On Network Coding for Interference Networks

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***Abstract*— We consider a finite-field model for the wireless broadcast and additive interference network (WBAIN), both in the presence and absence of fading. We show that the single-source unicast capacity (with extension to multicast) of a WBAIN with or without fading can be upper bounded by the capacity of an equivalent broadcast erasure network. We further present a coding strategy for WBAINs with i.i.d. and uniform fading based on random linear coding at each node that achieves a rate differing from the upper bound by no more than $O(1/q)$, where q is the field size. Using these results, we show that channel fading in conjunction with network coding can lead to large gains in the unicast (multicast) capacity as compared to no fading.**

I. Introduction

While network coding for broadcast networks has been the subject of much recent study [1], [2], [3], there remains a need to capture the interference nature of the wireless channel. In this work, we examine how network coding improves the throughput in a channel model that operates over a finite field but which incorporates both the interference and broadcast aspects of the wireless channel.

We consider a heterogeneous network composed of nodes that are connected by links that are either wireline or wireless. Unlike packets (symbols) traversing the wireline links, symbols that are transmitted by a wireless node are subject to the broadcast constraint that all links carry the same symbol. Further, if two or more wireless nodes transmit symbols to a particular wireless receiver node, the symbols being sent over the air are subject to both channel fading and additive interference, where all channel and network operations are assumed to occur over an appropriate finite field.

The finite-field channel model without fading has been used in [5], [6] to investigate the appropriateness of source-channel separation in various networks. Popular multiple access interference models include the collision multiple-access channel (MAC) such as Aloha [8], [9], where, when two or more transmit messages collide none of them get through, and Aloha with multi-packet reception [10], where the receiver can successfully decode one or more packets in a slot with some probability distribution. In this work, we consider an intermediate stance where the interference of two signals in a MAC, both of which are elements of a particular finite field, is modelled as *the sum of the signals* in the same finite field. Symbol loss due to noise is modelled by allowing random complete erasure of the received signal. Practical implications of such finite-field additive interference (for non-fading channels) are discussed in [6].

A. Main Contributions

(i) We consider a heterogeneous (wireline/wireless) network (directed graph) comprising of finite-field uniformly and independently distributed fading channels subject to broadcast and interference constraints, as well as random symbol erasure. We consider a single-source unicast, and discuss extensions to a multicast network. We derive an achievable rate using a random linear coding (RLC) strategy at each of the nodes, as well as an upper bound on the network capacity. We show that the bounds are tight asymptotically in the field size (i.e., the difference between the upper bound and the achievable rate scales as $O(1/q)$, where q is the field size).

(ii) We present example networks for which we explicitly compute the achievable rate with channel fading, as well as a tighter upper bound *without* fading. We show that the capacity with channel fading can be considerably larger (depending on the network topology) than that of the identical network graph but *without* fading.

We finally comment that the aim of employing the aforementioned finite-field model is to take a step in determining the capacity region of wireless networks operating over a general Gaussian channel with fading. The latter, as is well known, is a very challenging problem – for even simple network configurations, such as the single-relay channel or the interference channel, the capacity regions are not yet known. Hence, we consider a finite-field approximation of the general model, whose limit (under an appropriate distribution remapping) as the field size grows is the fading Gaussian channel. Even this simplification is not enough, as the capacity of a network of binary symmetric channels is not known, which is a special case of the finite-field approximation. Hence, we consider a further simplification of the model: instead of the additive noise term, we allow random complete erasure of the received signal. For this case, we are indeed able to obtain asymptotically tight bounds on the unicast (multicast) capacity. We expect that the insights obtained from the simplified model will aid in the understanding of the more general model.

II. System Model and Notation: WBAIN

We model a wireless network as a directed graph $G = (V, E)$, where V, $|V| = N$, is the set of all nodes in the network, and for each $v_i, v_j \in V$ such that node v_i can transmit to v_j, there is a directed edge (link) $(v_i, v_j) \in E$. In this work,

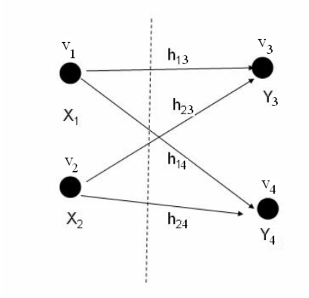

Fig. 1. Model of a wireless channel with broadcast and interference constraints in the presence of fading coefficients $h_{ij} \in \mathbb{F}_q$. Node $v_i, i = 1, 2$, is constrained to send the same codeword (chosen from $\mathbb{F}_q$) on its outgoing links. Receiver $v_j, j = 3, 4$ decodes the symbol $Y_j = h_{1j}X_1 + h_{2j}X_2$ with probability $1 - \epsilon_j$ and erasure symbol $\mathcal{E}$ with probability ϵ_j.

