Equivalence of LP Relaxation and Max-Product for Weighted Matching in General Graphs

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***Abstract*— Max-product belief propagation is a local, iterative algorithm to find the mode/MAP estimate of a probability distribution. While it has been successfully employed in a wide variety of applications, there are relatively few theoretical guarantees of convergence and correctness for general loopy graphs that may have many short cycles. Of these, even fewer provide exact "necessary and sufficient" characterizations.**

In this paper we investigate the problem of using max-product to find the maximum weight matching in an arbitrary graph with edge weights. This is done by first constructing a probability distribution whose mode corresponds to the optimal matching, and then running max-product. Weighted matching can also be posed as an integer program, for which there is an LP relaxation. This relaxation is not always tight. In this paper we show that

1) **If the LP relaxation is tight, then max-product always converges, and that too to the correct answer.**
2) **If the LP relaxation is loose, then max-product does not converge.**

This provides an exact, data-dependent characterization of max-product performance, and a precise connection to LP relaxation, which is a well-studied optimization technique. Also, since LP relaxation is known to be tight for bipartite graphs, our results generalize other recent results on using max-product to find weighted matchings in bipartite graphs.

I. INTRODUCTION

Message-passing algorithms, like Belief Propagation and its variants and generalizations, have been shown empirically to be very effective in solving many instances of hard/computationally intensive problems in a wide range of fields. These algorithms were originally designed for exact inference (i.e. calculation of marginals/max-marginals) in tree-structured probability distributions. Their application to general graphs involves replicating their iterative local update rules on the general graph. In this case however, there are no guarantees of either convergence or correctness in general.

Understanding and characterizing the performance of message-passing algorithms in general graphs remains an active research area. [1, 2] show correctness for graphs with at most one cycle. [3, 4] show that for gaussian problems the sum-product algorithm finds the correct means upon convergence, but does not always find the correct variances. [5, 6] show asymptotic correctness for random graphs associated with decoding. [7] shows that if max-product converges, then it is optimal in a relatively large "local" neighborhood.

In this paper we consider the problem of using max-product to find the maximum weight matching in an arbitrary graph with arbitrary edge weights. This problem can be formulated as an integer program, which has a natural LP relaxation. In this paper we prove the following

1) If the LP relaxation is tight, then max-product always converges, and that too to the correct answer.
2) If the LP relaxation is loose, then max-product does not converge.

Bayati, Shah and Sharma [8] were the first to investigate max-product for the weighted matching problem. They showed that if the graph is bipartite then max-product always converges to the correct answer. Recently, this result has been extended to b-matchings on bipartite graphs [9]. Since the LP relaxation is always tight for bipartite graphs, the first part of our results recover their results and can be viewed as the correct generalization to arbitrary graphs, since in this case the tightness is a function of structure as well as weights.

We would like to point out three features of our work:

1) It provides a *necessary and sufficient* condition for convergnce of max-product in arbitrary problem instances. There are very few non-trivial classes of problems for which there is such a tight characterization of message-passing performance.
2) The characterization is *data dependent*: it is decided based not only on the graph structure but also on the weights of the particular instance.
3) Tightness of LP relaxations is well-studied for broad classes of problems, making this chracterization promising in terms of both understanding and development of new algorithms.

Relations, similarities and comparisons between max-product and linear programming have been used/mentioned by several authors [10–12], and an exact characterization of this relationship in general remains an interesting endeavor. In particular, it would be interesting to investigate the implications of these results as regards elucidating the relationship between iterative decoding of channel codes and LP decoding [13].

II. WEIGHTED MATCHING AND ITS LP RELAXATION

A *matching* in a graph is a set of edges such that no two edges in the set are incident on the same node. Given a graph $G = (V, E)$, with non-negative weights w_e on the edges $e \in E$, the *weighted matching problem* is to find the matching M^* whose edges have the highest total weight. In this paper we find it convenient to refer to edges both as $e \in E$ and as (i, j), where $i, j \in V$.

