
- \( N \): noise power
Overview of AWGN transmitter and receiver

Learning objectives:
- Describe key components of the transmitter and receiver
Pulse shape design criterion

Learning objectives:

- Determine properties of the optimal receiver pulse-shape for the AWGN channel
Pulse shape design

- The $g_{rx}(t)$ that maximizes SNR is optimal
- Consider the received signal prior to sampling

$$y(t) = \sqrt{E_x}g_{rx}(t) * g_{tx}(t) * \sum_{m} s[m]\delta(t - mT) + g_{rx}(t) * v(t)$$

- Let $g(t) = g_{rx}(t) * g_{tx}(t)$
- After sampling

$$y[n] = \sqrt{E_x} \sum_{m} s[m]g((n - m)T) + \tilde{v}[n]$$

$$= \sqrt{E_x}s[n] + \sqrt{E_x} \sum_{m \neq n} s[m]g((n - m)T) + \tilde{v}[n]$$

$m=n$, desired component

$m \neq n$, intersymbol interference
Signal-to-interference-plus-noise ratio

- The quality of the receiver can be measured by the signal-to-interference-plus-noise ratio (SINR)
  - Signal energy
    \[ E \left| \sqrt{E_x} s[n] g(0) \right|^2 = E_x |g(0)|^2 \]
  - Noise energy
    \[ E \left| g_{rx}(t) \ast v(t) \right|_{nT}^2 = N_o \int |G_{rx}(f)|^2 df \]
  - Intersymbol interference interference energy
    \[ E \left| \sqrt{E_x} \sum_{m,m \neq n} s[m] g((n - m)T) \right|^2 = E_x \sum_{m \neq 0} |g(mT)|^2 \]
  - \[ SINR = \frac{E_x |g(0)|^2}{N_o \int |G_{rx}(f)|^2 df + E_x \sum_{m \neq n} |g(mT)|^2} \]

A good receiver maximizes the SINR.
SINR maximizing design (1)

\[
SINR = \frac{E_x |g(0)|^2}{N_0 \int |G_{tx}(f)|^2 df + E_x \sum_{m \neq n} |g(mT)|^2}
\]

- Use series of inequalities to find the SINR maximizing design

- First recall the Cauchy Schwartz inequality for functions

\[
\left| \int_{-\infty}^{\infty} a(t)b(t)dt \right|^2 \leq \int |a(t)|^2 dt \int |b(t)|^2 dt
\]

\[
b(t) = \alpha a(t) \quad \text{Functions need to be “parallel”}
\]

Complex scalar
SINR maximizing design (2)

- Using the Cauchy-Schwarz inequality

\[ |g(0)|^2 = \left| \int g_{rx}(-t)g_{tx}(t)dt \right|^2 \]

\[ \leq \int |g_{tx}(t)|^2 dt \int |g_{rx}(t)|^2 dt \]

\[ = \int |g_{rx}(t)|^2 dt \]

\[ = \int |G_{rx}(f)|^2 df \]

- Equality holds when \( g_{rx}(t) = \alpha g_{tx}^*(-t) \)

\textbf{Matched filter} \quad g_{rx}(t) = g_{tx}^*(-t) \quad \text{(take } \alpha = 1 \text{ for convenience)}
SINR maximizing design (3)

\[
SINR = \frac{E_x |g(0)|^2}{N_0 \int |G_{rx}(f)|^2 df + E_x \sum_{m \neq n} |g(mT)|^2} \leq \frac{E_x |g(0)|^2}{N_0 \int |G_{rx}(f)|^2 df}
\]

Positive, smallest value is zero

- Zero ISI condition

\[E_x \sum_{m \neq 0} |g(mT)|^2 = 0 \quad \text{with our normalization of } g_{tx}(t) \quad \Rightarrow \quad c = g(0) = \int |g_{rx}(t)|^2 dt = 1\]

- Such pulses are special, but they do exist: Nyquist pulse shapes
Pulse shape design criterion

Learning objectives:
- Determine properties of the optimal receiver pulse-shape for the AWGN channel
Nyquist Pulse Shapes

Learning objectives:

- Define Nyquist pulse criterion and determine if a given pulse shape satisfies that criterion.
- Describe sinc, raised cosine, and square root raised cosine filters
Nyquist pulse-shaping condition

- **Zero ISI condition** \( g(nT) = \delta[n] \) \( \Rightarrow g_d[n] \leftrightarrow G_d(e^{j2\pi f}) \)

- **From sampling theory** *
  \[
  g_d(e^{j2\pi f}) = \frac{1}{T} \sum_k G \left( \frac{f}{T} + \frac{k}{T} \right) = 1
  \]

  \[
  \sum_k G \left( f + \frac{k}{T} \right) = T
  \]
  *(putting \( f \) in Hertz here)*

*note may be some typos in the book eqs 4.63-4.65 in this section 😞
Classic pulse shape – the sinc

- Example: sinc function $g_{\text{sinc}}(t) = \text{sinc}(t/T)$

![Graph showing sinc function]

With the sinc function, the baseband bandwidth of the pulse shape is $1/2T$ and there is no overlap in the terms in $G(f)$ in (4.65). Other choice of $g_p(t)$ will have larger bandwidth and the aliases will add up so that equality is maintained in (4.65).