we restrict ourselves to directed acyclic graphs. Let $v_s \in V$ be the source node that wishes to transmit to destination $v_d \in N, v_d \neq v_s$. Further, let $\Gamma_O(v_i) \triangleq \{(v_i, v_j) | (v_i, v_j) \in A\}$ be the set of edges that leave node v_i. Correspondingly, $\Gamma_I(v_j) \triangleq \{(v_i, v_j) | (v_i, v_j) \in A\}$ is the set of edges incident on v_j. Hence, the out-degree and in-degree of any node v_j are $\delta_O(v_j) \triangleq |\Gamma_O(v_j)|$ and $\delta_I(v_j) \triangleq |\Gamma_I(v_j)|$, respectively.

Also, we model varying power constraints at various transmitters in the network by varying the entropy of the transmitted codewords at each transmitter. Let R_i be the rate at which $v_i \in V$ can inject packets. Then, we consider all codes to be subsets of the field $\mathbb{F}_q$ for any $\log q \geq \max_i\{R_i\}$. Thus, each codeword X_i transmitted by $v_i \in V$ must be an element of a subfield of $\mathbb{F}_q$, such that $H(X_i) = R_i$, $R_i \leq \log q$.

We assume that independent erasures occur at each receiver in the network. The broadcast nature of the wireless channel is modelled by constraining all outbound edges $\Gamma_O(v_i)$ to carry the same symbol $X_i \in \mathbb{F}_q$. After [5], [6], we model interference as addition in the field $\mathbb{F}_q$ as follows. Consider the simplest multiple access network where two wireless nodes v_i, v_j transmit simultaneously to receiver v_r such that $\Gamma_I(v_r) = \{(v_i, v_r), (v_j, v_r)\}$. Let $X_i, X_j \in \mathbb{F}_q$ be the codewords transmitted by v_i and v_j, respectively. Then, v_r receives $X_i + X_j$, where addition is in $\mathbb{F}_q$, with probability $1 - \epsilon_r$, and the erasure symbol $\mathcal{E}$ with probability ϵ_r. Erasure events are assumed to be independent across receivers. We also consider an extended model where we allow fading as well. In this model, each wireless link (v_i, v_j) in G has channel-gain coefficient h_{ij}, which are uniform i.i.d. over $\mathbb{F}_q$. Hence, for the simple MAC network above, v_r receives $h_{ir}X_i + h_{jr}X_j$ with probability $1 - \epsilon_r$ and $\mathcal{E}$ with probability ϵ_r and all arithmetic is performed in $\mathbb{F}_q$.

III. Asymptotically tight bounds on Capacity of WBAIN with fading

We next derive an upper bound and lower bound on the single-source unicast capacity of WBAIN. Let v_s and v_d denote the source node and the destination node, respectively. Also, let C_q denote the unicast capacity of WBAIN from v_s to v_d when operations and channel coefficients are in $\mathbb{F}_q$.

A. Upper bound

We first derive an upper bound on C_q. To do so, we first we define the following *graph transformation*. Given a heterogeneous network graph $G = (V, E)$ we define the transformation $\mathcal{T} : (V, E) \to (V', E')$ as follows. Initialize $V' = V$ and $E' = E$. For each node $v_r \in V$ that receives messages along wireless links in $\Gamma_I(v_r) \subset E$, create a node $v'_r \in V'$ and add a link (v'_r, v_r) of rate $R_r = (1 - q^{-\delta_I(v_r)}) \log q$ to E'. Set the erasure probability of edge (v'_r, v_r) as that of v_r in G, i.e., ϵ_r. Also, replace each wireless $(v_i, v_r) \in \Gamma_I(v_r)$ by an edge $(v_i, v'_r) \in E'$ with capacity R_i and erasure $1/q$ corresponding to the event that the edge may be in deep fade, viz. $h_{ij} = 0$. Each receiver v'_r in $T(G)$ receives separate signals over its incoming links, i.e., $T(G)$ has broadcast constraints but no interference.