Weighted matching can be written as the following integer program (IP):

$$\max \quad \sum w_e x_e$$
$$s.t. \sum_{j \in \mathcal{N}(i)} x_{ij} \leq 1 \quad \text{for all } i \in V \qquad (1)$$
$$x_e \in \{0,1\} \quad \text{for all } e \in E$$

The LP relaxation of the above problem is to replace the constraint $x_e \in \{0,1\}$ with the constraint $x_e \geq 0$. *This relaxation is in general not tight*, i.e. there might exist non-integer solutions with strictly higher value than any integral solution. It is known however that the LP relaxation is *always* tight for bipartite graphs: no matter what the edge weights, the bipartite-ness ensures tightness of the LP relaxation. If a graph is not bipartite, the tightness of the LP relaxation will depend on the edge weights: the same graph may have tightness for one set of weights and looseness for another set.

The dual of the above linear program is the *vertex cover* problem: minimize the total of the weights z_i that need to be placed on nodes so as to "cover" the edge weights: (DP)

$$\min \quad \sum z_i$$
$$s.t. \quad w_{ij} \leq z_i + z_j \quad \text{for all } (i,j) \in E$$
$$z_i \geq 0 \quad \text{for all } i$$

Lemma 1 (complimentary slackness): When the LP relaxation is tight, the optimal matching M^* and the optimal dual variables z and satisfy the following properties:

1) if $(i,j) \in M^*$ then $w_{ij} = z_i + z_j$
2) if $(i,j) \notin M^*$ then $w_{ij} \leq z_i + z_j$
3) if no edge in M^* is incident on node i, then $z_i = 0$
4) $z_i \leq \max_e w_e$ for all i

III. Background on the Max-Product Algorithm

The *factor graph* [14] of a probability distribution represents the conditional independencies of the distribution. The Max-Product (MP) algorithm is a simple, local, iterative message passing algorithm that can be used (in an attempt) to find the mode/MAP estimate of a probability distribution. Nodes and factors pass messages to each other, and nodes maintain "beliefs", which represent the max-marginals. When max-product is applied to problems involving general "loopy" graphs, one of the following three scenarios may result:

1) The algorithm may not converge.
2) The algorithm may converge, but to an incorrect answer.
3) The algorithm may converge to the correct answer.

As has been mentioned, here has been siginifcant work attempting to understand the properties of MP for loopy graphs. For the results in this paper, we will use the following two insights:

1) At any time, the belief of the max-product algorithm for a given variable corresponds to the belief at the root of the corresponding *computation tree* distribution [2] associated with that variable at that time. We describe what this computation tree distribution corresponds to for the weighted matching problem in the next section.
2) If max-product *does* converge, the resulting beliefs are optimal in a large "local" neighborhood [7]: let $\widehat{x}$ be the assignment as given by the converged max-product and $\widetilde{x}$ be any other assignment. If the variables assigned different values in $\widehat{x}$ and $\widetilde{x}$ form an induced graph containing at most one cycle in each component, then $p(\widehat{x}) \geq p(\widetilde{x})$.

IV. Max-Product for Weighted Matching

The problem of finding M^* can be formulated as the problem of finding the mode of a suitably (artifically) constructed probability distribution p. In fact, there are in general *several* ways to construct this distribution for the *same* instance of a graph G. We now present one construction[1].

Associate a binary variable $x_e \in \{0,1\}$ with each edge $e \in E$, and let

$$p(x) \quad = \quad \frac{1}{Z} \prod_{i \in V} \mathbf{1}_{\{\sum_{j \in \mathcal{N}(i)} x_{ij} \leq 1\}} \prod_{e \in E} e^{w_e x_e} \qquad (2)$$

Here $\mathcal{N}(i)$ represents the neighborhood of node i in G, and Z is a normalizing constant. The variable x_e can be interpreted as follows: $x_e = 1$ indicates that $e \in M^*$, while $x_e = 0$ indicates $e \notin M^*$. The term $\mathbf{1}_{\{\sum_{j \in \mathcal{N}(i)} x_{ij} \leq 1\}}$ enforces the cosntraint that of the edges incident to node i, at most one can be assigned the value "1". Thus, it is easy to see that $p(x) > 0$ if and only if the edges with $x_e = 1$ constitute a matching in G. Furthermore, the mode of p corresponds to the max-weight matching M^*.