The sinc pulse shaping filter has a number of problems in practice. Ideal implementations of $g_p(t)$ do not exist in analog in practice. Digital implementations require truncating the pulse shape, which is a problem since it decays in time with $1/t^2$, requiring a lot of memory. Further the sinc function is sensitive to sampling errors (not sampling at exactly the right point). For these reasons it is of interest to consider pulse shapes that have excess bandwidth, which means the baseband bandwidth of $g_p(t)$ is greater than $1/2T$, or equivalently the passband bandwidth is greater than $1/T$.
What is practically wrong with the sinc?

- The sinc pulse shaping filter has a number of problems in practice
  - Must be truncated in practice
  - Is non-causal, but can be made causal through sufficient delay
  - Difficult to approximate steep band edges
  - Wide main pulse implies lots of sensitivity to timing errors
  - Slow decay $\sim 1/t$
BPSK with sinc pulse shaping

\[ \sum_k s[k] \text{sinc}((t - kT)/T) \]

- BPSK signal using a sinc pulse shaping filter
  - Pulse has been truncated to 20 symbol periods
  - Zero ISI when sampled at the right points
Raised cosine pulse-shape (1)

\[ G_{rc}(f) = \begin{cases} 
\frac{T}{2} \left[ 1 + \cos \frac{\pi T}{\alpha} \left( |f| - \frac{1-\alpha}{2T} \right) \right], & 0 \leq |f| \leq \frac{(1 - \alpha)/2T}{2T} \\
0, & \frac{1-\alpha}{2T} \leq |f| < \frac{1+\alpha}{2T} 
\end{cases} \]

\[ g_{rc} = \frac{\sin \frac{\pi t}{T}}{\pi t/T} \cos(\pi \alpha t) \frac{\cos(\pi \alpha t)}{1 - 4\alpha^2 t^2/T^2} \]
Raised cosine pulse-shape (2)

\( \alpha \) is the rolloff factor
(often expressed as percentage excess bandwidth)

\( (1 + \alpha)/2T \) is the absolute bandwidth of the raised cosine pulse
\( (1 + \alpha)/2T \) is the Nyquist rate of \( x(t) \) with a raised cosine pulse shape

\( 1/T \) is the rate at which \( x(t) \) is sampled

\( \alpha = 0 \) is the sinc pulse

Width of main lobe controlled by \( \alpha \)

Decay at best \( \sim t^3 \)
Square-root raised cosine

- Recall $G(f) = G_{tx}(f)G_{rx}(f)$ and for matched filter $G_{tx}(f) = G_{tx}^*(-f)$
- Use the square-root raised cosine in practice for $g_{tx}(t)$ and $g_{rx}(t)$

$$g_{\text{sqrc}}(t) = \begin{cases} \frac{1}{\sqrt{T}} \left[ 1 + \alpha \left( \frac{4}{\pi} - 1 \right) \right], & t = 0, \\ \frac{\alpha}{\sqrt{2T}} \left[ (1 + \frac{2}{\pi}) \sin \left( \frac{\pi}{4\alpha} \right) + (1 - \frac{2}{\pi}) \cos \left( \frac{\pi}{4\alpha} \right) \right], & t = \pm \frac{T}{4\alpha}, \\ \frac{4\alpha}{\pi \sqrt{T}} \frac{\cos((1+\alpha)\pi t/T) + \sin((1-\alpha)\pi t/T)}{1-(4\alpha t/T)^2}, & \text{otherwise} \end{cases}$$
Noise with square root raised cosine

Sample noise because pulse shape is bandlimited

\[ C_{vv}[k] = \mathbb{E}_v[\hat{v}[n]\hat{v}^*[n+k]] \]

\[ = \mathbb{E}_v \left[ \int_{\tau_1} \int_{\tau_2} v(\tau_1)v^*(\tau_2)g_{\text{sqrc}}(nT-\tau_1)g_{\text{sqrc}}((n-k)T-\tau_2)\,d\tau_1\,d\tau_2 \right] \]

\[ = \sigma_v^2 \int_{\tau} g_{\text{sqrc}}(nT_s-\tau)g_{\text{sqrc}}((n-k)T-\tau)\,d\tau \]

\[ = \sigma_v^2 \int_{\tau} g_{\text{sqrc}}(\tau)g_{\text{sqrc}}(-kT-\tau)\,d\tau \]

But note that

\[ g_{\text{sqrc}}(t) = g_{\text{sqrc}}(-t). \]

Therefore

\[ C_{vv}[k] = \sigma_v^2 \int_{\tau} g_{\text{sqrc}}(\tau)g_{\text{sqrc}}(kT+\tau)\,d\tau. \]

\[ g_{\text{rc}}(t) = \int_{\tau} g_{\text{sqrc}}(t-\tau)g_{\text{sqrc}}(\tau)\,d\tau \]

\[ = \int_{\tau} g_{\text{sqrc}}(t+\tau)g_{\text{sqrc}}(\tau)\,d\tau \]

\[ C_{vv}[k] = \sigma_v^2 g_{\text{rc}}(kT) \]

\[ = \sigma_v^2 \delta[k] \]
Nyquist Pulse Shapes

Learning objectives:

- Define Nyquist pulse criterion and determine if a given pulse shape satisfies that criterion
- Describe sinc, raised cosine, and square root raised cosine filters