Note that in the above, all wireline links in E can be considered to be wireless links with only one outgoing node. This capacitated broadcast erasure network $\mathcal{T}(G)$ will be referred to as the *broadcast equivalent network* (BEN) corresponding to G.

The capacity of broadcast erasure networks has been shown to be given by a *generalized* min-cut value [1], [3]. We apply these results to the BEN $\mathcal{T}(G)$ to derive an upper bound on C_q. Specifically, for a cut $(S, \bar{S})$ in $\mathcal{T}(G)$, we define its value $V_{\mathcal{T}(G)}(S)$ as

$$V_{\mathcal{T}(G)}(S) \triangleq \sum_{\{i: v_i \in S, v_j \in \bar{S}, (v_i, v_j) \in E'\}} r_i \left(1 - \prod_{j \in \Gamma_O(v_i)} \epsilon_{ij}\right).$$

Theorem 1: The unicast capacity from source v_s to destination v_d in the directed acyclic network G consisting of links that are subject to broadcast and additive interference over the finite field $\mathbb{F}_q$, C_q, is upper bounded by $\bar{C}_q$, where

$$\bar{C}_q = \min_{(S, \bar{S}) \in \mathcal{S}(s,d)} V_{\mathcal{T}(G)}(S)$$

is the min-cut max-flow capacity of the BEN $\mathcal{T}(G)$, with $V_{\mathcal{T}(G)}(S)$ being the cut value for cut S.

We note that the capacity of a similarly constructed BEN provides an upper bound for the non-fading case as well.

B. Achievability

We next lower bound the unicast capacity of WBAIN with i.i.d. and uniform fading by constructing a coding strategy to achieve a rate of $R_q \triangleq \bar{C}_q(1 - \delta)(1 - O(1/q))$, for any $\delta > 0$ in the network. For brevity, we discuss the results for networks formed entirely of wireless links; they also hold for wireless/wireline heterogeneous networks in general.

To do so, we consider the BEN $\mathcal{T}(G)$ and derive the min-cut max-flow allocation on it using the approach of Lun et. al [3]. Then, it suffices to demonstrate that for any cut in the BEN, the WBAIN achieves the same flow rates at each of the input and output nodes with a loss of at most $O(1/q)$ from the corresponding flow on the BEN.

More specifically, we derive a flow allocation f_P over each s-d path P, $P = 1, 2, \ldots, \Lambda$ where Λ is the total number of s-d paths in $\mathcal{T}(G)$. Using the conformal realization theorem, for any $\delta > 0$, there exists a flow allocation [7] that satisfies

$$\bar{C}_q(1-\delta) = \sum_p f_P \quad (1)$$

$$\sum_{P:v_i \in f_P} f_P/(1-\epsilon_{\eta(v_i,P)}) \leq R_i(1-\delta)$$

where $\eta(v_i, P)$ is the next hop from i on path P.

We employ the same coding scheme as Lun et al. [3], [7]. Consider a coding epoch of Δ time units. Suppose that the source gets message packets at rate $\bar{C}_q(1-\delta)$. Given a collection of messages $\{a_1, a_2, \ldots, a_m\}$, we define Random Linear Combining (RLC) of these messages by $RLC(\{a_1, a_2, \ldots, a_m\}) \triangleq \sum_{i=1}^m \alpha_i a_i$ where each $\alpha_i, a_i \in \mathbb{F}_q$ and α_i's are chosen uniformly i.i.d. from $\mathbb{F}_q$. The source now generates such RLCs, and injects these RLCs at a Bernoulli process at total rate $\sum_P f_P/(1-\epsilon_{\eta(v_s,P))})$, where the rate on each path P is $f_P/(1-\epsilon_{\eta(v_s,P))})$. Similarly, each node v_i injects RLCs of its received messages at rate $\sum_{P:v_i\in f_P} f_P/(1-\epsilon_{\eta(v_i,P)})$.

Since each coded packet x is ultimately an RLC of the $a_i's$, we can express $x = \sum_{i=1}^m \beta_i a_i$ where $\beta_i \in \mathbb{F}_q$. This vector $\beta = (\beta_i)_{i=1}^m$ is called the *auxiliary encoding vector* for packet x. We can now think of each node in the network to be forwarding *innovative packets* (i.e. new linear combinations of messages that were not in the span of the existing codewords at each receiver) and hence, as done in [3], [7], it suffices to track the flow of innovative packets through the network.