The factor graph max-product involves messages between variables and factors. In our case the variables are the edges $(i,j) \in E$, and the factors are nodes $i \in V$. Thus at any time t there will be messages $m^t_{i \to (i,j)}$ from node (factor) i to edge (variable) (i,j), as well as messages $m^t_{(i,j) \to i}$. Each message will be a length-two vector of real numbers, indexed by 0 and 1. The message update rules can be simplified to the following:

$$\begin{aligned}
m^{t+1}_{(i,j) \to i}[1] &= e^{w_{ij}} m^t_{j \to (i,j)}[1] \\
m^{t+1}_{(i,j) \to i}[0] &= m^t_{j \to (i,j)}[0] \\
m^{t+1}_{i \to (i,j)}[1] &= \prod_{k \in \mathcal{N}(i)-j} m^t_{(k,i) \to i}[0] \\
m^{t+1}_{i \to (i,j)}[0] &= \max\{ \prod_{k \in \mathcal{N}(i)-j} m^t_{(k,i) \to i}[0] \; , \\
&\qquad \max_{k \in \mathcal{N}(i)-j} m^t_{(k,i) \to i}[1] \}
\end{aligned}$$

Also, at every time each edge (variable) maintains a belief vector $b^t_{(i,j)}$ as follows:

$$\begin{aligned}
b^t_{(i,j)}[0] &= m^t_{i \to (i,j)}[0] \; \times \; m^t_{j \to (i,j)}[0] \\
b^t_{(i,j)}[1] &= e^{w_{ij}} m^t_{i \to (i,j)}[1] \; \times \; m^t_{j \to (i,j)}[1]
\end{aligned}$$

[1] This construction is different from the one in [8], which had a pairwise model with variables corresponding to nodes in the graph. However, the results of this paper continue to hold when the construciton in [8] is modified to be applicable to general graphs

The p defined above can be used to find M^* as follows: first run max-product. At any time t and for each edge e there will be two beliefs $b_e^t[0]$ and $b_e^t[1]$. If max-product converges, assign to each variable the value (i.e. "0" or "1") that corresponds to the stronger belief. Then, declare the set of all edges set to "1" to be the max-product output.

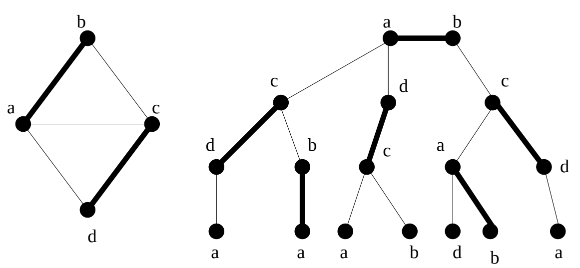
A. The Computation Tree for Weighted Matching

Our proofs rely on the computation tree interpretation [2, 15] of the Max-product beliefs. We now describe this interpretation when max-product is applied to p as given in (2).

For an edge e let $\overline{T}_e(k)$ be the *full depth-k computation tree rooted at e*. This is generated recursively: take $\overline{T}_e(k-1)$ and to each leaf v add as children a copy of each of the neighbors of v in G, except for the unique neighbor of v which is already present in $\overline{T}_e(k-1)$. Also, each new edge has the same weight as its copy in the original G. The recursion is started with the single-edge tree $\overline{T}_e(1) = e$, both of whose endpoints are leaves. This initial edge is the *root* of $\overline{T}_e$.