In the following, we will construct a scheme whereby the flow of innovation $\tilde{f}$ over the edges of the WBAIN will be (up to a difference of $O(1/q)$) equal to the information theoretic flow f over the BEN. Specifically, for any $\delta > 0$, we define $\tilde{f}_i \triangleq f_i(1-\delta)(1-O(1/q))$, $i = 1, 2, \ldots, \Lambda$. Observe from the definition of R_q and (1) that $R_q = \sum_{i=1}^{\Lambda} \tilde{f}_i$. To formalize the notion of an innovation, we will need to specify an ordered sequence of cuts which be proceed to do in the following paragraphs.

Since G is a Directed Acyclic Graph (DAG), without loss of generality, we can arrange the nodes in topological order with $v_s = v_0$, $v_d = v_N$, and for each $(v_i, v_j) \in E$, $i < j$. Each node v_i – starting with v_0 – creates RLC's of the all the data packets that it possesses (i.e. it has received over the edges in $\Gamma_I(v_i)$ for a node $i > 0$, whereas the packets a_m in case of node v_0) and sends them out over the edges $\Gamma_O(v_i)$ to the nodes in the next topological order.

To apply a min-cut max-flow, we next analyze flow across cuts in the the network. Since the nodes are arranged topologically, it suffices to considered N cuts defined as follows: A cut partitions the DAG, and is denoted by $(S_i, \bar{S}_i)$, $i = 0, 1, \ldots, N-1$, where for each i $S_i = \{v_0, v_1, \ldots v_i\}$, and $\bar{S}_i = \{v_0, v_1, \ldots, v_N\}\backslash S_i$. Since v_s and v_d are separated by these cuts, then, we require that each of these cuts transmit innovative packets on an average of R_q per time-slot. To show this, we will induce over the sequence of partitions $(S_i, \bar{S}_i)$ and show that if $\bar{C}_q(1-\delta)(1-O(1/q))\Delta$ innovative packets appear on the nodes at the left edges of each cut over an interval of Δ time slots, then we can transfer $\bar{C}_q(1-\delta)(1-O(1/q))$ packets to the right edge of the cut in Δ time-slots. Now, we can simply pipeline the packets to see that a steady state rate of $\bar{C}_q(1-\delta)(1-O(1/q))$ packets between source v_s and destination v_d is achievable.

Let $S_{v_j}(\tau)$ be the subspace formed by the auxiliary encoding vectors at node v_j up to (and not including) time τ. Given a cut, we can now formally associate the notion of an innovative packet as follows.

Definition 1: Innovative Packet: Consider a cut $(S, \bar{S})$. Suppose a packet x with auxiliary encoding vector β is received by node $v_j \in \bar{S}$ at time τ. Let $U_\tau(v_j) \triangleq \bigcup_{k\in\bar{S},k<j} \text{span}(S_{v_k}(\infty)) \cup \bigcup_{k\geq j} \text{span}(S_{v_k}(\tau))$. Then, x is *innovative* across cut $(S, \bar{S})$ if $\beta \notin U_\tau(v_j)$.

Further, observe that the rate of information across any cut in the WBAIN G is subject to the fading occurring at the edges that cross the cut. Due to the broadcast constraint imposed on the outgoing edges, it is also necessary for packets from each of the outgoing nodes to mix independently at the receiver nodes. For instance, if in a 2×2 cut of Figure 1, all h_{ij}'s are the same non-zero value, it can be seen that the rate across the cut is limited to $\log q$. However, if the vectors (h_{11}, h_{21}) and (h_{12}, h_{22}) are linearly independent, a rate of $2 \log q$ is achievable across the cut.

In the following lemma (which is a generalization of Lemma 1 in [4]), we will first consider the transmission of one innovation at one time-slot at a single receiver with multiple inputs. Subsequently, in Lemma 2, we will extend this result to consider innovations at multiple receivers on one side of a cut.

Lemma 1: Let each $u_i \in \Gamma_I(v_j)$ transmit messages X_i to node v_j over fading links $h_{ij} \in \mathbb{F}_q$ in a WBAIN, such that $P(h_{ij} = 0) = 1/q$. Further, let S_{u_i} be the subspace spanned by each u_i and S_{v_j} be the subspace spanned by v_j. Let $S_{v_j}^+$ be the subspace spanned by the linear combinations in v_j after the transmission is complete, then $P(\text{span}(S_{v_j}^+) > \text{span}(S_{v_j}) | \bigcup_{u_i\in\Gamma_I(v_j)} S_{u_i} \nsubseteq S_{v_j}) \geq (1 - \frac{2}{q})$.