Consider now the "full synchronous" max-product, where at each time every message in the network is updated. In this case the computation tree $T_e(k)$ for edge e at time k will be $\overline{T}_e(k)$. Alternatively, max-product may be executed asynchronously with only a subset of the messages updated in every time slot. In this case $T_e(k)$ will be a sub-tree of $\overline{T}_e(k)$. In either case, the computation tree interpretation states at time k we have $b_e^k[1] > b_e^k[0]$ if and only if the root of $T_e(k)$ is a member of a max-weight matching on the tree $T_e(k)$.

The figure below shows an example where on the left is G: the four-cycle $abcd$ and the chord ac, with a matching $M = \{(a,b),(c,d)\}$ depicted in bold. On the right is the computation tree $\overline{T}_{(a,b)}(4)$ which is the full tree of depth 4 rooted at edge (a,b). The bold edges depict the projection M_T of M onto $\overline{T}_{(a,b)}(4)$: an edge e in the tree is in M_T if and only if its copy in G is in M.



Lemma 2: Let M be a matching in G and $T_e(k)$ be a computation tree. Let M_T be the set of all copies in $T_e(k)$ of all edges in M. Then, M_T is a matching in $T_e(k)$. Also, if M is maximal in G, M_T is maximal in T_e.

Of course T_e will also contain other matchings that are not projections of matchings in G. Finally, we say that a (possibly not full) tree $T_e(k)$ is *full upto depth* k_1 if the full tree $\overline{T}_e(k_1)$ is contained in $T_e(k)$.

V. Equivalence of Max-Product and LP Relaxation

We are now ready to prove the main result of this paper: the equivalence of Max-Product and LP Relaxation. Before we proceed, we define the following terms

1) We say that the *LP relaxation is tight* if the linear program (LP) obtained by relaxing the integer program (1) has a unique optimal solution at which all values x_e are either 0 or 1.
2) We say that *max-product converges by step k* if the variable assignments (0 or 1) that maximize the beliefs at each node remain constant once the associated computation tree is full up to depth at least k. Note that this includes both synchronous and asynchronous message updates. We say that *max-product converges* if there exists some $k < \infty$ such that max-product converges by step k. Finally, we say that *max product converges to the correct answer* if the beliefs b_e at convergence are such that $b_e[1] > b_e[0]$ if and only if $e \in M^*$, and $b_e[1] < b_e[0]$ if and only if $e \notin M^*$

We also need to make some uniqueness assumptions. It is well-recognized that max-product may perform poorly in the presence of multiple optima, and that characterizing performance in this case is hard. For the rest of this paper we will assume the following:

- **A1** M^* is the unique optimal matching.
- **A2** The linear program always has a unique optimal solution. Note that this can be fractional, but it has to be unique.

A. Max-product is as Powerful as LP Relaxation

In this section we prove that if the LP relaxation is tight then Max-Product converges to the correct answer. Recall that when the LP is tight, part 2 of Lemma 1 says that if $(i,j) \notin M^*$ then $w_{ij} \le z_i + z_j$. The uniqueness assumptions **A1-2** further imply that the inequality is strict: $w_{ij} < z_i + z_j$. Another way of saying this is that there exists an $\epsilon > 0$ such that

$$w_{ij} \le z_i + z_j - \epsilon \quad \text{for all } (i,j) \notin M^* \tag{3}$$

Theorem 1: Consider a weighted graph G for which the LP relaxation is tight. Then max-product converges to the correct answer by step $\frac{2w_{max}}{\epsilon}$, where $w_{max} = \max_e w_e$ is the weight of the heaviest edge, and ϵ satisfies (3).

Proof:

Let M^* be the optimal matching on G. For max-product to be convergent and correct, we need that $b_e^t[1] > b_e^t[0]$ for all $e \in M^*$ and $b_e^t[1] < b_e^t[0]$ for all $e \notin M^*$, and for all t such that $T_e(t)$ is full upto depth $\frac{2w_{max}}{\epsilon}$.