To extend this result to a cut with more than one receiver on the right edge of the cut, we consider once again, the topological ordering of the nodes. For a cut $(S, \bar{S})$ with nodes u_i, $i = 1, 2, \ldots m$ on the left edge of the cut and v_j, $j = 1, 2, \ldots, n$ on the right edge of the cut, consider the transfer matrix $H = \{h_{ij}\}_{i=1 j=1}^{m,n}$ where h_{ij} are the channel coefficients chosen uniformly at random from $\mathbb{F}_q$ if $(u_i, v_j) \in E$, $h_{ij} = 0$ otherwise.

Note that since H is a random matrix whose elements change with each time-slot τ, rank$(H(\tau))$ is a random variable. Further, for any fixed τ let us define $\mathcal{H}$ as the event that for each $(u_i, v_j) \in E$, $h_{ij} \neq 0$.

Lemma 2: Consider a cut $(S, \bar{S})$ in G, with nodes labelled u_i, $i = 1, 2, \ldots, m$ on the left edge of the cut and nodes v_j, $j = 1, 2, \ldots, n$ on the right edge such that $i < l$ if u_i appears before u_l topologically, or v_i appears before v_l topologically (clearly all u_i's are topologically ordered before any v_j).

Further, let $\mathcal{B}$ denote the event that innovative packets are present at some distinct m_0 among the u_i's which are destined for m_0 distinct v_j's on the right of the cut. Formally, let us define the event $\mathcal{B}$ as follows: There exists a $m_0 > 0$, (with $m_0 \leq \min(m,n)$), *topologically ordered* collection of indices $\mathcal{J} = \{j_1, j_2, \ldots, j_{m_0}\}$, and the corresponding collection nodes $\{v_{j_1}, v_{j_2}, \ldots, v_{j_{m_0}}\}$, along with an (unordered) index set $\mathcal{K} = \{k_1, k_2, \ldots, k_{m_0}\}$, and the corresponding distinct collection of nodes $U_S = \{u_{k_1}, u_{k_2}, \ldots, u_{k_{m_0}}\}$, such that *(i)* $u_{k_l} \in \Gamma_I(v_{j_l})$, for all $l = 1, 2, \ldots, m_0$, and *(ii)* $S_{u_{k_l}} \not\subseteq \left[\bigcup_{x<l} S^+_{v_{j_x}}\right] \bigcup \left[\bigcup_{x \geq l} S_{v_{j_k}}\right]$ for all $l = 1, 2, \ldots, m_0$.

Then, we have $P(\bigcap_{l=1}^{m_0}\{v_{j_l}$ receives an innovative packet after transmission$\}|\mathcal{B}\cap\mathcal{H}) \geq 1 - O(1/q)$.

Sketch of Proof: Let us renumber the nodes $\{u_l, l = 1, 2, \ldots, m\}$ as $\{u_{k_1}, u_{k_2}, \ldots, u_{k_{m_0}}, u_{k_{m_0+1}}, \ldots, u_{k_m}\}$, where the first m_0 nodes correspond to those in U_S and the rest are arbitrarily assigned indices. Let each u_{k_i} have auxiliary coefficient vectors $g_w^{(i)}$, $w = 1, 2, \ldots l_i$. Of these, for each $u_{k_i} \in U_S$, let $\tilde{l}_i$ be the number of coefficient vectors that are innovative and not in span$(\bigcup_{l=1}^{m_0} v_{j_l})$, and let the corresponding coefficient vectors be $g_{w,\perp}^{(i)}$, $w = 1, 2, \ldots \tilde{l}_i$. Similarly, let $g_{w,\|}^{(i)}$, $w = 1, 2, \ldots, l_i^{\|}$ be the codeword vectors that are not innovative. Then we can write the codeword X_i broadcast by node $u_{k_i} \in U_S$ as $X_i = X_{i,\perp} + X_{i,\|}$ where $X_{i,\perp} \triangleq \sum_{w=1}^{\tilde{l}_i} \alpha_{i,w} g_{w,\perp}^{(i)}$ for coefficients $\alpha_{i,w} g_{w,\perp}^{(i)}$ chosen i.i.d. with uniform distribution on $\mathbb{F}_q$. It can be shown that

$$P(X_{i,\perp} > 0|\mathcal{H}\cap\mathcal{B}) > 1 - O(1/q^{\tilde{l}_i}) > 1 - O(1/q) \quad (2)$$

since $\tilde{l}_i \geq 1$ for all $u_{k_i} \in U_S$. Further, since by the definition of innovation, $g_{w,\perp}^{(i)}$ are different for $u_{k_i} \neq u_{k_j}, u_{k_i}, u_{k_j} \in U_S$, each $X_{i,\perp} \perp X_{j,\perp}$.

Recall that the nodes $\{v_{j_1}, v_{j_2}, \ldots, v_{j_{m_0}}\}$ are topologically ordered. We note that for the case of $\Gamma_I(v_{j_1})$ transmitting and additively interfering at v_{j_1}, conditioned on $\mathcal{H}\cap\mathcal{B}$, the probability that the auxiliary coefficient vector received at v_{j_1}, $Y_{j_1} \notin$ span$(S_{v_{j_1}})$ can be bounded below by $1 - O(1/q)$ by using Lemma 1.

Similarly, by considering the additive interference MAC from $\Gamma_I(v_{j_2})$ to v_{j_2} and using Lemma 1, $P(Y_{j_2} \notin$ span$(v_{j_2})) = 1 - O(1/q)$. However, by our definition of innovation in Definition 1, Y_{j_2} will be innovative at v_{j_2} only if $Y_{j_2} \notin$ span$(S_{v_{j_2}} \cup S^+_{v_{j_1}})$. Since span$(S^+_{v_{j_1}}) =$ span$(S_{v_{j_1}}) \cup Y_{j_1}$, we also require that $Y_{j_1} \perp Y_{j_2}$.

Let us define $Y_{j,\perp} \triangleq \sum_{i\in\Gamma_I(v_j)} h_{ij} X_{i,\perp}$. Then, it suffices to show that $Y_{j_1,\perp} \perp Y_{j_2,\perp}$. Conditioning on the event $\mathcal{H}\cap\mathcal{B}\cap\mathcal{A}$, where $\mathcal{A} \triangleq \{X_{i,\perp} > 0, u_{k_i} \in U_S\}$,

$$\begin{aligned} &P\left(Y_{j_1,\perp} \perp Y_{j_2,\perp}|\mathcal{H}\cap\mathcal{B}\right) \\ \geq\ & P(Y_{j_1,\perp} \perp Y_{j_2,\perp}|\mathcal{H}\cap\mathcal{B}\cap\mathcal{A}))P(\mathcal{A}|\mathcal{H}\cap\mathcal{B}) \\ =\ & P(Y_{j_1,\perp} \perp Y_{j_2,\perp}|\mathcal{H}\cap\mathcal{B}\cap\mathcal{A})(1 - O(1/q)) \end{aligned}$$

where the last relation follows from (2).

Also, since $X_{i,\perp} \perp X_{k,\perp}$, therefore conditioned on $\mathcal{H}\cap\mathcal{B}\cap\mathcal{A}$, $Y_{j_1,\perp} \perp Y_{j_2,\perp}$ if and only if the row vectors $(h_{i,j_1})_{i\in\Gamma_I(v_{j_1})}$ and $(h_{i,j_2})_{i\in\Gamma_I(v_{j_2})}$ of H are linearly independent. Using arguments similar to [4], [3] this can be shown to occur with probability $1 - O(1/q)$. Thus $P(Y_{j_1,\perp} \perp Y_{j_2,\perp}|\mathcal{H}\cap\mathcal{B}) \geq (1 - O(1/q))^2 = 1 - O(1/q)$. Therefore,

$$\begin{aligned} &P\left(Y_{j_2}\text{is not innovative at } v_{j_2}|\mathcal{H}\cap\mathcal{B}\right) \\ =\ & P\big(\{Y_{j_2} \in S_{v_{j_2}}\} \cup \{Y_{j_1} \perp Y_{j_2}\}^c|\mathcal{H}\cap\mathcal{B}\big) \\ \leq\ & P\big(\{Y_{j_2} \in S_{v_{j_2}}\}|\mathcal{H}\cap\mathcal{B}\big) + P(\{Y_{j_1} \perp Y_{j_2}\}^c|\mathcal{H}\cap\mathcal{B}) \\ \leq\ & O(1/q). \end{aligned}$$

This immediately leads to the result that

$$P(\bigcup_{i=1}^{2}\{Y_{j_i}\text{is not innovative at } v_{j_i}\}|\mathcal{H}\cap\mathcal{B}) \quad \leq \quad O(1/q)$$

We similarly progress inductively over the right edge by considering v_{j_3} where the additional requirement is that $Y_{j_3,\perp} \perp \{Y_{j_1,\perp}, Y_{j_2,\perp}\}$ and so on for all v_{j_k}, $k = 4, 5, \ldots, m_0$. This leads the desired result. ∎

Observe that in Lemma 2, in the presence of sufficient innovation rate at the left edge of the cut, the achievable rate across a cut is conditioned on the event $\mathcal{H}$.