So suppose that for such a t there exists an $e \notin M^*$ such that $b_e^t[1] > b_e^t[0]$. Then, there exists a matching M in $T_e(t)$ such that *(a)* the root $e \in M$, and *(b)* M has the largest weight among matchings on $T_e(t)$. Let M_T^* be the set of all edges in $T_e(t)$ that are copies of edges in M^*. By lemma 2, M_T^* is a maximal matching on $T_e(t)$. Also, the root $e \notin M_T^*$ by assumption.

The symmetric difference $M_T^* \triangle M$ consists of disjoint alternating paths in $T_e(t)$: each path will have every alternate edge in M_T^* and all other edges in M. Let P be the path that contains the root e. We now show that $w(P \cap M_T^*) > w(P \cap M)$.

Recall that the optimal dual solution assigns to each node i in G a "dual value" $z_i \geq 0$. Associate now with each node in $T_e(t)$ the dual value of its copy in G. Then, by Lemma 1 we have that $w_{ij} = z_i + z_j$ for each $(i,j) \in P \cap M_T^*$. Suppose now that neither endpoint of P is a leaf of $T_e(t)$. In this case, we have

$$w(P \cap M_T^*) \;=\; \sum_{i \in P} z_i$$

On the other hand, we know that (3) holds for each edge in $P \cap M$. Adding these up gives

$$w(P \cap M) \;\leq\; \sum_{i \in P} z_i \;-\epsilon |P \cap M|$$

By assumption, the root $e \in P \cap M$, so $|P \cap M| \geq 1$ and hence $w(P \cap M_T^*) > w(P \cap M)$ when no endpoints of P are leaves.

Suppose now that exactly one of the endpoints v of P is a leaf of $T_e(t)$. In this case, we have that

$$w(P \cap M_T^*) \;\;\geq\;\; \sum_{i \in P} z_i \;-\; z_v \;\geq\; \sum_{i \in P} z_i \;-\; w_{max}$$

where the last inequality follows from part 4 of Lemma 1 Also, $T_e(t)$ is assumed to be full up to depth k, so this implies that $|P \cap M| \geq \frac{k}{2}$. This means that

$$w(P \cap M) \;\leq\; \sum_{i \in P} z_i \;-\epsilon \frac{k}{2}$$

Now, since $k \geq \frac{2w_{max}}{\epsilon}$, this implies that $w(P \cap M_T^*) > w(P \cap M)$. The final case, where both endpoints of P are leaves, works out in the same way, except that now $|P \cap M| \geq k$ and $w(P \cap M_T^*) \geq \sum_{i \in P} z_i \;-\; 2w_{max}$.

Thus, in any case, we have that $w(P \cap M_T^*) > w(P \cap M)$. Consider now the set of edges $M - (P \cap M) + (P \cap M_T^*)$. This set forms a matching on $T_e(t)$, and has higher weight than M. This contradicts the choice of M, and so establishes that $b_e^t[1] < b_e^t[0]$ for all $e \notin M^*$. A similar contradiction argument can be used to establish that $b_e^t[1] > b_e^t[0]$ for all $e \in M^*$. This completes the proof. ∎

B. LP Relaxation is as Powerful as Max-product

In this section we prove that if the LP relaxation is loose then max-product does not converge to the correct answer. Before we do so however, we note that this implies a stronger result: that when LP is loose then in fact max-product does not converge at all.

Lemma 3: Consider the distribution $p(x)$ as given in (2). If Max-Product converges, then its output exactly corresponds to the true optimal matching M^*.

The proof of this lemma uses the "local optimality" result of Weiss and Freeman [7]. In particular, for p it turns out that local optimality implies global optimality. This means that it is not possible for max-product to converge to an incorrect answer: it will either not converge at all, or converge to M^*. We do not use this explicitly in the proofs below, but it strengthens the results as mentioned above.