Also, note that in the limit of large field $\mathbb{F}_q$, the probability that the transfer matrix $H = \{h_{ij}\}_{i=1,j=1}^{m,n}$ will have full rank tends to 1.

Proposition 1: $P(\mathcal{H}) = 1 - O(1/q)$.

Therefore, unconditioning on $\mathcal{H}$ and using the union bound, Proposition 1 implies that any innovative packet arriving at the right edge of a cut will be lost with probability at most $O(1/q)$.

Theorem 2: The rate R_q can be achieved over WBAIN with i.i.d. and uniform fading.

The following corollary now follows immediately from the above theorem and the definition of R_q and noting that $\sum_{i=1}^{\Lambda} f_i = \bar{C}_q(1-\delta)$ due to the max-flow min-cut theorem.

Corollary 1: (i) $R_q \leq C_q \leq \bar{C}_q$
(ii) $(\bar{C}_q - C_q)/\bar{C}_q \leq O(1/q)$.

We now provide a sketch of the proof for Theorem 2.

Let us define the sum-flow into a node $v_i \in V'$ in the BEN $\mathcal{T}(G)$ as $f_i^* \triangleq \sum_{P:(v_i',v_i)\in P} f_P$. For the corresponding node $v_i \in V$, let $\tilde{f}_i^* \triangleq \sum_{P:(v_i',v_i)\in P} \tilde{f}_P$. be the total innovative flow arriving at v_i.

Now using the per-time-slot innovation arguments from Lemma 2, we induce over the sequence of cuts $(S_i, \bar{S}_i)$. At each cut, we compare the net flow flows arriving at each node on the right edge of $(S_i, \bar{S}_i)$, with the net innovative flow arriving at the corresponding node on G, and in the following show that at each node, difference is $O(1/q)$.

Lemma 3: Given any cut $(S_m, \bar{S}_m)$, $m = 0, 1, 2, \ldots, N-1$ in the BEN $\mathcal{T}(G)$, let f_i^* be the flow arriving at each $v_i \in S$ and let f_j^* be the flow arriving at each $v_j \in \bar{S}_m$ under the min-cut max-flow allocation f on $\mathcal{T}(G)$. Then, for the same cut in the WBAIN, given that innovative packets are transmitted at rate $\tilde{f}_i^* = f_i^*(1-O(1/q))$ for all $v_i \in S$, each node $v_j \in \bar{S}$ receives the innovation rate of $\tilde{f}_j^*$ where $\tilde{f}_j^* = f_j^*(1-O(1/q))$.

This implies that if $\bar{C}_q(1-\delta)$ is the total flow supported across the cut $(S, \bar{S})$ in the BEN $\mathcal{T}(G)$, the corresponding cut in the WBAIN will support an innovative flow of $R_q = \bar{C}_q(1-\delta)(1-O(1/q))$.

Sketch of Proof: We will induce over the sequence $(S_r, \bar{S}_r)$, $r = 0, 1, 2, \ldots, N-1$.

Step 1 Consider cut $(S_0, \bar{S}_0)$, $S_0 = \{v_s\}$, $\bar{S}_0 = (V \backslash v_s)$ in the BEN $\mathcal{T}(G)$. Note that a rate of R_q is supported by flow f in $\mathcal{T}(G)$. Let flows f_j be directed along each (v_s, v_j), $\forall v_j \in \Gamma_O(v_s)$, such that $\sum f_j = \bar{C}_q$. This is a pure broadcast case with no interference. Thus, for the cut $(S_0, \bar{S}_0)$, the WBAIN performs exactly like the BEN with an additional $1/q$ erasure on each edge to account for the time-slots when $h_{sj} = 0$. Hence, the input flows are reduced by a $(1-O(1/q))$ fraction at each $v_i \in \Gamma_O(v_s)$ in the WBAIN. Thus $\tilde{f}_i^* = f_i^*(1-O(1/q))$.