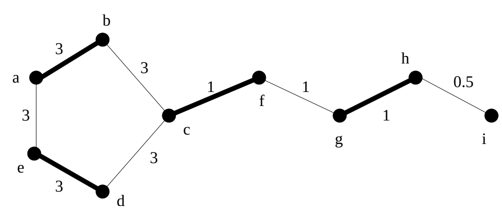
We now proceed with showing that max-product does not converge to the correct M^* when LP is loose. As a first step, we need a combinatorial characterization of when the LP relaxation is loose. We now make some definitions. We say that a node v is *saturated* by a matching M if there exists an edge $e \in M$ that is incident to v.

A *blossom* with respect to a matching M is an odd cycle C with $\frac{|C|-1}{2}$ edges in M.[2] Note that a blossom has a unique *base*: a node not saturated by any edge in $C \cap M$. A *stemmed blossom* B_1 (w.r.t M) is a blossom C, along with an alternating path (stem) P that starts at the base of C, and starts with an edge in M. Also, P should be such that the set $M - (P \cap M) + (P - M)$ remains a matching in G.

A *bad stemmed blossom* is one in which the edge weights satisfy

$$w(C \cap M) + 2w(P \cap M) \;<\; w(C - M) + 2w(P - M)$$

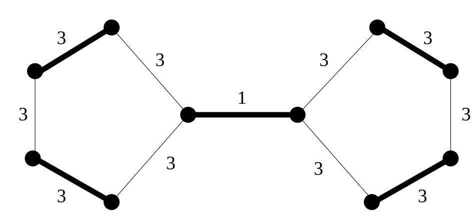
Note that it may well be the case that $|P| = 0$, in which case B_1 is just an odd cycle. The following is an example of a bad stemmed blossom. The bold edges are the ones in M, the numbers denote the weights of the corresponding edges, and the last node i has no edge of M incident on it. The blossom C in this case is the cycle $abcde$, and node c is its base. The path/stem P is $cfghi$.



A *blossom pair* B_2 is two blossoms C_1 and C_2 and an alternating path P between the bases of the two blossoms such that P begins and ends with edges in M. A *bad blossom pair* is one in which the edge weights satisfy

$$\begin{aligned} &w(C_1 \cap M) + w(C_2 \cap M) + 2w(P \cap M) \\ &\quad < \; w(C_1 - M) + w(C_2 - M) + 2w(P - M) \end{aligned}$$

The following is an example of a bad blossom pair.



[2] Blossoms were first defined in [16], which also provided the first efficient algorithm for weighted matching in arbitrary graphs.

The following proposition provides a combinatorial characterization of when the LP relaxation is loose, and is crucial to the proof of the subsequent theorem.

Proposition 1: If the LP relaxation is loose, then there exists a bad stemmed blossom, or a bad blossom pair, with respect to the optimal matching M^*.

Proof: In appendix.

We use the presence of these "bad" subgraphs in G to show that max-product does not converge to the correct answer. Before we do so, we need one additional lemma. This states that if max-product converges by step k to some matching M on G, then the optimal matching M_T on the computation tree looks like M in the neighborhood of the root.

Lemma 4: Suppose max-product converges to a matching M in G by step k. Consider any edge e, some $m \geq 1$ and a corresponding computation tree T_e which is full up to depth $k+m$. Let M_T be the max-weight matching on the tree. Then, for any edge $f \in T_e$ that is within distance m of the root e, $f \in M_T$ if and only if its copy f_1 in G is such that $f_1 \in M$.

Note that the above lemma also applies to the root e of the tree. We are now ready to state and prove the main result of this section. Recall that the belief b_e on an edge at convergence is incorrect if either $e \in M^*$ but $b_e[0] > b_e[1]$, or $e \notin M^*$ but $b_e[1] > b_e[0]$.

Theorem 2: Consider a weighted graph G for which the LP relaxation is loose. Then, the max-product beliefs do not converge to the correct M^*: for any given k, there exists a $k_1 \geq k$ and computation trees $T_e, e \in E$ such that each T_e is full upto depth k_1, but the beliefs on some of the edges are incorrect. Lemma 3 further implies that in fact in this case max-product does not converge at all.

Proof:

Let M^* be the max-weight matching on G. Since the LP relaxation is loose, by Prop. 1, there exists either a bad stemmed blossom or a bad blossom pair w.r.t. M^*. Suppose first that it contains a bad stemmed blossom B_1, and consider some $e \in C \cap M^*$ that is in the "blossom" part of B_1 (as opposed to the stem) and also in M^*. From the two nodes of e, make maximal alternating paths P_1 and P_2 that remain in B_1 and start out in opposite directions on C. For the stemmed blossom example above, if e is the edge (a, b) then the two paths will be $bcfghi$ and $aedcfghi$.

Let $d_1 = w(P_1 - M^*) - w(P_1 \cap M^*)$, and similarly d_2 for P_2. d_1 represents the change in the weight of the matching if each edge in P_1 were "switched", i.e. their membership in the matching was reversed from its original value. It is easy to see that

$$\begin{aligned} d_1 + d_2 - w(e) &= w(C - M^*) + 2w(P - M^*) \\ &\quad -w(C \cap M^*) - 2w(P \cap M^*) \end{aligned}$$

By assumption B_1 is a bad blossom and hence we have that $d_1 + d_2 - w(e) > 0$.

Suppose max-product converges to M^* by step k. Consider now the computation tree T_e which is full upto depth $k + |V|$, where $|V|$ is the number of nodes in G. Let M_T be the max-weight matching on T_e. Lemma 4 implies that M_T will be a projection of M^* in a distance-$|V|$ neighborhood of the root. Also, starting from the root e, each of P_1 and P_2 will have a unique copy, say R_1 and R_2 respectively, in T_e, with $|R_1|, |R_2| < |V|$. Since P_1 and P_2 are alternating w.r.t. M^*, it follows that R_1 and R_2 will be alternating with respect to M_T. Also, the set $S = R_1 \cup e \cup R_2$ forms an alternating path on T_e with respect to M_T, and this begins and ends in nodes unsaturated by M_T. Thus, M_T can be augmented by this path: the set $M_T - (S \cap M_T) + (S - M_T)$ will be a matching on T_e.

Also, the weight gain from doing this augmentation will be exactly $d_1 + d_2 - w(e)$, which we know is strictly positive. Thus, this shows that M_T is not the optimal matching on T_e, which contradicts the choice of M_T. This means that our assumption about max-product convergence to M^* is incorrect.

Thus, we see that if there exists a bad stemmed blossom w.r.t. M^* in G then max-product does not converge to M^*. A similar argument holds for the case of a bad blossom pair B_2, except that instead of paths P_1 and P_2 above we now have to look at alternating walks W_1 and W_2 that live in B_2 and are long enough. These walks can then be mapped to an augmenting path on T_e which strictly improves M_T, leading to a contradiction as was seen in the case of the paths P_1 and P_2. This completes the proof. ■

VI. Discussion

The results of this paper can be generalized to the case of perfect matchings, b-matchings and perfect b-matchings in general graphs, where similar results hold. In this paper max-product is shown to be as powerful as LP relaxation, but it would be more interesting to outline a direct *operational* link between max-product and a linear programming algorithm. As an example, [8] shows that for bipartite matching max-product has an operational correspondance with the auction algorithm [17]. Also, the form of the message update equations suggests that it can be implemented via an equivalent message passing update rule between just the nodes of the graph G, instead of having messages go from nodes to edges and vice versa.

More generally, it would be interesting to see if the ideas presented in this paper could be used/genealized to show connections between linear programming and belief propagation in other applications.

Acknowledgements

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Appendix

Proof of Proposition 1

We now show that if the LP relaxation is loose then there exists in the graph either a bad stemmed blossom or a bad blossom pair, with respect to the optimal matching M^*. Let x be the optimal (fractional) solution to the LP relaxation.