Step 2 For cuts $(S_r, \bar{S}_r)$, $r \geq 1$, let f_i^* be the flow into a node v_i on the left edge of the cut as part of the max-flow min-cut solution over $\mathcal{T}(G)$, and let $\tilde{f}_i^*$ be the corresponding flow into a node u_i in G such that $\tilde{f}_i^* = f_i^*(1-O(1/q))$. We will denote nodes on the left edge by u_i, $i = 1, 2, \ldots, m$ and on the right edge by v_j, $j = 1, 2, \ldots, n$.

Now, from the conformal realization theorem [7] applied to $\mathcal{T}(G)$, we arrive at a schedule $\Sigma(\tau)$, $\tau = 1, 2 \ldots, \Delta$ for cut $(S_r, \bar{S}_r)$, where new packets arrive at $m_0(\tau)$ nodes at the left edge of $(S_r, \bar{S}_r)$ and an appropriate set of $m_1(\tau)$ nodes on the right receive a new packet, such that if $\{f_i^*\}$ are the average rates of flow to nodes $\{u_i\}$ on the left edge of $(S_r, \bar{S}_r)$, then each v_j on the right edge of the cut will receive an average flow of f_j^*.

Employing a probabilistic coupling, we arrive at a schedule $\Sigma'(\tau)$ for the WBAIN. Next applying Lemma 2 and Proposition 1 at each time instant $\tau = 1, 2, \ldots, \Delta$ it can be shown that the net average rates of flow at each v_j on the right edge of the cut will be $\tilde{f}_j^*(1-O(1/q))$.

Recall from our hypothesis that $\tilde{f}_i^* = f_i^*(1-O(1/q))$ at each u_i. Thus the net average rates of flow at the right edge will be thinned by another $1-O(1/q)$ factor resulting in a net average flow rate of $\tilde{f}_j^*(1-O(1/q))^2 = \tilde{f}_j^*(1-O(1/q))$. The induction hypothesis is thus proved. ∎

To prove Theorem 2, observe that since f is a feasible and optimal (max) flow on $\mathcal{T}(G)$ and we have from the Lemma 3 that the rate across any cut in $\mathcal{T}(G)$ can be achieved across the corresponding cut in the WBAIN G.

Hence, if $\bar{C}_q(1-\delta)$ is the max-flow min-cut capacity achievable in $\mathcal{T}(G)$, a flow rate of $R_q = \bar{C}_q(1-\delta)(1-O(1/q))$ is achievable in the WBAIN G.

IV. Capacity Gain due to Fading

We illustrate the gain in network capacity due to fading diversity by analyzing the capacity of the heterogenous network given in Figure 2 under fading and non-fading cases. Specifically, we compare the unicast capacity from S_1 to D_1 under fading with an upper bound for the non-fading case. The source S_1 is connected to each of its outgoing nodes with wireline links of rate R_1, the nodes on the left edge of the cut

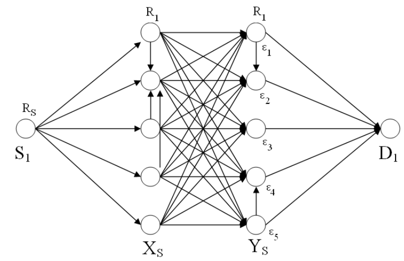

Fig. 2. Capacity across the cut in the DAG above, $R_S = 10R_1 < \log q$ Nodes are labelled with erasure probabilities ϵ_i.

$(S, \bar{S})$ transmit over wireless links to the nodes on the cut's right edge, and the latter transmit to D_1 over wireline links, each of rate R_1.

Suppose that R_1 and q are such that the cut $(S, \bar{S})$ is the bottleneck cut (for instance, $R_1 = \log q$). Then, from Corollary 1, the capacity of the unicast from S_1 to D_1 under uniform i.i.d. fading is $R_1 \sum_{i=1}^{5}(1-\epsilon_i)(1-O(1/q))$. In contrast, if the links crossing the cut have no fading, direct computation yields the capacity of the cut to be $R_1(1-\prod_{j=1}^{5}\epsilon_i)$.

Thus, for small erasure probabilities, approximately 5-fold increase in the capacity is afforded by fading diversity in the example network. Clearly, gains will be higher for graphs with larger bottleneck bipartite subgraphs embedded in them.

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