Let E' be the set of all edges e such that either *(a)* $e \in M^*$, or *(b)* $e \notin M^*$ and $x_e > 0$. Then, E' will contain at least one edge $e \notin M^*$, because if all $e \notin M^*$ had $x_e = 0$ then the LP would be tight. Let $G' = (V, E')$ be the subgraph of G having only the edges in E'. An *cycle augmentation* is any even cycle in which every alternate edge is in M^*. A *path augmentation* is any path in which every alternate edge is in M^*, and which begins and ends in nodes unsaturated by M^*. For any augmentation A, we have that $M^* - (A \cap M^*) + (A - M^*)$ is also a matching in G'. Thus, if M^* is the unique max-weight matching it has to be that $w(A \cap M^*) > w(A - M^*)$.

Lemma 5: G' cannot contain any augmentations: cycles or paths.

Proof: Let A be an augmentation in G'. By assumption, $x_e > 0$ for all $e \in A - M^*$, which implies that $x_e < 1$ for all $e \in A \cap M^*$. Thus, there exists some $\epsilon > 0$ such that decreasing each $x_e, e \in A - M^*$ by ϵ and increasing each $x_e, e \in A \cap M^*$ by ϵ represents a valid new feasible point for the LP. The weight of this new point exceeds the weight of x by $\epsilon(w(A \cap M^*) - w(A - M^*)) > 0$. However this contradicts the optimality of x, and thus G' cannot contain any augmentation. ∎

Let S be the longest alternating sequence of edges in G', and let v_1 and v_2 be its endpoints. By the lemma above, both cannot be unsaturated. We say that v_1 or v_2 is a *saturated leaf* if it is saturated by M^* and there exist no edges in $G' - M^*$ incident on it. Also, note that an endpoint is saturated if and only if its corresponding edge in S is also in M^*.

The fact that S is the longest sequence means that it cannot be extended further beyond v_1 and v_2. This implies that one of the following cases must occur:

1) Both v_1 and v_2 are both saturated leaves
   In this case, the constraints at v_1 and v_2 are loose. So, there exists an ϵ such that if all $x_e, e \in S - M^*$ are decreased by ϵ and all $x_e, e \in S \cap M^*$ are increased by ϵ then the new solution remains feasible. This new solution will have strictly higher weight than x, which is a contradiction. Thus this case cannot occur.
2) v_1 is a saturated leaf and v_2 is unsaturated.
   An ϵ-perturbation argument like the one above can be used to show that this case too cannot occur.
3) v_1 is saturated by M^*. but is not a leaf. v_2 is either unsaturated, or a saturated leaf.
   Since S cannot be extended, it has to be that all edges in $G' - M^*$ incident to v_1 have other endpoints in S. Let e be one such edge. Then, $e \cap S$ forms a stemmed blossom: the resulting cycle has to be odd, and the remaining part of S will be a stem whose endpoint is v_2. Note that in this case it has to be that the constraint at v_2 is loose.
4) Both v_1 and v_2 are saturated by M^*, but are not leaves. Applying the above blossom argument to both v_1 and v_2 yields the existence of a blossom pair.

Thus if the LP relaxation is loose then there exists a stemmed blossom or a blossom pair. Now all that is remaining to show is that they are "bad". Let B_1 be a stemmed blossom in G', consisting of blossom C and stem P. Then, there exists some $\epsilon > 0$ such that if $x_e, e \in C \cap M^*$ is increased by ϵ, $x_e, e \in C - M^*$ is decreased by ϵ, $x_e, e \in P \cap M^*$ is increased by 2ϵ, and $x_e, e \in C - M^*$ is decreased by 2ϵ, then the new solution remains feasible for the LP. Also, the new solution weighs

$$\epsilon \left[w(C \cap M^*) + 2w(P \cap M^*) - w(C - M^*) - 2w(P - M^*) \right]$$

more than x. For x to be the unique optimal of the LP, this has to be strictly negative and thus any stemmed blossom B_1 is bad. A similar argument shows that any blossom pair is bad. This finishes the proof of the proposition. ∎

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